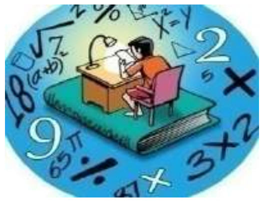


DIAMOND

MATHEMATICS

PUC II YEAR

CHAPTERWISE MOST PROBABLE QUESTIONS WITH ANSWERS

SPECIAL FEATURES:

1. CHAPTER WISE DETALED IMPORTANT QUESTIONS AND SOLUTIONS IN ORDER.
2. CHAPTERWISE DETAILED SOLUTIONS OF PREVIOUS ANNUAL AND SUPPLEMENTARY EXAMINTION QUESTIONS.
3. SYLLABUS BLUE PRINT WITH QUESTION PAPER DESIGN.
4. DESIGNED TO HELP STUDENTS TO SCORE HIGEST MARKS IN MATHEMATICS.
5. PROVIDES ESSENTIAL QUESTIONS FOR PRACTICE TO BUILD CONFIDENCE.
6. STEP-BY-STEP SOLUTIONS PROVIDED FOR COMPLETE AND QUICK REFERENCE FOR SELF ASSESSMENT.

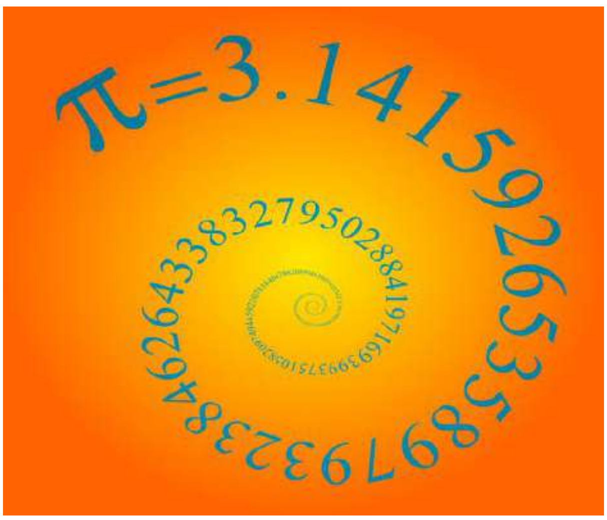
ABOVE 95
SCORE
GUARANTED

2021-2022

OVER 720
SOLVED
PROBLEMS

Rajesh Gull
M.Sc., B.Ed.

K. Sainath
M.Sc.



PREPARED BY:

1. RAJESH V. GULL M.Sc., B.Ed. Lecturer in Mathematics, BLDEA's P.U. College, Jamkhandi & Basavajyoti P.U. College, Jamkhandi. (9742356643)

2. K. Sainath M.Sc. Lecturer in Mathematics, BLDEA's P.U. College, Jamkhandi. (9491007234)

Special Note:

1. These are the most Important likely questions for Annual Examination. Question number may subject to change in the Annual Question paper but they may appear with different question numbers on the same concept. If you practice all the questions given in this book, you will surely score **90 to 100** marks in Annual Examination.

2. If any mistakes are their please suggest us.

DESIGN OF QUESTION PAPER

Sl.No	Chapter	Marks
1	Relations and Functions	11(1+2+3+5)
2	Inverse Trigonometric Functions	8(1+2+2+3)
3	Matrices	9(1+3+5)
4	Determinants	12(1+2+4+5)
5	Continuity and Differentiability	20(1+2+2+3+3+4+5)
6	Applications of derivative	10(2+3+5)
7	Integrals	22(1+2+2+3+3+5+6)
8	Application of integrals	8(3+5)
9	Differential Equations	10(2+3+5)
10	Vectors	11(1+2+2+3+3)
11	3-Dimensional Geometry	11(1+2+3+5)
12	Linear Programming Problem	7(1+6)
13	Probability	11(1+2+3+5)

SCHEMATIC REPRESENTATION OF QUESTION PAPER PATTERN

PART-A
Answer all 10 Questions
$1\times10=10M$

1. Definitions, problems on Relation or function or binary operation.
2. Domain or Range of I.T.Fs or Problems on I.T.F.
3. Definition of Types of matrices or simple problems.
4. Determinant problem.
5. Differentiation of Composite Functions or Continuity.
6. Indefinite integral problem.
7. Vectors definitions or Vector problems.
8. 3D problems
9. Definitions of LPP
10. Conditional probability problem.

PART-B
Answer any 10Q out of 14Q
$2\times10=20M$

11. Relation & Functions problems or Binary operation problem.
12. Inverse Trigonometric Functions (I.T.F) problems.
13. Inverse Trigonometric Functions (I.T.F) problems.
14. Area of triangle, find k or equation of line by using determinant.
15. Differentiation of Implicit functions.
16. Differentiation of Logarithmic functions.
17. Application of Derivative Problems.
18. Indefinite integral problems.
19. Definite integral problems.
20. Differential equation problems.
21. Vector Problems.
22. Vector Problems.
23. 3-D geometry problems
24. Probability problems.

PART-C
Answer any 10Q out of 14Q
$3\times10=30M$

25. Function problems.
26. I.T.F Problems.
27. Finding A^{-1} by elem. transformation, express matrix as a sum of symmetric and skew symmetric or basic proofs.
28. Parametric differentiation problems.
29. Rolle's or mean value theorem (MVT) problems.
30. Application of derivative problems.
31. Integration or definite integral problems.
32. Integration or definite integral problems.
33. Area bounded by the curve problems.
34. Differential equation problems.
35. Vector Problems.
36. Vector Problems.
37. 3-D geometry problems
38. Probability problems(using Baye's formula)

PART-D
Answer any 6Q out of 10Q
$5\times6=30M$

39. Function problems.
40. Matrix problems.
41. Solving system of equation by matrix method.
42. Second derivative problems.
43. Application of derivative problems.
44. Standard integrals proof & Problems.
45. Problem on area of the region bounded by the curves.
46. Solution of linear differential equation.
47. 3-D geometry standard theorems proofs.
48. Probability Problems.

PART-E
Answer any 1Q out of 2Q
$10\times1=10M$

49. (a) Linear programming problem (LPP). (6M)
    (b) Problems on properties of determinant. (4M)
50. (a) Proof on properties of definite integrals & related problems. (6M)
    (b) Continuity Problems. (4M)

BLUE PRINT

Sl.No	CHAPTER	PART-A	PART-B	PART-C	PART-D	PART-E		TOTAL MARKS
		1-mark	2-mark	3-mark	5-mark	6 M	4 M	
1	Relation & Functions	√	√	√	√	-----	-----	11
2	Inverse Trigonometric Functions	√	√√	√	------	-----	-----	8
3	Matrices	√	------	√	√	-----	-----	9
4	Determinants	√	√	------	√	-----	√	12
5	Continuity And Differentiability	√	√√	√√	√	-----	√	20
6	Application of Derivatives	------	√	√	√	-----	-----	10
7	Integrals	√	√√	√√	√	√	-----	22
8	Applications of Integrals	------	-------	√	√	-----	-----	08
9	Differential equations	------	√	√	√	-----	-----	10
10	Vector algebra	√	√√	√√	-----	-----	-----	11
11	Three Dimensional Geometry	√	√	√	√	-----	-----	11
12	Linear Programming Problems	√	-----	-----	-----	√	-----	07
13	Probability	√	√	√	√	-----	-----	01
	Total possible questions	10	14	14	10	2	2	150

TARGET 100/100 MARKS IN MATHEMATICS

Sl.No	CHAPTER	PART-A	PART-B	PART-C	PART-D	PART-E		TOTAL MARKS
		1-Mark	2-Mark	3-Mark	5-Mark	6 M	4 M	
1	Relation & Functions	√	√	√	√	-----	-----	11
2	Inverse Trigonometric Functions	√	√√	√	------	-----	-----	8
3	Matrices	√	------	√	√	-----	-----	9
4	Determinants	√	√	------	√	-----	√	12
5	Continuity And Differentiability	√	√√	√√	√	-----	√	20
6	Application of Derivatives	------	X	√	√	-----	-----	08
7	Integrals	√	√√	√√	√	√	-----	22
8	Applications of Integrals	------	-------	X	X	-----	-----	00
9	Differential equations	------	√	√	√	-----	-----	10
10	Vector algebra	√	√√	√√	-----	-----	-----	11
11	Three Dimensional Geometry	√	√	√	√	-----	-----	11
12	Linear Programming Problems	√	-----	-----	-----	√	-----	07
13	Probability	√	X	X	X	-----	-----	01
	Total possible questions	10	12	12	8	2	2	130
	Target	10	10	10	6	1	1	100

TARGET 90 ABOVE MARKS IN MATHEMATICS

Sl.No	CHAPTER	PART-A	PART-B	PART-C	PART-D	PART-E		TOTAL MARKS
		1-Mark	2-Mark	3-Mark	5-Mark	6 M	4 M	
1	Relation & Functions	√	√	√	√	-----	-----	11
2	Inverse Trigonometric Functions	√	√√	√	------	-----	-----	8
3	Matrices	√	------	√	√	-----	-----	9
4	Determinants	√	√	------	√	-----	√	12
5	Continuity And Differentiability	√	√√	√√	√	-----	√	20
6	Application of Derivatives	------	X	X	√	-----	-----	05
7	Integrals	√	X	X	√	√	-----	12
8	Applications of Integrals	------	-------	X	X	-----	-----	00
9	Differential equations	------	√	X	X	-----	-----	02
10	Vector algebra	√	√√	√√	-----	-----	-----	11
11	Three Dimensional Geometry	√	√	√	√	-----	-----	11
12	Linear Programming Problems	√	-----	-----	-----	√	-----	07
13	Probability	√	X	X	X	-----	-----	01
	Total possible questions	10	10	8	7	2	2	109
	Target	10	10	8	6	1	1	094

TARGET 70 TO 90 MARKS IN MATHEMATICS

Sl.No	CHAPTER	PART-A	PART-B	PART-C	PART-D	PART-E		TOTAL MARKS
		1-Mark	2-Mark	3-Mark	5-Mark	6 M	4 M	
1	Relation & Functions	√	√	√	√	-----	-----	11
2	Inverse Trigonometric Functions	√	√√	√	------	-----	-----	8
3	Matrices	√	------	√	√	-----	-----	9
4	Determinants	√	√	------	√	-----	√	12
5	Continuity And Differentiability	√	√√	√√	√	-----	√	20
6	Application of Derivatives	------	X	X	√	-----	-----	05
7	Integrals	X	X	X	X	√	-----	06
8	Applications of Integrals	------	-------	X	X	-----	-----	00
9	Differential equations	------	√	X	X	-----	-----	02
10	Vector algebra	√	√√	√√	-----	-----	-----	11
11	Three Dimensional Geometry	√	X	X	√	-----	-----	06
12	Linear Programming Problems	√	-----	-----	-----	√	-----	07
13	Probability	0	X	X	X	-----	-----	00
	Total possible questions	08	09	7	6	2	2	097
	Target	08	09	7	6	1	1	087

TARGET 50 TO 70 MARKS IN MATHEMATICS

Sl.No	CHAPTER	PART-A	PART-B	PART-C	PART-D	PART-E		TOTAL MARKS
		1-Mark	2-Mark	3-Mark	5-Mark	6 M	4 M	
1	Relation & Functions	√	X	X	√	-----	-----	06
2	Inverse Trigonometric Functions	√	√√	√	------	-----	-----	8
3	Matrices	√	------	√	√	-----	-----	9
4	Determinants	√	√	------	√	-----	√	12
5	Continuity And Differentiability	√	√√	√√	√	-----	√	20
6	Application of Derivatives	------	X	X	√	-----	-----	05
7	Integrals	X	X	X	X	X	-----	00
8	Applications of Integrals	------	-------	X	X	-----	-----	00
9	Differential equations	------	√	X	X	-----	-----	02
10	Vector algebra	√	√√	√√	-----	-----	-----	11
11	Three Dimensional Geometry	√	X	X	√	-----	-----	06
12	Linear Programming Problems	√	-----	-----	-----	√	-----	07
13	Probability	X	X	X	X	-----	-----	00
	Total possible questions	08	08	06	6	1	2	086
	Target	08	06	05	5	1	1	070

TARGET PASS MARKS IN MATHEMATICS

Sl.No	CHAPTER	PART-A	PART-B	PART-C	PART-D	PART-E		TOTAL MARKS
		1-Mark	2-Mark	3-Mark	5-Mark	6 M	4 M	
1	Relation & Functions	X	X	X	√	-----	-----	05
2	Inverse Trigonometric Functions	X	√√	√	------	-----	-----	07
3	Matrices	X	------	√	√	-----	-----	8
4	Determinants	X	√	------	√	-----	X	07
5	Continuity And Differentiability	X	√√	√√	√	-----	X	15
6	Application of Derivatives	------	X	X	√	-----	-----	05
7	Integrals	X	X	X	X	X	-----	00
8	Applications of Integrals	------	-------	X	X	-----	-----	00
9	Differential equations	------	√	X	X	-----	-----	02
10	Vector algebra	X	√√	√√	-----	-----	-----	10
11	Three Dimensional Geometry	X	X	X	√	-----	-----	05
12	Linear Programming Problems	X	-----	-----	-----	√	-----	06
13	Probability	X	X	X	X	-----	-----	00
	Total possible questions	00	08	06	6	1	0	070
	Target	00	06	05	5	1	0	058

UNIT-1
RELATIONS AND FUNCTIONS

$\left[Total\ marks: 11,\ Q.No-1(1M),\ 11(2M)\ 25(3M)\ \&39(5M)\right]$

Question No: 1 (1M)

1. Define a reflexive relation on a set.
Ans: A relation R in a set A is called reflexive, if $(a,a)\in R,$ for every $a\in A$.

2. Define symmetric relation.
Ans: A relation R in a set A is called symmetric, if $(a,b)\in R \Rightarrow (b,a)\in R$ for every $a,b\in A$.

3. Define Transitive relation.
Ans: A relation R in a set A is called transitive, if $(a,b)\in R \& (b,c)\in R \Rightarrow (a,c)\in R$ for every $a,b,c\in A$

4. Define equivalence relation.
Ans: A relation R is said to be equivalence relation if R is reflexive, symmetric and transitive.

5. Define one-one (Injective) function.
Ans: A function $f: X\to Y$ is said to be one-one (injective), if the image of distinct element of X under f are distinct, i.e., for every $x_1,x_2\in X, f(x_1)=f(x_2)\Rightarrow x_1=x_2$.

6. Define an onto (Surjective) function.
Ans: A function $f: X\to Y$ is said to be onto (Surjective), if every element of Y is the image of some element of X under f, i.e., for every $y\in Y$ there exist an element $x\in X$ such that $f(x)=y$.

7. Define bijective function. **(2018-M)**
Ans: A function $f: X\to Y$ is said to be bijective, if f is one-one and onto.

8. Define Binary Operation on a set.
Ans: A binary operation * on set A is a function $*: A\times A\to A$. We denoted by a*b.

9. A relation R on $A=\{1,2,3\}$ defined by $R=\{(1,1),(1,2),(3,3)\}$ is not symmetric why?
Ans: R is not symmetric because $(1,2)\in R\ but\ (2,1)\notin R$ **(2014-M)**

10. A relation R on $A=\{1,2,3\}$ defined by $R=\{(1,1),(2,2),(3,3)\}$ is reflexive?
Ans: $(1,1)\in R,(2,2)\in R,(3,3)\in R$ So R is reflexive

11. Examine whether the operation is binary or not on Z defined by $a*b=a+ab$
Ans: Let $a,b\in Z$ then $a*b=a+ab\in Z$
$(\because 7,-1\in Z \text{ then } 7*(-1)=7+7(-1)=0\in Z)$
$\therefore *$ is binary operation.

12. Examine whether the operation is binary or not on Z^+ defined by $a*b=a+ab$
Ans: Let $a,b\in Z^+$ then $a*b=a+ab\in Z^+$
$(\because 7,1\in Z^+ \text{ then } 7*1=7+7(1)=14\in Z^+)$,
$\therefore *$ is binary operation.

13. Examine whether the operation is binary or not on Z defined by $a*b=a-b$.
Ans: Let $a,b\in Z$ then $a*b=a-b\in Z$
$(\because 7,-1\in Z \text{ then } 7*(-1)=7-(-1)=8\in Z)$,
$\therefore *$ is binary operation.

14. On Z^+, the set of non-negative integers $*$ is defined by $a*b=a^b$. Is $*$ a binary operation?
Ans: Let $a,b\in Z^+$ then $a*b=a^b\in Z^+$
$(\because 7,2\in Z^+ \text{ then } 7*2=7^2=49\in Z^+)$,
$\therefore *$ is binary operation.

15. An operation $*$ on Z^+ (except of all non-negative integer) is defined as $a*b=|a-b|\ \forall a,b\in Z^+$, is $*$ is a binary operation on Z^+ ? **(2016-J)**
Ans: Let $a,b\in Z^+$ then $a*b=|a-b|\in Z^+$
$(\because 2,7\in Z^+ \text{ then } 2*7=|2-7|=5\in Z^+)$,
$\therefore *$ is binary operation.

16. Let $*$ be a binary operation on N given by $a*b=HCF$ of a & b find $22*2$
Ans: $22*2=HCF of\ 22\ \&\ 2=2$

17. Let $*$ be a binary operation on N given by $a*b=LCM$ of a & b find $20*16$.
Ans: $20*16=LCM\ of\ 20\ \&\ 16=80$ **(2017-M)**

18. Let $*$ be a binary operation defined on the set of natural number given by $a*b=LCM$ of a & b. find
(i) $5*7$ **(2015-J, 2019-J, 2020-M)** **(ii)** $7*5$

Ans: (i) $5*7 = LCM\ of\ 5\&7 = 35$

(ii) $7*5 = LCM\ of\ 7\&5 = 35$

19. Let $*$ be a binary operation defined on the set of rational number Q defined by $a*b = ab+1$. Prove that $*$ is commutative. **(2014-J)**

Ans: Let

$a*b = ab+1 = ba+1 = b*a. \quad \therefore * \text{ is commutative}$

$(7*6 = 7(6)+1 = 43 \ \ \& \ 6*7 = 6(7)+1 = 43)$

20. Give an example of a relation which is symmetric only.

Ans: Let A= {1,2,3} , R defined on set A as

$R = \{(1,2),(2,1)\}$

21. Give an example of a relation which is reflexive, symmetric and transitive.

Ans: Let A= {1,2,3} , R defined on set A as

$R = \{(1,1),(2,2),(3,3)\}$

22. Give an example of a relation which is reflexive and symmetric but not transitive.

Ans: Let A= {1,2,3} , R defined on set A as

$R = \{(1,1),(2,2),(3,3),(2,3),(3,2),(3,1),(1,3)\}$

23. Give an example of a relation which is reflexive and transitive but not symmetric.

Ans: Let A= {1,2,3} , R defined on set A as

$R = \{(1,1),(2,2),(3,3),(2,3),(3,2),(3,1),(2,1)\}$

24. Give an example of a relation which is symmetric and transitive, but not reflexive.

Ans: Let A= {1,2,3} , R defined on set A as

$R = \{(1,1),(2,2),(2,3),(3,2)\}$

25. Give an example of a relation which is symmetric, but not transitive and not reflexive.

Ans: Let A= {1,2,3} , R defined on set A as

$R = \{(2,3),(3,2)\}$

26. Give an example of a relation which is not reflexive, neither symmetric nor transitive.

Ans: Let A= {1,2,3} , R defined on set A as

$R = \{(2,3),(3,1),(1,2)\}$

27. If $f(x) = x+7 \ \ and \ \ g(x) = x-7$ where $f(x)$ and $g(x)$ are real functions.

Find $f \circ g(x)$

Ans: $fog(x) = f(g(x)) = f(x-7) = x-7+7 = x$

$\therefore fog(x) = x$

28. On N Define * by $a*b = 2^{ab}$ show that * is commutative.

Ans: Let $a*b = 2^{ab} = 2^{ba} = b*a$,

$(3*1 = 2^{3(1)} = 8 \ \ \& \ 1*3 = 2^{1(3)} = 8), \quad * \text{ iscommutative.}$

29. On Z define * by $a*b = \dfrac{ab}{3}$, show that * is associative.

Ans: Let $(a*b)*c = \left(\dfrac{ab}{3}\right)*c = \dfrac{\frac{ab}{3}c}{3} = \dfrac{abc}{9}$

$\& \ a*(b*c) = a*\left(\dfrac{bc}{3}\right) = \dfrac{a\frac{bc}{3}}{3} = \dfrac{abc}{9}$

$(a*b)*c = a*(b*c), \qquad \therefore * \text{ is associative.}$

30. On N Define * by $a*b = LCM$ of a and b, is commutative? Justify

Ans: Let $a*b = LCM\ of\ a\&b = LCM\ of\ b\&a = b*a$

$(4*2 = LCM\ of\ 4\&2 = 4, 2*4 = LCM\ of\ 2\&4 = 4)$

$\therefore * \text{ is commutative}$

31. On Q Define * by $a*b = ab^2$, verify whether * is commutative.

Ans: Let $7,2 \in Q \ then\ 7*2 = 7(2)^2 = 28$

$but\ 2*7 = 2(7)^2 = 98, \therefore 7*2 \neq 2*7,$

$\therefore *$ is not commutative

32. Show that the function $f: Z \to Z \ \ given \ \ f(x) = x^2$ is not bijective.

Ans: $f(1) = 1^2 = 1, f(-1) = (-1)^2 = 1,$

$f(1) = f(-1) \ but \ 1 \neq -1, \ \therefore f \text{ is not one} - \text{one}$

range of $f(x) = Z^+ \neq co-domain(Z),$

$\therefore \ f \text{ is not onto}$

$\therefore \ f \text{ is not bijective}$

33. Show that the function $f: Z \to Z \ \ given \ \ f(x) = 2x+3$ is one-one.

Ans: Let
$x_1, x_2 \in Z, f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3$

$\Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2, \therefore f(x)$ is $1-1$.

34. Show that the function $f: R \to R$ given $f(x) = 4x - 1$ is one-one.

Ans: Let $x_1, x_2 \in R, f(x_1) = f(x_2) \Rightarrow 4x_1 - 1 = 4x_2 - 1$
$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2, \therefore f(x)$ is $1-1$.

35. Find the number of binary operations on a set $\{a,b\}$.

Ans: A $*$ is binary operation on a set $\{a,b\}$ is a function from $\{a,b\} \times \{a,b\} \to \{a,b\}$

i.e., $\{(a,a),(a,b),(b,a),(b,b)\} \to \{a,b\}$,

$\therefore$ total number of binary operations $= 2^4 = 16$.

36. Find the number of all one-one functions from set $A = \{1,2,3\}$ to itself.

Ans: Here number of elements in set A =3,
$\therefore$ number of one-one functions$= 3! = 6$.

37. Let $*$ be a binary operation defined on Q by $a * b = \frac{ab}{4}$ find the identity element.

Ans: Let $a * e = a \Rightarrow \frac{ae}{4} = a \Rightarrow e = 4$,

$a * 4 = 4 * a = \frac{a4}{4} = a.$

$\therefore$ identity element $e = 4$. **(2015-M, 2017-J)**

38. On $R - \{-1\}$ define by $a * b = \frac{a}{b+1}$, verify whether * is commutative?

Ans: $1 * 2 = \frac{1}{2+1} = \frac{1}{3}$ *but* $2 * 1 = \frac{2}{1+1} = 1$

$1 * 2 \neq 2 * 1.$ $\therefore$ $*$ *is not commutative*

39. Let $f: R \to R$ be the function defined by $f(x) = 2x - 3 \in R$ .write f^{-1}.

Ans: Let $f(x) = y \Rightarrow 2x - 3 = y \Rightarrow x = \frac{y+3}{2}$

$x = f^{-1}(y) \Rightarrow f^{-1}(y) = \frac{y+3}{2} \therefore f^{-1}(x) = \frac{x+3}{2}$

40. State with reason whether the function $h: \{2,3,4,5\} \to \{7,9,11,13\}$ has inverse with $h = \{(2,7),(3,9),(4,11),(5,13)\}$.

Ans: Here h is one-one (every element of domain have unique image) and onto (range of $h =$ codomain) then inverse exists

41. Let * be a binary operation defined on the set of rational numbers by, $a * b = a - b$, find the identity element.

Ans: If $a * e = a \Rightarrow a - e = a \Rightarrow e = 0$

$\& if \ e * a = a \Rightarrow e - a = a \Rightarrow e = 2a,$

$\therefore a * e \neq e * a$, *no identity element*

42. Let * be a binary operation defined on the set of rational numbers by, $a * b = a^2 + b^2$ find the identity element.

Ans: If $a * e = a \Rightarrow a^2 + e^2 = a,$

$let\ a = -2 \Rightarrow (-2)^2 + e^2 = 4 + e^2 = -2,$

$\Rightarrow e^2 = -6 \Rightarrow e = \sqrt{-6} \notin Q$
$\therefore$ *no identity element*

43. Let * be a binary operation defined on the set of rational numbers by, $a * b = a + ab$ find the identity element.

Ans: If $a * e = a \Rightarrow a + ae = a \Rightarrow e = 0$

$\& if \ e * a = a \Rightarrow e + ea = a \Rightarrow e = \frac{a}{1+a},$

$\therefore a * e \neq e * a$, *no identity element*

44. Let * be a binary operation defined on the set of rational numbers by, $a * b = (a - b)^2$ find the identity element.

Ans: If $a * e = a \Rightarrow (a - e)^2 = a, let\ a = -2 \Rightarrow (-2 - e)^2 \neq -2,$

$\therefore$ *no identity element*

45. Let * be a binary operation defined on the set of rational numbers by, $a * b = ab^2$ find the identity element.

Ans: If $a * e = a \Rightarrow ae^2 = a \Rightarrow e^2 = 1 \Rightarrow e = \pm 1,$
but identity is unique. $\therefore$ *no identity element*

46. An operation $*$ on Z^+ (except of all non-negative integer) is defined as $a * b = a - b \ \forall a, b \in Z^+$, is $*$ a binary operation on Z^+ ? **(2016-M)**

Ans: Let $1,2 \in Z^+$ but $1*2 = 1-2 = -1 \notin Z^+$, $\therefore *$ is not binary operation.

47. The relation R on set $A = \{1,2,3\}$ is defined as $R = \{(1,1),(2,2),(3,3),(1,2),(2,3)\}$ is not transitive. Why? **(2018-J)**

Ans: $(1,2) \in R \,\&\, (2,3) \in R \, but \, (1,3) \notin R.$

48.On N the set of natural numbers, $a*b = L.C.M$ of a & b. Find the identity for $*$ in N.

Ans: WKT $LCM\ of\ a \,\&\, 1 = a = LCM\ of\ 1 \,\&\, a$

$i.e\, a*1 = a = 1*a, \qquad \therefore identity\ \ e = 1$

49. Is the Relation $R = \{(2,1),(1,2),(2,2)\}$ defined on the set $A = \{1,2,3\}$ transitive?

Ans: Not transitive, because $(1,2) \in R \,\&\, (2,1) \in R \, but \, (1,1) \notin R$

50. Determine whether the relation R in the set $A = \{1,2,3,4,5,6\}$, $R = \{(x,y): y\ is\ divisible\ by\ x\}$ is reflexive.

Ans: R is reflexive. Because every element divisible by itself, so $(1,1) \in R, (2,2) \in R, (3,3) \in R, (4,4) \in R, (5,5) \in R, (6,6) \in R.$

51. Let $A = \{0,1,2,3\}$ and defined a relation R on A as follows $R = \{(0,0,),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\}$ is reflexive? Symmetric? Transitive?

Ans: R is reflexive, symmetric and not transitive.

52. For the set $A = \{1,2,3\}$ define a relation R in the set A as follows $R = \{(1,1),(2,2),(3,3),(1,3)\}$ write the ordered pair to be added to R to make it the smallest equivalence relation?

Ans: (3,1)

53. Let R be the equivalence relation in the set Z of integers given by $R = \{(a,b): 2\ divides\ a-b\}$ write the equivalence class of $[0]$.

Ans: Equivalence class of $[0] = \{0, \pm2, \pm4, \pm6, ...\} = even\ integers$

54. If $f = \{(5,2),(6,3)\}$, $g = \{(2,5),(3,6)\}$ write fog.

Ans: $fog = \{(2,2),(3,3)\}$.

55. If $f = \{(5,2),(6,3)\}$ & $g = \{(2,5),(3,6)\}$ write the range of f and g.

Ans: Range of $f = \{2,3\}$, Range of $g = \{5,6\}$.

56. Set A has 3 elements and the set B has 4 elements. Then the number of injective mapping that can be defined from A to B is ?

Ans: $4C_3 \times 3! = 24$ $\left(\because\ n(B) C_{n(A)} \times n(A)!\right)$

57. Write down the identity relation on the set $A = \{x,y,z\}$

Ans: $R = \{(x,x),(y,y),(z,z)\}$

58. Consider set $A = \{1,2,3\}$, write the smallest equivalence relation on R.

Ans: $R = \{(1,1),(2,2),(3,3)\}$

59.Consider the set A containing n elements. Then the total number of injective functions from A onto itself is ?

Ans: The total number of injective functions = $n!$

60. Consider the set $A = \{1,2,3\}$ and the relation $R = \{(1,2),(1,3)\}$, R is a transitive relation ?.

Ans: R is Transitive relation.

Question No: 11 (2M)

1. Show that function $f: N \to N$ given by $f(1) = f(2) = 1$ & $f(x) = x - 1$ for $x > 2$ is onto but not one – one. **(2014-J)**

Ans: Given

$f(1) = f(2) = 1$ but $1 \neq 2$ $\therefore$ f is not one – one.

And

$f(3) = 3-1 = 2, f(4) = 4-1 = 3, f(5) = 5-1 = 4$............

Then range of f(x)=N=co-domain.

$\therefore$ f is onto

2. Show that the function $f: N \to N$ given by $f(x) = 2x$ is one-one but not onto.

Ans: Let $x_1, x_2 \in N, f(x_1) = f(x_2)$ **(2019-M)**

$\Rightarrow x_1 = x_2 \quad \Rightarrow 2x_1 = 2x_2 \quad \therefore f$ is one – one

Let $y \in N, f(x) = y \quad \Rightarrow 2x = y$

$$\Rightarrow x = \frac{y}{2} \notin N \left(\because if\ y = 1 \Rightarrow x = \frac{1}{2} \notin N \right)$$

$\therefore$ f is not onto.

3. If $f: R \to R$ defined by $f(x) = 1 + x^2$, then S.T f is neither one-one nor onto. **(2017-M)**

Ans: Let $f(1) = 1 + (1)^2 = 2$

$f(-1) = 1 + (-1)^2 = 2$ but $1 \neq -1$

$\therefore$ f is not one – one

Let $y \in R, f(x) = y$

$\Rightarrow 1 + x^2 = y$

$$\Rightarrow x = \sqrt{y-1} \notin R \left(\because if\ y = 0 \Rightarrow x = \sqrt{-1} \notin R \right)$$

$\therefore$ f is not onto.

4. Show that function $f: R \to R$, defined as $f(x) = x^2$ is neither one-one nor onto.

Ans: Let $f(1) = (1)^2 = 1$

$f(-1) = (-1)^2 = 1$ but $1 \neq -1$

$\therefore$ f is not one – one

Let $y \in R, f(x) = y \quad \Rightarrow x^2 = y$

$$\Rightarrow x = \sqrt{y} \notin R \left(\because if\ y = -1 \Rightarrow x = \sqrt{-1} \notin R \right)$$

$\therefore$ f is not onto.

5. Show that $f: A \to B$ & $g: B \to C$ are one – one then $g \circ f: A \to C$ is also one-one. **(2015-M, 2017-J)**

Ans: Given

$f: A \to B$ & $g: B \to C$ are one – one

Let $x_1, x_2 \in A, gof(x_1) = gof(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2) \quad (\because g$ is one – one$)$

$\Rightarrow x_1 = x_2 \quad (\because f$ is one – one$)$

$\therefore$ gof is one – one

6. If $f: R \to R$ is given by $f(x) = (3 - x^3)^{\frac{1}{3}}$ then find $(f \circ f)(x)$. **(2015-J)**

Ans: $(fof)(x) = f(f(x)) = f\left((3 - x^3)^{\frac{1}{3}} \right)$

$$= \left(3 - \left((3 - x^3)^{\frac{1}{3}} \right)^3 \right)^{\frac{1}{3}} = (x^3)^{\frac{1}{3}} = x$$

$\therefore (fof)(x) = x.$

7. If function $f: R \to R$ & $g: R \to R$ are given by $f(x) = |x|$ & $g(x) = [x]$ (where $[x]$ is greatest integer function) Find $f \circ g\left(-\frac{1}{2}\right)$ & $g \circ f\left(-\frac{1}{2}\right)$. **(2016-M)**

Ans:

$$fog\left(-\frac{1}{2}\right) = f(g(-0.5)) = f([-0.5]) = f(-1) = |-1| = 1$$

$$gof\left(-\frac{1}{2}\right) = g(f(-0.5)) = g(|-0.5|) = g(0.5) = [0.5] = 0$$

$$fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2) = \cos 3x^2$$

8. Find $g \circ f$ & $f \circ g$ if $f: R \to R$ and $g: R \to R$ are given by $f(x) = cosx$ and $g(x) = 3x^2$

Ans: **(2016-J, 2020-M)**

$$gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$$

$$fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2) = \cos 3x^2$$

9. Find $g \circ f$ & $f \circ g$ if $f(x) = 8x^3$ & $g(x) = x^{\frac{1}{3}}$ **(2019-J)**

Ans: $gof(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = ((2x)^3)^{\frac{1}{3}} = 2x$

$$fog(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$$

10. Define binary operation on a set, verify whether the operation $*$ defined on Z by $a*b = ab+1$ is binary or not.

Ans: A binary operation * on set A is a function $*: A \times A \to A$. We denoted by a*b.

Let $a,b \in Z$ then $a*b = ab+1 \in Z$

$(\because 7,-1 \in Z$ then $7*(-1) = 7(-1)+1 = -6 \in Z)$

$\therefore *$ is binary operation.

11. Verify whether the operation * defined on Q by $a * b = \frac{ab}{2}$ is associative or not. **(2014-M, 2018-M)**

Ans: Let $(a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{\frac{ab}{2}c}{2} = \frac{abc}{4}$

$a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{a\frac{bc}{2}}{2} = \frac{abc}{4}$

$(a*b)*c = a*(b*c)$,

$\therefore *$ is associative.

12. A binary operation $\wedge$ on a set $\{1,2,3,4,5\}$ defined by $a \wedge b = \min(a,b)$ write the operation table for operation $\wedge$.

Ans: Given $a \wedge b = \min(a,b)$

Λ	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

$1 \wedge 2 = \min(1,2) = 1$

$1 \wedge 3 = \min(1,3) = 1 .... so\ on$

13. Define binary operation on a set. Verify whether the operation * defined on Q set of rational numbers by $a*b = ab+1$, $\forall a,b \in Q$ is binary or not. **(2018-J)**

Ans: A binary operation * on set A is a function $*: A \times A \to A$. We denoted by a*b.

Let $a,b \in Q$ then $a*b = ab+1 \in Q$

$(\because 7,-1 \in Q$ then $7*(-1) = 7(-1)+1 = -6 \in Q)$

$\therefore *$ is binary operation.

Question No: 25 (3M)

1. Show that the relation R in the set Z of integers given by $R = \{(a,b): 2\ divides\ a-b\}$ is an equivalence relation. **(2014-J)**

Ans:

Reflexive: WKT 2 divides $a-a=0$

$\Rightarrow (a,a) \in R, \forall a \in Z$

$\therefore R$ is reflexive

Symmetric: Let $(a,b) \in R \Rightarrow$ 2 divides $a-b$

$\Rightarrow$ 2 divides b-a $\Rightarrow (b,a) \in R$

∴ R is symmetric.

Transitive:

Let $(a,b) \in R$ & $(b,c) \in R$

$\Rightarrow$ 2 divides $a-b$ & 2 divides $b-c$

$\Rightarrow$ 2 divides $a-b+b-c$

$\Rightarrow$ 2 divides $a-c$ $\Rightarrow (a,c) \in R$

∴ R is transitive

∴ R is equivalence relation.

2. Prove that the relation R in the set of integers Z defined by $R = \{(x,y): x-y\ is\ an\ integer\}$ is an equivalence relation. **(2015-J)**

Ans:

Reflexive: WKT $x-x=0$ is an integer.

$\Rightarrow (x,x) \in R, \forall x \in Z$ ∴ R is reflexive

Symmetric:

Let $(x,y) \in R \Rightarrow$ x-y is an integer.

$\Rightarrow$ y-x is also an integer

$\Rightarrow (y,x) \in R$ ∴ R is symmetric.

Transitive: Let $(x,y) \in R$ & $(y,z) \in R$

$\Rightarrow$ $x-y$ is an integer & $y-z$ is an integer.

$\Rightarrow$ $x-y+y-z$ is also an integer.

$\Rightarrow$ $x-z$ is an integer.

$\Rightarrow (x,z) \in R$ ∴ R is transitive

∴ R is equivalence relation.

3. Show that the relation R in the set $A = \{x: x \in Z, 0 \leq x \leq 12\}$ given by $R = \{(a,b): |a-b| is multiple of\ 4\}$ is an equivalence relation. **(2016-M)**

Ans:

Reflexive: WKT $|x-x|$=0 is a multiple of 4

$\Rightarrow (x,x) \in R, \forall x \in A$ ∴ R is reflexive.

Symmetric: Let $(x,y) \in R \Rightarrow |x-y|$ is a multiple of 4 $\Rightarrow |y-x|$ is a multiple of 4.

$\Rightarrow (y,x) \in R$ ∴ R is symmetric.

Transitive: Let $(x,y) \in R \& (y,z) \in R$

$\Rightarrow |x-y|$ is a multiple of 4

$\& |y-z|$ is a multiple of 4

$\Rightarrow x-y+y-z$ is also a multiple of 4

$\Rightarrow |x-z|$ is a multiple of 4

$\Rightarrow (x,z) \in R$ ∴ R is transitive

∴ R is equivalence relation.

4. Show that R in the set $A = \{1,2,3,4,5\}$ given by $R = \{(a,b): |a-b| is\ even\}$ is an equivalence relation. **(2016-J, 2018-M)**

Ans: Reflexive: WKT $|a-a|$=0 is even

$\Rightarrow (a,a) \in R, \forall a \in A$ ∴ R is reflexive.

Symmetric: Let $(a,b) \in R \Rightarrow |a-b|$ is even.

$\Rightarrow |b-a|$ is even. $\Rightarrow (b,a) \in R$

∴ R is symmetric.

Transitive: Let $(a,b) \in R$ & $(b,c) \in R$

$\Rightarrow |a-b|$ is even & $|b-c|$ is even

$\Rightarrow a-b+b-c$ is also even

$\Rightarrow |a-c|$ is even. $\Rightarrow (a,c) \in R$

∴ R is transitive

∴ R is equivalence relation.

5. Show that the relation R in R (set of real nor) is defined as $R = \{(a,b): a \leq b\}$ is reflexive & transitive but not symmetric. **(2017-M)**

Ans: Reflexive: WKT $x \leq x \rightarrow x = x$

$\Rightarrow (x,x) \in R, \forall x \in R$

∴ R is reflexive.

Symmetric: Let $(1,2) \in R \Rightarrow 1 \leq 2$

but $2 \not\leq 1 \Rightarrow (2,1) \notin R$ ∴ R is not symmetric.

Transitive: Let $(a,b) \in R \& (b,c) \in R$

$\Rightarrow a \leq b \& b \leq c \quad \Rightarrow a \leq b \leq c$

$\Rightarrow a \leq c \Rightarrow (a,c) \in R$ ∴ R is transitive

∴ R is reflexive & transitive but not symmetric.

6. Check whether the relation R defined in the set $\{1,2,3,4,5,6\}$ as $R = \{(a,b): b = a+1\}$ is reflexive, symmetric & transitive. **(2017-J)**

Ans: Given $R = \{(a,b): b = a+1\}$

$\Rightarrow R = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

Reflexive: $(1,1),(2,2),(3,3),.........(6,6) \notin R$

$\therefore R\ is\ not\ reflexive$.

Symmetric: $(1,2) \in R\ but\ (2,1) \notin R$

$\therefore R\ is\ not\ symmetric.$

Transitive: $(1,2) \in R \& (2,3) \in R\ but\ (1,3) \notin R$

$\therefore R\ is\ not\ transitive.$

∴ R is neither reflexive, nor symmetric, nor transitive.

7. Show that the relation R on the set of integers given by $R = \{(a,b): 5\ divides\ (a-b)\}$ is equivalence relation.

Ans: Reflexive: Wkt 5 divides $a-a=0$

$\Rightarrow (a,a) \in R, \forall a \in Z \quad \therefore R\ is\ reflexive$.

Symmetric: Let $(a,b) \in R \Rightarrow$ 5 divides $a-b$

$\Rightarrow$ 5 divides $b-a$ $\Rightarrow (b,a) \in R$

∴ R is symmetric.

Transitive: Let $(a,b) \in R \& (b,c) \in R$

$\Rightarrow$ 5 divides $a-b$ & 5 divides $b-c$

$\Rightarrow$ 5 divides $a-b+b-c$

$\Rightarrow$ 5 divides $a-c$ $\Rightarrow (a,c) \in R$

∴ R is transitive

∴ R is equivalence relation.

8. Determine whether each of the following relation are reflexive, symmetric and transitive.

(i) Relation R on set $A = \{1,2,3,.............13,14\}$ defined as $R = \{(x,y): 3x - y = 0\}$

(ii) Relation R on set $A = \{1,2,3,4,5,6\}$ defined as $R = \{(x,y): y\ is\ divisible\ by\ x\}$

Ans: **(i)** Given relation we can written as

$R=\{(x,y): y=3x\}$

$R=\{(1,3),(2,6),(3,9),(4,12)\}$

Reflexive:

$(1,1),(2,2),(3,3),.........(14,14)\notin R$

$\therefore$ *R is not reflexive*

Symmetric: $(1,3)\in R$ *but* $(3,1)\notin R$

$\therefore$ *R is not symmetric.*

Transitive: $(1,3)\in R \,\&\, (3,9)\in R$ *but* $(1,9)\notin R$

$\therefore$ *R is not transitive.*

$\therefore$ R is neither reflexive, nor symmetric, nor transitive.

(ii). Given relation

$R=\{(x,y): y \text{ is divisible by } x\}$

Reflexive: WKT every number divisible by itself. $\Rightarrow (x,x)\in R$ $\therefore$ R is reflexive.

Symmetric: Let $(1,2)\in R \Rightarrow 2$ is divisible by 1 but 1 is not divisible by 2 $\Rightarrow (2,1)\notin R$

$\therefore$ *R is not symmetric.*

Transitive: Let $(x,y)\in R \,\&\, (y,z)\in R$

$\Rightarrow y \text{ is divisible by } x \,\&\, z \text{ is divisible by } y$

$\Rightarrow 3 \text{ is divisible by } 1 \,\&\, 6 \text{ is divisible by } 3$

$\Rightarrow z \text{ is divisible by } x \quad \Rightarrow (x,z)\in R$

$\therefore$ R is transitive.

$\therefore$ R is reflexive & transitive but not symmetric.

9. Check whether the relation in R in R of real numbers defined by $R=\{(a,b): a\le b^3\}$ is Reflexive, symmetric or transitive. **(2019-M)**

Ans: Given $R=\{(a,b): a\le b^3\}$

Reflexive: WKT $\frac{1}{2}\not\le\left(\frac{1}{2}\right)^3 \Rightarrow \left(\frac{1}{2},\frac{1}{2}\right)\notin R$

$\therefore$ *R is not reflexive* **.**

Symmetric: Let $(1,7)\in R \Rightarrow 1\le 7^3$

but $7\not\le 1^3 \Rightarrow (7,1)\notin R$ $\therefore$ *R is not symmetric.*

Transitive: Let $(3,3/2)\in R$ & $(3/2,6/5)\in R$

$\Rightarrow 3\le(3/2)^3$ & $3/2\le(6/5)^3$

but $3\not\le(6/5)^3 \Rightarrow (3,6/5)\notin R$

$\therefore$ R is not transitive.

10. Check whether the relation in R in R of real numbers defined by $R=\{(a,b): a\le b^2\}$ is Reflexive, symmetric or transitive.

Ans: Given $R=\{(a,b): a\le b^2\}$

Reflexive: WKT $\frac{1}{2}\not\le\left(\frac{1}{2}\right)^2 \Rightarrow \left(\frac{1}{2},\frac{1}{2}\right)\notin R$

$\therefore$ *R is not reflexive*

Symmetric: Let $(1,7)\in R \Rightarrow 1\le 7^2=49$

but $7\not\le 1^2 \Rightarrow (7,1)\notin R$ $\therefore$ *R is not symmetric.*

Transitive: Let $(10,5)\in R$ & $(5,3)\in R$

$10\le(5)^2=25$ & $5\le(3)^2=9.$

but $10\not\le(3)^2=9 \Rightarrow (10,3)\notin R$

$\therefore$ R is not transitive.

11. Show that the relation R in the set Z of integers given by $R=\{(x,y): |x-y| \text{ is even}\}$ is an equivalence relation.

Ans:

Reflexive: WKT $|x-x|=0$ is even.

$\Rightarrow (x,x)\in R, \forall x\in Z$ $\therefore$ R is reflexive.

Symmetric: Let $(x,y)\in R \Rightarrow |x-y|$ is even.

$\Rightarrow |y-x|$ is even. $\Rightarrow (y,x)\in R$

$\therefore$ R is symmetric.

Transitive: Let $(x,y)\in R \,\&\, (y,z)\in R$

$\Rightarrow |x-y|$ is even $\&\, |y-z|$ is even

$\Rightarrow x-y+y-z$ is also even

$\Rightarrow |x-z|$ is even. $\Rightarrow (x,z)\in R$

$\therefore$ R is transitive

$\therefore$ R is equivalence relation.

12. Given that $f(x)=8x^3$ & $g(x)=x^{\frac{1}{3}}$ show that $f\circ g\ne g\circ f$.

Ans: $gof(x)=g(f(x))=g(8x^3)=(8x^3)^{\frac{1}{3}}=\left((2x)^{3}\right)^{\frac{1}{3}}=2x$

$fog(x)=f(g(x))=f\left(x^{\frac{1}{3}}\right)=8\left(x^{\frac{1}{3}}\right)^3=8x$

$\therefore gof\ne fog$

13. Find $g\circ f$ and $f\circ g$ if $f: R\to R$ and $g: R\to R$ are given by $f(x)=cos x$ & $g(x)=3x^2$. Show that $g\circ f\ne f\circ g$. **(2014-M, 2018-J)**

Ans:

$gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$

$fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2) = \cos 3x^2$

$\therefore gof \neq fog$

14. If $*$ is a binary operation defined on Q, the set of all real numbers by $a*b = \frac{ab}{2}$ show that $*$ is a commutative & associative. Also find identity element.

Ans: **Commutative:**

Let $a*b = \frac{ab}{2} = \frac{ba}{2} = b*a$

$\therefore$ $*$ is commutative.

Associative:

$$Let\ (a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{\frac{ab}{2}c}{2} = \frac{abc}{4}$$

$$\&\quad a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{a\frac{bc}{2}}{2} = \frac{abc}{4}$$

$(a*b)*c = a*(b*c)$. $\therefore *$ is associative.

Identity element:

$$Let\quad a*e = a \Rightarrow \frac{\not{a}e}{2} = \not{a} \Rightarrow \boxed{e=2}$$

$$e*a = a \Rightarrow \frac{e\not{a}}{2} = \not{a} \Rightarrow \boxed{e=2}$$

15. Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$ is an equivalence relation. **(2020-M)**

Ans:

Reflexive: WHT, T_1 is similar to T_1

$\Rightarrow (T_1, T_1) \in R \quad \therefore R$ is reflexive

Symmetric:

Let $(T_1, T_2) \in R \Rightarrow T_1$ is similar to T_2

$\Rightarrow T_2$ is similar to $T_1 \quad \Rightarrow (T_2, T_1) \in R$

$\therefore R$ is symmetric

Transitive:

Let $(T_1, T_2) \in R \ \& \ (T_2, T_3) \in R$

$\Rightarrow T_1$ is similar to T_2 & T_2 is similar to T_3

$\Rightarrow T_1$ is similar to $T_3 \Rightarrow (T_1, T_3) \in R$

$\therefore R$ is transitive

$\therefore R$ is equivalence relation.

Question No: 39 (5M)

1. Prove that the function $f: N \to Y$ defined by $f(x) = 4x + 3$ where $Y = \{y : y = 4x + 3, x \in N\}$ is invertible. Also write the inverse of $f(x)$. **(2014-M, 2019-M)**

Ans: Given function $f: N \to Y$ defined by $f(x) = 4x + 3$

One-one: Let $x_1, x_2 \in N$, $f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + \not{3} = 4x_2 + \not{3} \Rightarrow \not{4}x_1 = \not{4}x_2$

$\Rightarrow x_1 = x_2 \quad \therefore$ f is one-one.

Onto: Let $y \in Y$, $f(x) = y$

$\Rightarrow 4x + 3 = y \Rightarrow 4x = y - 3$

$\Rightarrow x = \frac{y-3}{4} \in N, \forall y \in Y$

Now $f(x) = f\left(\frac{y-3}{4}\right) = \not{4}\left(\frac{y-3}{\not{4}}\right) + 3 = y - \not{3} + \not{3} = y$

$\forall y \in Y$ there exist $x \in N$ such that $f(x) = y$

$\therefore$ f is onto. $\therefore$ f is one-one and onto.

Then f is invertible.

To find f^{-1}: Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\Rightarrow \frac{y-3}{4} = f^{-1}(y) \quad (or) \quad \boxed{f^{-1}(x) = \frac{x-3}{4}}$.

2. Prove that the function $f: R \to R$ defined by $f(x) = 4x + 3$, is invertible and find the inverse of f. **(2015-J, 2017-J , 2019-J, 2020-M)**

Ans: Given function $f: R \to R$ defined by $f(x) = 4x + 3$

One-one: Let $x_1, x_2 \in R$, $f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + \not{3} = 4x_2 + \not{3} \Rightarrow \not{4}x_1 = \not{4}x_2$

$\Rightarrow x_1 = x_2 \quad \therefore$ f is one-one.

Onto: Let $y \in R$, $f(x) = y$

$\Rightarrow 4x + 3 = y \Rightarrow 4x = y - 3$

$\Rightarrow x = \frac{y-3}{4} \in R, \forall y \in R$

Now $f(x) = f\left(\frac{y-3}{4}\right) = \not{4}\left(\frac{y-3}{\not{4}}\right) + 3 = y - \not{3} + \not{3} = y$

$\forall y \in R$ there exist $x \in R$ such that $f(x) = y$

$\therefore$ f is onto $\therefore$ f is one-one and onto.

Then f is invertible.

To find f^{-1}: Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\Rightarrow \frac{y-3}{4} = f^{-1}(y) \quad (or) \quad \boxed{f^{-1}(x) = \frac{x-3}{4}}$.

3. Prove that the function $f: N \to Y$ defined by $f(x) = x^2$ where $Y = \{y: y = x^2, x \in N\}$ is invertible. Also find the inverse of f. **(2014-J)**

Ans: Given function $f: N \to Y$ defined by $f(x) = x^2$

<u>One-one</u>: Let $x_1, x_2 \in N, f(x_1) = f(x_2)$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = x_2$ $\quad \therefore$ f is one-one.

<u>Onto</u>: Let $y \in Y$, $f(x) = y$

$\Rightarrow x^2 = y \Rightarrow x = \sqrt{y} \in N, \forall y \in Y.$

Now $f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = y$

$\forall y \in Y$ *there exist* $x \in N$ *such that* $f(x) = y$

$\therefore$ f is onto. $\therefore$ f is one-one and onto.
Then f is invertible.

<u>To find f^{-1}</u>: Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\Rightarrow \sqrt{y} = f^{-1}(y)$ $\quad (or)$ $\boxed{f^{-1}(x) = \sqrt{x}}$.

4. Let R_+ be the set of all non-negative real numbers. Show that the function $f: R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$ is invertible & write the inverse of. **(2015-M, 2016-J, 2018-M, 2018-J)**

Ans: Given function $f: R_+ \to [4, \infty)$ defined by $f(x) = x^2 + 4$

<u>One-one</u>: Let $x_1, x_2 \in R_+, f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + \not{4} = x_2^2 + \not{4} \Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = x_2$ $\quad \therefore$ f is one-one.

<u>Onto</u>: Let $y \in [4, \infty)$, $f(x) = y$

$\Rightarrow x^2 + 4 = y \Rightarrow x^2 = y - 4$

$\Rightarrow x = \sqrt{y-4} \in R_+, \forall y \in [4, \infty).$

Now

$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$

$\forall y \in [4, \infty)$ *there exist* $x \in R_+$ *such that* $f(x) = y$

$\therefore$ f is onto. $\therefore$ f is one-one and onto.
Then f is invertible.

<u>To find f^{-1}</u>: Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\Rightarrow \sqrt{y-4} = f^{-1}(y)$ $\quad (or)$ $\boxed{f^{-1}(x) = \sqrt{x-4}}$.

5. Let $f: N \to R$ be defined by $f(x) = 4x^2 + 12x + 15$ Show that $: N \to S$, where S is the range of function is invertible. Also find the inverse of f. **(2016-M, 2017-M)**

Ans: Given function $f: N \to R$ defined by $f(x) = 4x^2 + 12x + 15$

$\Rightarrow f(x) = (2x)^2 + 2(2x)(3) + 9 + 6$

$\Rightarrow f(x) = (2x)^2 + 2(2x)(3) + (3)^2 + 6$

$\Rightarrow f(x) = (2x+3)^2 + 6$

<u>One-one</u>: Let $x_1, x_2 \in R_+, f(x_1) = f(x_2)$

$\Rightarrow (2x_1 + 3)^2 + \not{6} = (2x_2 + 3)^2 + \not{6}$

$\Rightarrow (2x_1 + 3)^{\not{2}} = (2x_2 + 3)^{\not{2}}$

$\Rightarrow 2x_1 + \not{3} = 2x_2 + \not{3} \Rightarrow \not{2}x_1 = \not{2}x_2$

$\Rightarrow x_1 = x_2$ $\therefore$ f is one-one.

<u>Onto</u>: Let $y \in R$, $f(x) = y$

$\Rightarrow (2x+3)^2 + 6 = y \Rightarrow (2x+3)^2 = y - 6$

$\Rightarrow 2x + 3 = \sqrt{y-6} \Rightarrow 2x = \sqrt{y-6} - 3$

$\Rightarrow x = \dfrac{\sqrt{y-6} - 3}{2} \in N, \forall y \in R$

Now $f(x) = f\left(\dfrac{\sqrt{y-6}-3}{2}\right) = \left(\not{2}\left(\dfrac{\sqrt{y-6}-3}{\not{2}}\right) + 3\right)^2 + 6$

$= (\sqrt{y-6} - \not{3} + \not{3})^2 + 6 = y - \not{6} + \not{6} = y$

$\forall y \in R$ *there exist* $x \in R$ *such that* $f(x) = y$

$\therefore$ f is onto. $\therefore$ f is one-one and onto.
Then f is invertible.

<u>To find f^{-1}</u>: Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\Rightarrow \dfrac{\sqrt{y-6}-3}{2} = f^{-1}(y)$ (or) $\boxed{f^{-1}(x) = \dfrac{\sqrt{x-6}-3}{2}}$.

6. Consider $f: R \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$ show that f is invertible with $f^{-1} = \left\{\dfrac{\sqrt{y+6}-1}{3}\right\}$

Ans: Given function $f: R \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$

$\Rightarrow f(x)=(3x)^2+2(3x)1+1^2-1^2-5$

$\Rightarrow f(x)=(3x)^2+6x+(1)^2-(1)^2-5$

$\Rightarrow f(x)=(3x+1)^2-6$

One-one: Let $x_1,x_2\in R, f(x_1)=f(x_2)$

$\Rightarrow (3x_1+1)^2-6=(3x_2+1)^2-6$

$\Rightarrow (3x_1+1)^2=(3x_2+1)^2 \Rightarrow 3x_1+1=3x_2+1$

$\Rightarrow 3x_1=3x_2 \Rightarrow x_1=x_2$ ∴ f is one-one.

Onto: Let $y\in[-5,\infty)$, $f(x)=y$

$\Rightarrow (3x+1)^2-6=y \Rightarrow (3x+1)^2=y+6$

$\Rightarrow 3x+1=\sqrt{y+6} \Rightarrow 3x=\sqrt{y+6}-1$

$\Rightarrow x=\frac{\sqrt{y+6}-1}{3}\in R, \forall y\in[-5,\infty)$

$Now\ f(x)=f\left(\frac{\sqrt{y+6}-1}{3}\right)=\left(3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right)^2-6$

$=\left(\sqrt{y+6}-1+1\right)^2-6=y+6-6=y$

$\forall y\in[-5,\infty)$ *there exist* $x\in R$ *such that* $f(x)=y$

∴ f is onto. ∴ f is one-one and onto.

Then f is invertible.

To find f^{-1}: Let $f(x)=y\Rightarrow x=f^{-1}(y)$

$\Rightarrow \frac{\sqrt{y+6}-1}{3}=f^{-1}(y)$ (or) $\boxed{f^{-1}(x)=\frac{\sqrt{x+6}-1}{3}}$.

7. Let $f:R\to R$ be defined by $f(x)=10x+7$. show that f is invertible. Find the inverse of f.

Ans: Given function f: R → R defined by f(x) = 10x + 7

One-one: Let $x_1,x_2\in R, f(x_1)=f(x_2)$

$\Rightarrow 10x_1+7=10x_2+7$

$\Rightarrow 10x_1=10x_2 \Rightarrow x_1=x_2$

∴ f is one-one.

Onto: Let $y\in R$, $f(x)=y$

$\Rightarrow 10x+7=y \Rightarrow 10x=y-7$

$\Rightarrow x=\frac{y-7}{10}\in R, \forall y\in R$

Now

$f(x)=f\left(\frac{y-7}{10}\right)=10\left(\frac{y-7}{10}\right)+7=y-7+7=y$

$\forall y\in R$ *there exist* $x\in R$ *such that* $f(x)=y$

∴ f is onto. ∴ f is one-one and onto.
Then f is invertible.

To find f^{-1}: Let $f(x)=y\Rightarrow x=f^{-1}(y)$

$\Rightarrow \frac{y-7}{10}=f^{-1}(y)$ (or) $\boxed{f^{-1}(x)=\frac{x-7}{10}}$.

8. Prove that the function $f:R-\left\{\frac{-4}{3}\right\}\to R$ is defined by $f(x)=\frac{4x}{3x+4}$ is invertible. Also find the inverse of f.

Ans: Given function $f:R-\left\{\frac{-4}{3}\right\}\to R$ is defined by $f(x)=\frac{4x}{3x+4}$

One-one: Let $x_1,x_2\in R-\left\{\frac{-4}{3}\right\}, f(x_1)=f(x_2)$

$\Rightarrow \frac{4x_1}{3x_1+4}=\frac{4x_2}{3x_2+4}$

$\Rightarrow 4x_1(3x_2+4)=4x_2(3x_1+4)$

$\Rightarrow 12x_1x_2+16x_1=12x_1x_2+16x_2$

$\Rightarrow 16x_1=16x_2 \Rightarrow x_1=x_2$

∴ f is one-one.

Onto: Let $y\in R$, $f(x)=y$

$\Rightarrow \frac{4x}{3x+4}=y \Rightarrow 4x=3xy+4y$

$\Rightarrow 4x-3xy=4y \Rightarrow x(4-3y)=4y$

$\Rightarrow x=\frac{4y}{4-3y}\in R-\left\{\frac{-4}{3}\right\}, \forall y\in R.$

$\forall y\in R$ *there exist* $x\in R-\left\{\frac{-4}{3}\right\}$ *such that* $f(x)=y$

∴ f is onto. ∴ f is one-one and onto.
Then f is invertible.

To find f^{-1}: Let $f(x)=y\Rightarrow x=f^{-1}(y)$

$\Rightarrow \frac{4y}{4-3y}=f^{-1}(y)$ (or) $\boxed{f^{-1}(x)=\frac{4x}{4-3x}}$.

9. Let R be the set of real numbers and $f : R \to R$ be the function defined by $f(x) = 4x + 5$. Show that f is invertible and find f^{-1}.

Ans: Given function f: R → R defined by f(x) = 4x + 5

<u>One-one</u>: Let $x_1, x_2 \in R, f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + \cancel{5} = 4x_2 + \cancel{5} \quad \Rightarrow \cancel{4}x_1 = \cancel{4}x_2$

$\Rightarrow x_1 = x_2$ ∴ f is one-one.

<u>Onto</u>: Let $y \in R,\ f(x) = y$

$\Rightarrow 4x + 5 = y \Rightarrow 4x = y - 5$

$\Rightarrow x = \frac{y-5}{4} \in R, \forall y \in R$

Now

$$f(x) = f\left(\frac{y-5}{4}\right) = \cancel{4}\left(\frac{y-5}{\cancel{4}}\right) + 5 = y - \cancel{5} + \cancel{5} = y$$

$\forall y \in R$ *there exist* $x \in R$ *such that* $f(x) = y$

∴ f is onto. ∴ f is one-one and onto.

Then f is invertible.

<u>To find f^{-1}</u>: Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\Rightarrow \frac{y-5}{4} = f^{-1}(y) \qquad (or) \quad \boxed{f^{-1}(x) = \frac{x-5}{4}}.$$

UNIT-2
INVERSE TRIGONOMETRIC FUNCTIONS

$\left[\text{Total marks}: 8,\ \ Q.No-2(1M), 12(2M), 13(2M), 26(3M)\right]$

Question No: 2 (1M)

1. Write the domain of $f(x)=\sin^{-1}x$.

Ans: $[-1,1]$ **(2016-J)**

2. Write the domain of $f(x)=\cos^{-1}x$.

Ans: $[-1,1]$. **(2014-M)**

3. Write the domain of $f(x)=\tan^{-1}x$.

Ans: R.

4. Write the domain of $f(x)=\cot^{-1}x$

Ans: R.

5. Write the domain of $f(x)=co\sec^{-1}x$

Ans: $R-(-1,1)$.

6. Write the domain of $f(x)=\sec^{-1}x$

Ans: $R-(-1,1)$.

7. Write the domain of $f(x)=\sin^{-1}2x$

Ans: $\left[\frac{-1}{2},\frac{1}{2}\right]$

8. Write the range of $f(x)=\sin^{-1}x$ other than $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ *in* $[0,2\pi]$

Ans: $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$

9. Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Ans:

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{4}\quad\left(\because\frac{\pi}{4}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\right)$

10. Find the principal value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Ans: $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)\qquad\frac{13\pi}{6}\notin[0,\pi]$

$\Rightarrow\cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{6}\right)\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}\quad\left(\because\frac{\pi}{6}\in[0,\pi]\right)$

11. Find the principle value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Ans: $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}\quad\left(\because\frac{\pi}{6}\in[0,\pi]\right)$

12. Find the principle value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Ans: $\sin^{-1}\left(-\frac{1}{2}\right)=-\sin^{-1}\left(\frac{1}{2}\right)$ **(2014-J)**

$=-\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)=-\frac{\pi}{6}.\quad\left(\because-\frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

13. Find the principle value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$

Ans: $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)\qquad\frac{3\pi}{5}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow\sin^{-1}\left(\sin\frac{3\pi}{5}\right)=\sin^{-1}\left(\sin\left(\pi-\frac{2\pi}{5}\right)\right)$

$=\sin^{-1}\left(\sin\left(\frac{2\pi}{5}\right)\right)=\frac{2\pi}{5}\quad\left(\because\frac{2\pi}{5}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

14. Find the principal value of $\tan^{-1}(-1)$.

Ans:

$\tan^{-1}(-1)=-\tan^{-1}(1)=-\tan^{-1}\left(\tan\left(\frac{\pi}{4}\right)\right)=-\frac{\pi}{4}$

15. Find the principal value of $\cot^{-1}(-\sqrt{3})$.

Ans:

$\cot^{-1}(-\sqrt{3})=\pi-\cot^{-1}(\sqrt{3})=\pi-\cot^{-1}\left(\cot\left(\frac{\pi}{6}\right)\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}.$

16. Find the principal value of $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Ans: $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{4}$

17. Find the principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Ans: $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\pi-\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **(2019-J)**

$=\pi-\cot^{-1}\left(\cot\left(\frac{\pi}{3}\right)\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}.$

18. Find the principal value of $\tan^{-1}\left(\tan\frac{13\pi}{6}\right)$

Ans: $\tan^{-1}\left(\tan\frac{13\pi}{6}\right)$ $\quad \frac{13\pi}{6}\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\Rightarrow \tan^{-1}\left(\tan\frac{13\pi}{6}\right)=\tan^{-1}\left(\tan\left(2\pi+\frac{\pi}{6}\right)\right)$

$=\tan^{-1}\left(\tan\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}.$ $\quad \left(\because \frac{\pi}{6}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$

19. Find the principal value of $\cos^{-1}\left(\frac{1}{2}\right)$

Ans: **(2019-M)**

$\cos^{-1}\left(\frac{1}{2}\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)=\frac{\pi}{3}$ $\quad \left(\because \frac{\pi}{3}\in[0,\pi]\right)$

20. Find the principal value of $\operatorname{cosec}^{-1}(2)$.

Ans: $\operatorname{cosec}^{-1}(2)=\operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}$

21. Find the principal value of $\sin^{-1}(\sin(-600))$

Ans: $\sin^{-1}(\sin(-600))$ $\quad \left(\because -600\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

$\Rightarrow \sin^{-1}(\sin(-600))=-\sin^{-1}(\sin(600))$

$=-\sin^{-1}\left(\sin\left(600X\frac{\pi}{180}\right)\right)=-\sin^{-1}\left(\sin\left(\frac{10\pi}{3}\right)\right)$

$=-\sin^{-1}\left(\sin\left(4\pi-\frac{2\pi}{3}\right)\right)=-\sin^{-1}\left(\sin\left(-\frac{2\pi}{3}\right)\right)$

$=\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ $\quad \left(\because \frac{2\pi}{3}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

$=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)=\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)=\frac{\pi}{3}\left(\because \frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

22. Find the value of

$\cos^{-1}\left(\cos\frac{3\pi}{2}\right)+\sin^{-1}\left(\sin\frac{3\pi}{2}\right)$

Ans: $\cos^{-1}\left(\cos\left(\pi+\frac{\pi}{2}\right)\right)+\sin^{-1}\left(\sin\left(\pi+\frac{\pi}{2}\right)\right)$

$\Rightarrow \cos^{-1}\left(\cos\left(\frac{\pi}{2}\right)\right)+\sin^{-1}\left(-\sin\left(\frac{\pi}{2}\right)\right)$

$\Rightarrow \cos^{-1}\left(\cos\left(\frac{\pi}{2}\right)\right)-\sin^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right)$

$\Rightarrow \frac{\pi}{2}-\frac{\pi}{2}=0$ $\quad \left(\because \frac{\pi}{2}\in[0,\pi]\ \&\ \frac{\pi}{2}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

23. Find the value of $\cot\left(\tan^{-1}x+\cot^{-1}x\right)$

Ans: $\cot\left(\tan^{-1}x+\cot^{-1}x\right)=\cot\left(\frac{\pi}{2}\right)=0$

24. Find the value of

$\cot\left(\tan^{-1}a+\cot^{-1}a\right),\ a\in R$

Ans: $\cot\left(\tan^{-1}a+\cot^{-1}a\right)=\cot\left(\frac{\pi}{2}\right)=0$

25. If $\tan^{-1}x+\cot^{-1}\sqrt{3}=\frac{\pi}{6}$. Find the value of x.

Ans: $\tan^{-1}x+\cot^{-1}\sqrt{3}=\frac{\pi}{6}$ $\quad \Rightarrow \tan^{-1}x+\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

$\Rightarrow \tan^{-1}\left(\frac{x+\frac{1}{\sqrt{3}}}{1-x\left(\frac{1}{\sqrt{3}}\right)}\right)=\frac{\pi}{6}$

$\Rightarrow \left(\frac{\frac{\sqrt{3}x+1}{\sqrt{3}}}{\frac{\sqrt{3}-x}{\sqrt{3}}}\right)=\tan\left(\frac{\pi}{6}\right)\Rightarrow \frac{\sqrt{3}x+1}{\sqrt{3}-x}=\frac{1}{\sqrt{3}}$

$\Rightarrow 3x+\sqrt{3}=\sqrt{3}-x \Rightarrow 4x=0 \Rightarrow \boxed{x=0}$.

26. Write the value of x for which

$2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ holds. **(2015-M)**

Ans: $x\geq 0$

27. Write the set of values of x for which

$2\cos^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ holds.

Ans: $\frac{1}{\sqrt{2}}\leq x\leq 1$

28. Write the set of values of x for which

$2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ holds.

Ans: $-\frac{1}{\sqrt{2}}\leq x\leq\frac{1}{\sqrt{2}}$

29. Write the set of values of x for which

$3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right)$ holds.

Ans: $x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

30. Write the set of values of x for which

$3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right)$ holds.

Ans: $x\in\left[\frac{1}{2},1\right]$

31. Write the set of values of x for which $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ holds.

Ans: $|x|<\frac{1}{\sqrt{3}}$

32. Write the set of values of x for which $\tan^{-1}\left(\frac{1}{x}\right)=\cot^{-1}x$ holds. **(2017-J)**

Ans: $x>0$

33. Find the principal value of x, if $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

Ans: $\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=\sin^{-1}(1)$

$\Rightarrow\sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}\quad\Rightarrow\boxed{x=\frac{1}{5}}$

$\left(\because\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right)$

34. Find the value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ **(2015-J)**

Ans: $\sin^{-1}\left(\frac{2\pi}{3}\right)\quad\left(\because\frac{2\pi}{3}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

$\Rightarrow\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)=\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)=\frac{\pi}{3}\left(\because\frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

35. Find the value of $\cos\left(\sec^{-1}x+\operatorname{cosec}^{-1}x\right),\ |x|\geq1$. **(2016-M)**

Ans: $\cos\left(\sec^{-1}x+\operatorname{cosec}^{-1}x\right)=\cos\left(\frac{\pi}{2}\right)=0.$

$\left(\because\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}\right)$

36. Find the principle value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ **(2017-M)**

Ans: $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)=-\operatorname{cosec}^{-1}\left(\sqrt{2}\right)$

$=-\operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\frac{\pi}{4}\right)\right)=-\frac{\pi}{4}$

37. Write the principle value branch of $\cos^{-1}(x)$. **(2018-M)**

Ans: $[0,\pi]$

38. Write the range of $y=\cos^{-1}x$ **(2018-J)**

Ans: $[0,\pi]$

39. Write the principle branch of
$(i)\ f(x)=\sin^{-1}(x)\quad(ii)\sec^{-1}(x)$ **(2020-M)**

$(iii)\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\quad(iv)\ \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

$(v)\sec^{-1}(-2)\quad(vi)\ \cos^{-1}\cos\left(\frac{5\pi}{3}\right)\ (vii)\tan^{-1}\left(-\sqrt{3}\right)$

Ans: $(i)\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\qquad(ii)[0,\pi]-\left\{\frac{\pi}{2}\right\}$

$(iii)\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\pi-\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}.$

$(iv)\ \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{6}.$

$(v)\sec^{-1}(-2)=\pi-\sec^{-1}(2)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

$(vi)\ \cos^{-1}\cos\left(\frac{5\pi}{3}\right)\qquad\left(\frac{5\pi}{3}\notin[0,\pi]\right)$

$\Rightarrow\cos^{-1}\cos\left(\frac{5\pi}{3}\right)=\cos^{-1}\cos\left(2\pi-\frac{\pi}{3}\right)$

$=\cos^{-1}\cos\left(\frac{\pi}{3}\right)=\frac{\pi}{3}.\quad\left(\because\frac{\pi}{3}\in[0,\pi]\right)$

$(vii)\tan^{-1}\left(-\sqrt{3}\right)=-\tan^{-1}\left(\sqrt{3}\right)=-\frac{\pi}{3}$

40. If $\tan^{-1}x=\frac{\pi}{10}$ for some $x\in R$ then the value of $\cot^{-1}x$ is ?

Ans: $\tan^{-1}x=\frac{\pi}{10}\Rightarrow\frac{\pi}{2}-\cot^{-1}x=\frac{\pi}{10}$

$\left(\because\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right)$

$\Rightarrow\cot^{-1}x=\frac{\pi}{2}-\frac{\pi}{10}=\frac{4\pi}{10}=\frac{2\pi}{5}.$

41. The domain of the function $y=\sin^{-1}\left(-x^2\right)$ is ?

Ans: $[-1,1]$

Question No: 12 & 13 (2M Each)

1. Prove that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2},\quad x\in[-1,1]$

(2015-J, 2019-M)

Ans: $Let\ \sin^{-1}x=y --------(1)$

$\Rightarrow x=\sin y \ \Rightarrow x=\cos\left(\frac{\pi}{2}-y\right)$

$\Rightarrow \cos^{-1}x=\frac{\pi}{2}-y \ \Rightarrow \cos^{-1}x+y=\frac{\pi}{2}$

$\Rightarrow \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}. \qquad (\because from\ (1))$

2. Prove that $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\ x\in R$

(2014-J)

Ans: $Let\ \tan^{-1}x=y \quad \to(1)$

$\Rightarrow x=\tan y \ \Rightarrow x=\cot\left(\frac{\pi}{2}-y\right)$

$\Rightarrow \cot^{-1}x=\frac{\pi}{2}-y \Rightarrow \cot^{-1}x+y=\frac{\pi}{2}$

$\Rightarrow \cot^{-1}x+\tan^{-1}x=\frac{\pi}{2}. \qquad (\because from\ (1))$

3. P.T $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\cos^{-1}x,\ \frac{1}{\sqrt{2}}\le x\le 1$

(2016-M, 2017-M)

Ans: $L.H.S=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$put\ x=\cos y \ \Rightarrow y=\cos^{-1}x \ --(1)$

$=\sin^{-1}\left(2\cos y\sqrt{1-\cos^2 y}\right)$

$=\sin^{-1}\left(2\cos y\sqrt{\sin^2 y}\right)$

$=\sin^{-1}(2\sin y\cos y)=\sin^{-1}(\sin 2y)$

$=2y$

$=2\cos^{-1}x=R.H.S \qquad (\because from(1))$

$\therefore \sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\cos^{-1}x, \quad \frac{1}{\sqrt{2}}\le x\le 1$

4. P.T $3\sin^{-1}x=\sin^{-1}(3x-4x^3),\ x\in\left[\frac{-1}{2},\frac{1}{2}\right].$

(2016-J)

Ans: $R.H.S=\sin^{-1}(3x-4x^3)$

$put\ x=\sin y \ \Rightarrow y=\sin^{-1}x \quad \to(1)$

$\Rightarrow \sin^{-1}(3\sin y-4\sin^3 y)=\sin^{-1}(\sin 3y)$

$\Rightarrow 3y=3\sin^{-1}x=L.H.S \qquad (\because from(1))$

$\therefore 3\sin^{-1}x=\sin^{-1}(3x-4x^3),\ x\in\left[\frac{-1}{2},\frac{1}{2}\right]$

5. S.T $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\quad x\ge 0$.

(2017-J)

Ans: $R.H.S=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$put\ x=\tan y \quad \Rightarrow y=\tan^{-1}x \ \to(1)$

$\Rightarrow \cos^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right)=\cos^{-1}(\cos 2y)$

$\Rightarrow 2y=2\tan^{-1}x=L.H.S \qquad (\because from(1))$

$\therefore 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\quad x\ge 0$

6. P.T $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x,\ for\ \frac{-1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

(2015-M)

Ans: $L.H.S=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$put\ x=\sin y \ \Rightarrow y=\sin^{-1}x \ ----(1)$

$=\sin^{-1}\left(2\sin y\sqrt{1-\sin^2 y}\right)$

$=\sin^{-1}\left(2\sin y\sqrt{\cos^2 y}\right)=\sin^{-1}(2\sin y\cos y)$

$=\sin^{-1}(\sin 2y)=2y$

$=2\sin^{-1}x=R.H.S \qquad (\because from(1))$

$\therefore \sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x, \quad -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

7. Find the value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ **.(2016-J)**

Ans: $\sin^{-1}\sin\left(\frac{2\pi}{3}\right) \qquad \left(\because \frac{2\pi}{3}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

$\Rightarrow \sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)=\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)=\frac{\pi}{3}$

8. Find the value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$

(2017-J, 2020-M)

Ans: $\sin^{-1}\sin\left(\frac{3\pi}{5}\right)$ $\left(\because \frac{3\pi}{5}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

$$\Rightarrow \sin^{-1}\left(\sin\left(\pi-\frac{2\pi}{5}\right)\right)=\sin^{-1}\left(\sin\left(\frac{2\pi}{5}\right)\right)=\frac{2\pi}{5}$$

9. Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ **(2018-J)**

Ans: $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ $\frac{13\pi}{6}\notin[0,\pi]$

$$\Rightarrow \cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{6}\right)\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}$$

10. Write the simplest form of $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, $0<x<\pi$ **(14-M, J, 2018-J)**

Ans: $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$

$$=\tan^{-1}\left(\sqrt{\tan^2\frac{x}{2}}\right)=\tan^{-1}\left(\tan\frac{x}{2}\right)=\frac{x}{2}.$$

11. Write $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$, $0<x<\pi$ in the simplest form. **(2015-J, 2018-M)**

Ans:

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=\tan^{-1}\left(\frac{\cos x\left(1-\frac{\sin x}{\cos x}\right)}{\cos x\left(1+\frac{\sin x}{\cos x}\right)}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+1.\tan x}\right)=\tan^{-1}(1)-\tan^{-1}(\tan x)=\frac{\pi}{4}-x$$

12. Write the simplest form of $\tan^{-1}\left[\frac{3\cos x-4\sin x}{4\cos x+3\sin x}\right]$ *if* $\frac{3}{4}\tan x<-1$ **(2016-M)**

Ans: $\tan^{-1}\left[\frac{3\cos x-4\sin x}{4\cos x+3\sin x}\right]$

divided by Nr and Dr by bcosx

$$=\tan^{-1}\left(\frac{\frac{3}{4}-\tan x}{1+\frac{3}{4}.\tan x}\right)=\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}(\tan x)$$

$$=\tan^{-1}\left(\frac{3}{4}\right)-x.$$

13. Write the simplest form of $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]$ *if* $\frac{a}{b}\tan x<-1$.

Ans: $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]$

divided by Nr and Dr by bcosx

$$=\tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}.\tan x}\right)=\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}(\tan x)=\tan^{-1}\left(\frac{a}{b}\right)-x.$$

14. Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$, $x>1$ in the simplest form. **(2019-M, 2019-J)**

Ans: $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$

put $x=\sec\theta \quad \Rightarrow \theta=\sec^{-1}x --------(1)$

$$\cot^{-1}\left(\frac{1}{\sqrt{\sec^2\theta-1}}\right)=\cot^{-1}\left(\frac{1}{\sqrt{\tan^2\theta}}\right)=\cot^{-1}\left(\frac{1}{\tan\theta}\right)$$

$$=\cot^{-1}(\cot\theta)=\theta=\sec^{-1}x. \quad (\because \textit{from}(1))$$

15. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$ then find the value of x. **(2018-M)**

Ans:

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}1$$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\frac{1}{5}=\frac{\pi}{2}-\cos^{-1}x \quad \left(\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right)$$

$$\Rightarrow \sin^{-1}\frac{1}{5}=\sin^{-1}x \quad \Rightarrow \boxed{x=\frac{1}{5}}.$$

16. Evaluate $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right\}$.**(2014-M)**

Ans: $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right\}=\sin\left\{\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right\}$

$$=\sin\left\{\frac{\pi}{3}+\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)\right\}=\sin\left\{\frac{\pi}{3}+\frac{\pi}{6}\right\}=\sin\left(\frac{\pi}{2}\right)=1.$$

17. P.T $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{31}{17}\right)$

(2015-M, 2020-M(3M))

Ans: $L.H.S = 2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$$= \tan^{-1}\left(\frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{1}{1-\frac{1}{4}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right)=\tan^{-1}\left(\frac{31}{17}\right)=R.H.S.$$

$$\therefore 2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{31}{17}\right)$$

18. P. T $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$ **(2015-J)**

Ans: $L.H.S = \tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$

$$= \tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}.\frac{7}{24}}\right)=\tan^{-1}\left(\frac{\frac{48+77}{264}}{\frac{264-14}{264}}\right)$$

$$= \tan^{-1}\left(\frac{125}{250}\right)=\tan^{-1}\left(\frac{1}{2}\right)=R.H.S.$$

$$\therefore \tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}.$$

19. Find the value of

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$$

Ans: $Let\ \tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$= \tan^{-1}\left(\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\frac{x}{y}.\frac{x-y}{x+y}}\right)=\tan^{-1}\left(\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y+x(x-y))}{y(x+y)}}\right)$$

$$= \tan^{-1}\left(\frac{x^2+\cancel{xy}-\cancel{yx}+y^2}{\cancel{yx}+y^2+x^2-\cancel{xy}}\right)=\tan^{-1}\left(\frac{x^2+y^2}{y^2+x^2}\right)$$

$$= \tan^{-1}(1)=\frac{\pi}{4}.$$

20. Solve the equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\quad (x>0)$$ **(2017-M)**

Ans: $Let\ \tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$

$$\Rightarrow \tan^{-1}(1)-\tan^{-1}(x)=\frac{1}{2}\tan^{-1}(x)$$

$$\Rightarrow \frac{\pi}{4}=\tan^{-1}(x)+\frac{1}{2}\tan^{-1}(x)$$

$$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}(x)\ \Rightarrow \frac{\pi}{6}=\tan^{-1}(x)$$

$$\Rightarrow x=\tan\left(\frac{\pi}{6}\right)\Rightarrow \boxed{x=\frac{1}{\sqrt{3}}}.$$

21. Prove that

$\cot^{-1}(-x)=\pi-\cot^{-1}x,\ \forall\ x\in R$ **(2020-M)**

Ans: $Let\ \cot^{-1}(-x)=y\ \rightarrow(1)$

$$\Rightarrow -x=\cot y\ \Rightarrow x=-\cot y$$

$$\Rightarrow x=\cot(\pi-y)\ \Rightarrow \cot^{-1}x=\pi-y$$

$$\Rightarrow \cot^{-1}x=\pi-\cot^{-1}(-x)\quad (\because from(1))$$

$$\Rightarrow \cot^{-1}(-x)=\pi-\cot^{-1}(x).$$

<u>Question No: 26 (3M)</u>

1. P. T $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ *where* $xy<1$.

Ans: **(2018-M)**

$Let\ \tan^{-1}x=\theta\ \&\ \tan^{-1}y=\phi\ then\ \tan\theta=x\ \&\ \tan\phi=y$

$$Now\ \tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta.\tan\phi}$$

$$\Rightarrow \tan(\theta+\phi)=\frac{x+y}{1-xy}$$

$$\Rightarrow \theta+\phi=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \boxed{\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)}.$$

2. P. T $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$ *where* $xy<1$.

Ans:

$Let\ \tan^{-1}x=\theta\ \&\ \tan^{-1}y=\phi\ then\ \tan\theta=x\ \&\ \tan\phi=y$

Now $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta.\tan\phi}$

$$\Rightarrow \tan(\theta-\phi)=\frac{x-y}{1+xy}$$

$$\Rightarrow \theta-\phi=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\Rightarrow \boxed{\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)}.$$

3. P. T $\tan^{-1}x+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right),\ |x|<\dfrac{1}{\sqrt{3}}$

(2014-J)

Ans: *Let* $x=\tan\theta\Rightarrow\theta=\tan^{-1}x \quad ------(1)$

$$R.H.S=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 3\theta)=3\theta=3\tan^{-1}x \quad (\because from(1))$$

$$=\tan^{-1}x+2\tan^{-1}x=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=L.H.S.$$

$$\therefore \tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

4. Find the value x if

$\tan^{-1}\left(\dfrac{x-1}{x-2}\right)+\tan^{-1}\left(\dfrac{x+1}{x+2}\right)=\dfrac{\pi}{4}$. **(2015-M)**

Ans: *Given* $\tan^{-1}\left(\dfrac{x-1}{x-2}\right)+\tan^{-1}\left(\dfrac{x+1}{x+2}\right)=\dfrac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}.\frac{x+1}{x+2}}\right)=\frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)}}{\frac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}}=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x^2+2x-x-2+x^2-2\Rightarrow x+x-2}{x^2-4-x^2+1}=1$$

$$\Rightarrow \frac{2x^2-4}{-3}=1 \qquad \Rightarrow 2x^2-4=-3$$

$$\Rightarrow 2x^2=4-3 \qquad \Rightarrow 2x^2=1$$

$$\Rightarrow x^2=\frac{1}{2}\Rightarrow \boxed{x=\pm\frac{1}{\sqrt{2}}}.$$

5. S. T $\tan^{-1}\left(\dfrac{1}{2}\right)+\tan^{-1}\left(\dfrac{2}{11}\right)+\tan^{-1}\left(\dfrac{4}{3}\right)=\dfrac{\pi}{2}$.

(2015-J, 2016-J, 2017-J)

Ans: $L.H.S=\tan^{-1}\left(\dfrac{1}{2}\right)+\tan^{-1}\left(\dfrac{2}{11}\right)+\tan^{-1}\left(\dfrac{4}{3}\right)$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}.\frac{2}{11}}\right)+\tan^{-1}\left(\frac{4}{3}\right)=\tan^{-1}\left(\frac{\frac{11+4}{22}}{\frac{22-2}{22}}\right)+\tan^{-1}\left(\frac{4}{3}\right)$$

$$=\tan^{-1}\left(\frac{15}{20}\right)+\tan^{-1}\left(\frac{4}{3}\right)=\tan^{-1}\left(\frac{3}{4}\right)+\cot^{-1}\left(\frac{3}{4}\right)$$

$$\left(\because \tan^{-1}(x)=\cot^{-1}\left(\frac{1}{x}\right)\right)$$

$$=\frac{\pi}{2}=R.H.S \quad \left(\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right)$$

6. Solve $\tan^{-1}(2x)+\tan^{-1}(3x)=\dfrac{\pi}{4}$, $x>0$

(2016-M, 2018-J, and 2019-J)

Ans: *Given* $\tan^{-1}(2x)+\tan^{-1}(3x)=\dfrac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-2x.3x}\right)=\frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1-6x^2}=\tan\frac{\pi}{4}\Rightarrow\frac{5x}{1-6x^2}=1$$

$$\Rightarrow 5x=1-6x^2$$

$$\Rightarrow 6x^2+5x-1=0\Rightarrow 6x^2+6x-x-1=0$$

$$\Rightarrow 6x(x+1)-(x+1)=0$$

$$\Rightarrow (x+1)(6x-1)=0 \quad \Rightarrow x=-1(or)\,x=\frac{1}{6}$$

$(x=-1 \text{ does not satisfy the equation})$

$$\therefore \boxed{x=\frac{1}{6}}.$$

7. Solve for x, if $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)=\dfrac{1}{2}\tan^{-1}x,\ x>0$

Ans: *Let* $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)=\dfrac{1}{2}\tan^{-1}x$

$$\Rightarrow \tan^{-1}(1)-\tan^{-1}(x)=\frac{1}{2}\tan^{-1}(x)$$

$$\Rightarrow \frac{\pi}{4}=\tan^{-1}(x)+\frac{1}{2}\tan^{-1}(x)$$

$$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}(x) \Rightarrow \frac{\pi}{6}=\tan^{-1}(x)$$

$$\Rightarrow x=\tan\left(\frac{\pi}{6}\right)\Rightarrow \boxed{x=\frac{1}{\sqrt{3}}}.$$

8. P.T $3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right),\ x\in\left[\frac{1}{2},1\right]$. **(2014-M)**

Ans: $R.H.S=\cos^{-1}\left(4x^3-3x\right)$

$$put\ x=\cos\theta \Rightarrow \theta=\cos^{-1}x---(1)$$

$$RHS=\cos^{-1}\left(4\cos^3\theta-3\cos\theta\right)=\cos^{-1}(\cos 3\theta)$$

$$=3\theta=3\cos^{-1}x=L.H.S \quad (\because from(1))$$

$$\therefore 3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right),\ x\in\left[\frac{1}{2},1\right]$$

9. Write $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $x\neq 0$ in the simplest form. **(2017-M)**

Ans: *Given* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

$$put\ x=\tan\theta \Rightarrow \theta=\tan^{-1}x \quad -----(1)$$

$$=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)=\tan^{-1}\left(\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right)=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x. \quad (\because from(1))$$

10. Prove that

$$\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}}\le x\le 1$$

Ans: $L.H.S=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$put\ x=\cos 2\theta \Rightarrow \theta=\frac{1}{2}\cos^{-1}x-----(1)$$

$$=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)=\tan^{-1}\left(\frac{\sqrt{2}(\cos\theta-\sin\theta)}{\sqrt{2}(\cos\theta+\sin\theta)}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan\theta}{1+1.\tan\theta}\right)=\tan^{-1}(1)-\tan^{-1}(\tan\theta)$$

$$=\frac{\pi}{4}-\theta=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x.=R.H.S \quad (\because from(1))$$

11. Prove that

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2},\ x\in\left(0,\frac{\pi}{4}\right)$$

Ans: *Consider* $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}=\frac{2+2\sqrt{\cos^2 x}}{2\sin x}$$

$$=\frac{2(1+\cos x)}{2\sin x}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\cot\frac{x}{2}$$

$$L.H.S=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=R.H.S$$

12. P.T $\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$ **(2019-M)**

Ans: $L.H.S=\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)$

Use

$$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$=\cos^{-1}\left(\frac{4}{5}.\frac{12}{13}-\sqrt{1-\left(\frac{4}{5}\right)^2}.\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$$

$$=\cos^{-1}\left(\frac{48}{65}-\sqrt{\frac{25-16}{25}}.\sqrt{\frac{169-144}{169}}\right)$$

$$=\cos^{-1}\left(\frac{48}{65}-\frac{3}{5}.\frac{5}{13}\right)=\cos^{-1}\left(\frac{48-15}{65}\right)$$

$$=\cos^{-1}\left(\frac{33}{65}\right)=R.H.S$$

13. Solve for x, if $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

Ans: $Given\ \sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

$put\ x=\sin\theta \quad \rightarrow (1)$

$$\sin^{-1}(1-\sin\theta)-2\sin^{-1}(\sin\theta)=\frac{\pi}{2}$$

$$\sin^{-1}(1-\sin\theta)-2\theta=\frac{\pi}{2}$$

$$\sin^{-1}(1-\sin\theta)=\frac{\pi}{2}+2\theta$$

$$1-\sin\theta=\sin\left(\frac{\pi}{2}+2\theta\right)$$

$$1-\sin\theta=\cos 2\theta \Rightarrow \cancel{1}-\sin\theta=\cancel{1}-2\sin^2\theta$$

$$2\sin^2\theta-\sin\theta=0 \ \Rightarrow \sin\theta(2\sin\theta-1)=0$$

$$\sin\theta=0\ \ (or)\ \ \sin\theta=\frac{1}{2}$$

$$\Rightarrow x=0 \quad (or) \quad x=\frac{1}{2} \quad (\because from(1))$$

$$\left(x=\frac{1}{2}\ does\ not\ satisfy\ equation\right)$$

$$\therefore \boxed{x=0}.$$

14. Solve for x, if

$$2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$$

Ans: $Given\ 2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\Rightarrow \frac{\cancel{2}\cos x}{1-\cos^2 x}=\frac{\cancel{2}}{\sin x}\Rightarrow \frac{\cos x}{\sin^2 x}=\frac{1}{\sin x}$$

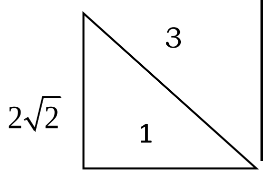
$$\Rightarrow \frac{\cos x}{\sin x}=1\Rightarrow \cot x=1\Rightarrow \boxed{x=\frac{\pi}{4}}.$$

15. Prove that $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Ans:

$$L.H.S=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$=\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{1}{3}\right)\right)$$



$$=\frac{9}{4}\cos^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)=R.H.S.$$

$$\because pytm(3)^2=1+x^2\Rightarrow x^2=8\Rightarrow x=2\sqrt{2}$$

$$\&\text{using } \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$

UNIT-3 MATRICES

$\left[Total\ marks:9,\ Q.No-3(1M),\ 26(3M)\ \&\ 40(5M)\right]$

Question No: 3 (1M)

1. Define a matrix.

Sol: A matrix is an ordered rectangular array of numbers or functions.

2. Define Diagonal matrix. **(2014 J)**

Sol: A square matrix is said to be diagonal matrix if all its non-diagonal elements are zero, that is a matrix $B=\left[b_{ij}\right]_{m\times m}$ is said to be a diagonal matrix if $b_{ij}=0,\ when\ i\neq j$

3. Define Scalar matrix. **(2014 M, 2015 J, 2016 J, 2019 M)**

Sol: A diagonal matrix is said to be scalar matrix if its diagonal elements are equal, that is, a square matrix $B=\left[b_{ij}\right]_{m\times m}$ is said to be scalar matrix if $b_{ij}=0,\ when\ i\neq j$ and $b_{ij}=k,\ when\ i=j$ for some constant k.

4. Define Symmetric matrix.

Sol: A square matrix $A=\left[a_{ij}\right]$ is said to be symmetric if $A'=A$, that is $\left[a_{ji}\right]=\left[a_{ij}\right]$ for all possible values of i and j.

5. Define skew-symmetric matrix.

Sol: A square matrix $A=\left[a_{ij}\right]$ is said to be skew symmetric if $A'=-A$, that is $\left[a_{ji}\right]=-\left[a_{ij}\right]$ for all possible values of i and j.

6. Define identity matrix.

Sol: A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an identity matrix.

7. Define invertible matrix.

Sol: If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that $AB=BA=I$ then B is called the inverse matrix of A and it is denoted by A^{-1}.

8. Find the number of all possible matrices of order 3x3 with each entry 0 or 1.

Sol: Total number of elements in 3×3 matrix is 9, each entry 0 or 1 so there are 2 elements. Total number of matrices is $2^9=512$

9. If A matrix is of order 1×3 and B is a matrix of order 3×1. Find the order of the matrix AB.

Sol: 1×1

10. Give an example of a matrix which is both symmetric and skew symmetric matrix.

Sol: Null matrix or Zero matrix of order $n\times n$

11. If A is square matrix satisfying $A^2=I$ then what is the inverse of A?

Sol: Inverse of A is A itself

12. If A is a matrix of order 3X4 and the order of AB is 3X3, what is the order of B.

Sol: 4×3

13. If X and Y are matrices of order 2×3 what is the order of 7X-5Y.

Sol: 2×3

14. If a matrix has 6 elements, then write all possible order it can have.

Sol: $1\times 6,\ 6\times 1,\ 2\times 3,\ 3\times 2$

15. If a matrix has 13 elements, write the all-possible orders it can have?

Sol: $1\times 13,\ 13\times 1$

16. If a matrix has 5 elements, then write all possible order it can have. **(2018-J, 2020-M)**

Sol: $1\times 5,\ 5\times 1$

17. Construct a 2x3 matrix $A=\left[a_{ij}\right]$ whose elements are given by $a_{ij}=\left|i-j\right|$

Sol: $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ $given\ a_{ij}=\left|i-j\right|$

$a_{11}=|1-1|=0, a_{12}=|1-2|=1, a_{13}=|1-3|=2$

$a_{21}=|2-1|=1, a_{22}=|2-2|=0, a_{23}=|2-3|=1$

$\therefore A=\begin{bmatrix}0&1&2\\1&0&1\end{bmatrix}$

18. Construct a 2x2 matrix $A=\left[a_{ij}\right]$ whose elements are given by $a_{ij}=\frac{i}{j}$ **(2017 J)**

Sol: $A=\begin{bmatrix}1&1/2\\2&1\end{bmatrix}$

19. Construct a 2x2 matrix $A=\left[a_{ij}\right]$ whose elements are given by $a_{ij}=\frac{1}{2}|-3i+j|$

Sol: $A=\begin{bmatrix}1&1/2\\5/2&2\end{bmatrix}$ **(2015- M, 2019-J)**

20. Construct a 2x2 matrix $A=\left[a_{ij}\right]$ whose elements are given by $a_{ij}=2i+j$

Sol: $A=\begin{bmatrix}3&4\\5&6\end{bmatrix}$

21. Construct a 2x2 matrix $A=\left[a_{ij}\right]$ whose elements are given by $a_{ij}=(i+j)^2$

Sol: $A=\begin{bmatrix}4&9\\9&16\end{bmatrix}$

22. Construct a 2x2 matrix $A=\left[a_{ij}\right]$ whose elements are given by $a_{ij}=\frac{i-j}{2}$

Sol: $A=\begin{bmatrix}0&-1/2\\1/2&0\end{bmatrix}$

23. Construct a 2x2 matrix $A=\left[a_{ij}\right]$ whose elements are given by $a_{ij}=i+j$

Sol: $A=\begin{bmatrix}2&3\\3&4\end{bmatrix}$ **(2017 M, 2018 M)**

24. Construct a 2x2 matrix $A=\left[a_{ij}\right]$ whose elements are given by $a_{ij}=i^2-j^2$

Sol: $A=\begin{bmatrix}0&-3\\3&0\end{bmatrix}$

25. If $\begin{bmatrix}x+2&y-3\\0&4\end{bmatrix}$ is a scalar matrix find x & y. **(2016 M)**

Sol: If a matrix is scalar then diagonal elements are equal and non-diagonal elements are zero

$x+2=4\Rightarrow x=2$ & $y-3=0\Rightarrow y=3$

26. If $\begin{bmatrix}5-x&2y-8\\2&3\end{bmatrix}$ is a symmetric matrix. Find x and y.

Sol: If A is symmetric then $A'=A$

$\begin{bmatrix}5-x&2\\2y-8&3\end{bmatrix}=\begin{bmatrix}5-x&2y-8\\2&3\end{bmatrix}$

$5-x=5-x\Rightarrow x=any\ real\ no$ & $2y-8=2\Rightarrow y=5$

27. Name the matrix $A=\begin{bmatrix}0&0&5\\0&5&0\\5&0&0\end{bmatrix}$

Sol: It's a Square matrix.

28. If $A=\begin{bmatrix}-2\\4\\5\end{bmatrix}$ & $B=\begin{bmatrix}1&3&-6\end{bmatrix}$ find AB.

Sol: $AB=\begin{bmatrix}-2\\4\\5\end{bmatrix}\begin{bmatrix}1&3&-6\end{bmatrix}=\begin{bmatrix}-2&-6&12\\4&12&-24\\5&15&-30\end{bmatrix}$

Question No: 26 (3M)

1. Find the value of x, y & Z in the following matrices

(i) $\begin{bmatrix}x+y&2\\5+z&xy\end{bmatrix}=\begin{bmatrix}6&2\\5&8\end{bmatrix}$ **(ii)** $\begin{bmatrix}x+2y&2\\4&x+y\end{bmatrix}=\begin{bmatrix}3&2\\4&1\end{bmatrix}$

Sol: (i) $x+y=2$ & $xy=8$

On solving we get

$x=2$ & $y=4$ or $x=4$ & $y=2$,

$5+z=5\Rightarrow z=0$

(ii) $x+2y=3$ & $x+y=1$

On solving we get $x=-1$ & $y=2$

2. If $X+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ *and* $X-Y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$ then find matrix X & Y.

Sol: adding given two equations

$$(X+Y)+(X-Y)=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}+\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

$$\Rightarrow 2X=\begin{bmatrix}10 & 0\\ 2 & 8\end{bmatrix}\Rightarrow X=\begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}$$

Substitute X in given equation, we get

$$Y=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$$

3. For any square matrix 'A' with real number prove that $A+A'$ is a symmetric matrix & $A-A'$ is a skew symmetric matrix. **(2014-J).**

Sol: If A is a square matrix with real entries

Let $B=A+A'$ apply transpose on both side

$B'=(A+A')'$

$B'=A'+(A')'$ $\quad\left(wkt\ (A+B)'=A'+B'\right)$

$B'=A'+A$ $\quad\left(wkt\ (A')'=A\right)$

$B'=A+A'$

$B'=B$ $\quad\therefore B=A+A'$ *is symmetric*

Let $C=A-A'$ apply transpose on both side

$C'=(A-A')'$

$C'=A'-(A')'$ $\quad\left(wkt\ (A-B)'=A'-B'\right)$

$C'=A'-A$ $\quad\left(wkt\ (A')'=A\right)$

$C'=-A+A'=-(A-A')$

$C'=-C$ $\quad\therefore C=A-A'$ *is skew symmetric*

4. For the matrix $A=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}$ verify that

(a) $A+A'$ is symmetric & **(b)** $A-A'$ is skew symmetric

Sol: (a) $A+A'=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}+\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}=\begin{bmatrix}2 & 11\\ 11 & 14\end{bmatrix}--(1)$

$$(A+A')'=\begin{bmatrix}2 & 11\\ 11 & 14\end{bmatrix}------(2)$$

From (1) & (2) $(A+A')'=A+A'$

$\therefore A+A'$ *is symmetric matrix*

(b) $A-A'=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}-\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}--(3)$

$$(A-A')'=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}------(4)$$

From (3) & (4) $(A-A')'=-(A-A')$

$\therefore A-A'$ *is skew symmetric matrix*

5. Express $A=\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}$ as a sum of symmetric & skew symmetric matrix. **(2015 J).**

Sol: $A=\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}$ & $A'=\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}$

WKT $A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$

Let $P=\frac{1}{2}(A+A')=\frac{1}{2}\left\{\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}+\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}\right\}$

$=\frac{1}{2}\begin{bmatrix}6 & 6\\ 6 & -2\end{bmatrix}=\begin{bmatrix}3 & 3\\ 3 & -1\end{bmatrix}$ $\quad\boxed{\therefore P'=P}$

P is a symmetric matrix

Let $Q=\frac{1}{2}(A-A')=\frac{1}{2}\left\{\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}-\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}\right\}$

$=\frac{1}{2}\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$ $\quad\boxed{\therefore Q'=-Q}$

Q is a skew-symmetric matrix

$$\therefore A=P+Q=\begin{bmatrix}3 & 3\\ 3 & -1\end{bmatrix}+\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}=\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}=A$$

6. If A & B are symmetric matrices of the same order then S.T AB is symmetric if and only if $AB=BA$. **(2017 M).**

Sol: Let A and B are symmetric matrices of same order *i.e.,* $A=A'$ & $B=B'$

Let AB is symmetric $(AB)'=AB$

$B'A'=AB$ $\quad\Rightarrow BA=AB$

Therefore, A and B are commute.
Conversely, Let A and B are commute

$BA = AB$ We have $(AB)' = B'A'$

$(AB)' = BA \Rightarrow (AB)' = AB,$

$\therefore AB$ is Symmetric.

7. If $F(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$ then show that $F(x)F(y)=F(x+y)$. **(2020- M).**

Sol: $F(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$ and

$$F(y)=\begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$F(x+y)=\begin{bmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{bmatrix}--(1)$$

$$F(x)F(y)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$F(x)F(y)=\begin{bmatrix}\cos x.\cos y-\sin x.\sin y & -\cos x.\sin y-\sin x.\cos y & 0\\ \sin x.\cos y+\cos x.\sin y & -\sin x.\sin y+\cos x.\cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$F(x)F(y)=\begin{bmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{bmatrix}----(2)$$

From (1) and (2) $F(x)F(y)=F(x+y)$

8. Find $\frac{1}{2}(A+A')$ & $\frac{1}{2}(A-A')$, when

$$A=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$$

Sol: Now,

$$\frac{1}{2}(A+A')=\frac{1}{2}\left(\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}+\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\frac{1}{2}(A-A')=\frac{1}{2}\left(\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}-\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}\right)$$

$$=\frac{1}{2}\begin{bmatrix}0 & 2a & 2b\\ -2a & 0 & 2c\\ -2b & -2c & 0\end{bmatrix}=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$$

9. Find A^{-1} by elementary operations

(i) $A=\begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}$ **(2018 M)** **(ii)** $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

(iii) $A=\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}$ **(2014-M)**

(iv) $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$ **(2019-M, 19-J)**

(v) $A=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$ **(2016 M)**

(vi) $A=\begin{bmatrix}3 & -1\\ -4 & 2\end{bmatrix}$ **(2018-J)**

Sol: (i) $A=\begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}$

By elementary row operation
we have $A = IA$

$$\begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}A$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A$$

$I = A^{-1}A$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

By elementary row operation we have $A = IA$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A$$

$R_2 \to -1/2\, R_2$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} A$$

$I = A^{-1}A$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

(iii) $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

By elementary row operation we have $A = IA$

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$R_2 \to (1/5)\, R_2$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2/5 & 1/5 \end{bmatrix} A$$

$R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix} A$$

$I = A^{-1}A$

$$\therefore A^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

(iv) $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

By elementary row operation we have $A = IA$

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$R_2 \to (-1/5)\, R_2$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2/5 & -1/5 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix} A$$

$I = A^{-1}A$

$$\therefore A^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$

(v) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

By elementary row operation we have $A = IA$

$$\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix}1 & -1\\ 0 & 5\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}A$$

$R_2 \to (1/5)R_2$

$$\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2/5 & 1/5\end{bmatrix}A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}3/5 & 1/5\\ -2/5 & 1/5\end{bmatrix}A$$

$I = A^{-1}A$

$$A^{-1}=\begin{bmatrix}3/5 & 1/5\\ -2/5 & 1/5\end{bmatrix}$$

(vi) $A=\begin{bmatrix}3 & -1\\ -4 & 2\end{bmatrix}$

By elementary row operation we have $A = IA$

$$\begin{bmatrix}3 & -1\\ -4 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix}1 & -1\\ -4 & 2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 0 & 1\end{bmatrix}A$$

$R_2 \to R_2 + 4R_1$

$$\begin{bmatrix}1 & -1\\ 0 & -2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ -4 & -3\end{bmatrix}A$$

$R_2 \to (-1/2)R_2$

$$\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 2 & 3/2\end{bmatrix}A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1/2\\ 2 & 3/2\end{bmatrix}A$$

$$I = A^{-1}A \Rightarrow A^{-1}=\begin{bmatrix}1 & 1/2\\ 2 & 3/2\end{bmatrix}$$

Question No: 40 (5M)

1. If $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}, B=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}$ *and* $C=\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$ calculate AC, BC & $(A+B)C$ verify that $(A+B)C = AC+BC$ **(2014-M)**

Sol: $A+B=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}+\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}=\begin{bmatrix}3 & 2\\ 3 & 4\end{bmatrix}$

$$(A+B)C=\begin{bmatrix}3 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}=\begin{bmatrix}3+4 & 3+6\\ 3+8 & 3+12\end{bmatrix}=\begin{bmatrix}7 & 9\\ 11 & 15\end{bmatrix}--(1)$$

$$AC=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}=\begin{bmatrix}1+4 & 1+6\\ 2+2 & 2+3\end{bmatrix}=\begin{bmatrix}5 & 7\\ 4 & 5\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}=\begin{bmatrix}2+0 & 2+0\\ 1+6 & 1+9\end{bmatrix}=\begin{bmatrix}2 & 2\\ 7 & 10\end{bmatrix}$$

$$AC+BC=\begin{bmatrix}5 & 7\\ 4 & 5\end{bmatrix}+\begin{bmatrix}2 & 2\\ 7 & 10\end{bmatrix}=\begin{bmatrix}7 & 9\\ 11 & 15\end{bmatrix}--(2)$$

From (1) and (2) $(A+B)C = AC+BC$

2. If $A=\begin{bmatrix}2 & 3\\ 4 & -5\end{bmatrix}, B=\begin{bmatrix}3 & 8\\ 11 & 21\end{bmatrix}$ *and* $C=\begin{bmatrix}7 & 13\\ 5 & 19\end{bmatrix}$ Calculate BA, CA & $(B+C)A$ verify that $(B+C)A = BA+CA$

Sol: $B+C=\begin{bmatrix}3 & 8\\ 11 & 21\end{bmatrix}+\begin{bmatrix}7 & 13\\ 5 & 19\end{bmatrix}=\begin{bmatrix}10 & 21\\ 16 & 40\end{bmatrix}$

$$(B+C)A=\begin{bmatrix}10 & 21\\ 16 & 40\end{bmatrix}\begin{bmatrix}2 & 3\\ 4 & -5\end{bmatrix}$$

$$=\begin{bmatrix}20+84 & 30-105\\ 32+160 & 48-200\end{bmatrix}=\begin{bmatrix}104 & -75\\ 192 & -152\end{bmatrix}--(1)$$

$$BA=\begin{bmatrix}3 & 8\\ 11 & 21\end{bmatrix}\begin{bmatrix}2 & 3\\ 4 & -5\end{bmatrix}=\begin{bmatrix}38 & -31\\ 106 & -72\end{bmatrix}$$

$$\& \ CA=\begin{bmatrix}7 & 13\\ 5 & 19\end{bmatrix}\begin{bmatrix}2 & 3\\ 4 & -5\end{bmatrix}=\begin{bmatrix}66 & -44\\ 86 & -80\end{bmatrix}$$

Now,

$$BA+CA=\begin{bmatrix}38 & -31\\ 106 & -72\end{bmatrix}+\begin{bmatrix}66 & -44\\ 86 & -80\end{bmatrix}=\begin{bmatrix}104 & -75\\ 192 & -152\end{bmatrix}--(2)$$

From, (1) and (2) $(B+C)A = BA+CA$

3. If $A=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}$ $B=\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix}$ $and\ C=\begin{bmatrix}2\\-2\\3\end{bmatrix}$

calculate $AC,\ BC\ and\ (A+B)\ C.$

also verify that $(A+B)\ C\ =\ AC+BC.$

(2016-J, 2017-J, 2018-M, 2019-J)

Sol: Now

$$A+B=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}+\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix}=\begin{bmatrix}0&7&8\\-5&0&10\\8&-6&0\end{bmatrix}$$

$$(A+B)C=\begin{bmatrix}0&7&8\\-5&0&10\\8&-6&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-14+24\\-10+0+30\\16+12+0\end{bmatrix}=\begin{bmatrix}10\\20\\28\end{bmatrix}--(1)$$

Now,

$$AC=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-12+21\\-12+0+24\\14+16+0\end{bmatrix}=\begin{bmatrix}9\\12\\30\end{bmatrix}\ \&$$

$$BC=\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-2+3\\2+0+6\\2-4+0\end{bmatrix}=\begin{bmatrix}1\\8\\-2\end{bmatrix}$$

$$AC+BC=\begin{bmatrix}9\\12\\30\end{bmatrix}+\begin{bmatrix}1\\8\\-2\end{bmatrix}=\begin{bmatrix}10\\20\\28\end{bmatrix}----(2)$$

$\therefore$ From (1) and (2) $(A+B)\ C\ =\ AC+BC.$

4. If $A=\begin{bmatrix}1&2&-3\\5&0&2\\1&-1&1\end{bmatrix},B=\begin{bmatrix}3&-1&2\\4&2&5\\2&0&3\end{bmatrix}\ and\ C=\begin{bmatrix}4&1&2\\0&3&2\\1&-2&3\end{bmatrix}$

then compute that $(A+B)\ \&\ (B-C)$

Also verify that $A+(B-C)=(A+B)-C$

(2015 J, 2020-M)

Sol: $A+B=\begin{bmatrix}4&1&-1\\9&2&7\\3&-1&4\end{bmatrix}\ \&\ B-C=\begin{bmatrix}-1&-2&0\\4&-1&3\\1&2&0\end{bmatrix}$

$$(A+B)-C=\begin{bmatrix}4&1&-1\\9&2&7\\3&-1&4\end{bmatrix}-\begin{bmatrix}4&1&2\\0&3&2\\1&-2&3\end{bmatrix}=\begin{bmatrix}0&0&-3\\9&-1&5\\2&1&1\end{bmatrix}--(1)$$

$$A+(B-C)=\begin{bmatrix}1&2&-3\\5&0&2\\1&-1&1\end{bmatrix}+\begin{bmatrix}-1&-2&0\\4&-1&3\\1&2&0\end{bmatrix}=\begin{bmatrix}0&0&-3\\9&-1&5\\2&1&1\end{bmatrix}--(2)$$

From (1) and (2) $A+(B-C)=(A+B)-C$

5. If $A=\begin{bmatrix}1&1&-1\\2&0&3\\3&-1&2\end{bmatrix},B=\begin{bmatrix}1&3\\0&2\\-1&4\end{bmatrix}\&C=\begin{bmatrix}1&2&3&-4\\2&0&-2&1\end{bmatrix}$

Prove that $(AB)\ C\ =\ A\ (BC)$

Sol:

$$AB=\begin{bmatrix}1&1&-1\\2&0&3\\3&-1&2\end{bmatrix}\begin{bmatrix}1&3\\0&2\\-1&4\end{bmatrix}=\begin{bmatrix}1+0+1&3+2-4\\2+0-3&6+0+12\\3+0-2&9-2+8\end{bmatrix}=\begin{bmatrix}2&1\\-1&18\\1&15\end{bmatrix}$$

$$(AB)C=\begin{bmatrix}2&1\\-1&18\\1&15\end{bmatrix}\begin{bmatrix}1&2&3&-4\\2&0&-2&1\end{bmatrix}=\begin{bmatrix}2+2&4+0&6-2&-8+1\\-1+36&-2+0&-3-36&4+18\\1+30&2+0&3-30&-4+15\end{bmatrix}$$

$$=\begin{bmatrix}4&4&4&-7\\35&-2&-39&22\\31&2&-27&11\end{bmatrix}--(1)$$

Now,

$$BC=\begin{bmatrix}1&3\\0&2\\-1&4\end{bmatrix}\begin{bmatrix}1&2&3&-4\\2&0&-2&1\end{bmatrix}=\begin{bmatrix}1+6&2+0&3-6&-4+3\\0+4&0+0&0-4&0+2\\-1+8&-2+0&-3-8&4+4\end{bmatrix}$$

$$=\begin{bmatrix}7&2&-3&-1\\4&0&-4&2\\7&-2&-11&8\end{bmatrix}$$

$$A(BC)=\begin{bmatrix}1&1&-1\\2&0&3\\3&-1&2\end{bmatrix}\begin{bmatrix}7&2&-3&-1\\4&0&-4&2\\7&-2&-11&8\end{bmatrix}$$

$$=\begin{bmatrix}7+4-7&2+0+2&-3-4+11&-1+2-8\\14+0+21&4+0-6&-6+0-33&-2+0+24\\21-4+14&6+0-4&-9+4-22&-3-2+16\end{bmatrix}$$

$$=\begin{bmatrix}4&4&4&-7\\35&-2&-39&22\\31&2&-27&11\end{bmatrix}--(2)$$

From (1) and (2) $(AB)\ C\ =\ A\ (BC)$

6. If $A=\begin{bmatrix}1&2&3\\3&-2&1\\4&2&1\end{bmatrix}$ verify that $A^3-23A-40I=O$

where O is zero matrix of order 3x3.

Sol: $A=\begin{bmatrix}1&2&3\\3&-2&1\\4&2&1\end{bmatrix}$ **(2014-J, 2015-M, 2019-M)**

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+12 & 2-4+6 & 3+2+3 \\ 3-6+4 & 6+4+2 & 9-2+1 \\ 4+6+4 & 8-4+2 & 12+2+1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 19+2+42 & 4+24+18 & 8+16+45 \\ 39-2+14 & 12-24+6 & 24-16+15 \\ 76+2+14 & 16+24+6 & 32+16+15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Now $LHS = A^3 - 23A - 40I$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - \begin{bmatrix} 23 & 46 & 69 \\ 69 & -46 & 23 \\ 92 & 46 & 23 \end{bmatrix} - \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3X3} = RHS$$

7. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ P. T $A^3 - 6A^2 + 7A + 2I = O$. **(2017 M)**

Sol: $A^2 = A \times A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 4+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$LHS = A^3 - 6A^2 + 7A + 2I$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} + \begin{bmatrix} -30 & 0 & -48 \\ -12 & -24 & -30 \\ -48 & 0 & -78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = RHS$$

8. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$ verify that $(AB)' = B'A'$.**(2016 M)**

Sol: $A' = \begin{bmatrix} -2 & 4 & 5 \end{bmatrix}$ & $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$

$$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

then $(AB)' = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$ ---(1)

Now,

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}\begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} \text{---}(2)$$

From (1) and (2) $(AB)' = B'A'$

9. If $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$ verify that $(AB)' = B'A'$. **(2018 J)**

Sol: $A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}$ & $B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$Now,\ AB=\begin{bmatrix}1\\-4\\3\end{bmatrix}\begin{bmatrix}-1 & 2 & 1\end{bmatrix}=\begin{bmatrix}-1 & 2 & 1\\4 & -8 & -4\\-3 & 6 & 3\end{bmatrix}$$

$$then\ \left(AB\right)'=\begin{bmatrix}-1 & 4 & -3\\2 & -8 & 6\\1 & -4 & 3\end{bmatrix}---(1)$$

$$B'A'=\begin{bmatrix}-1\\2\\1\end{bmatrix}\begin{bmatrix}1 & -4 & 3\end{bmatrix}=\begin{bmatrix}-1 & 4 & -3\\2 & -8 & 6\\1 & -4 & 3\end{bmatrix}---(2)$$

From (1) *and* (2) $(AB)'=B'A'$

10. If $A=\begin{bmatrix}3 & \sqrt{3} & 2\\4 & 2 & 0\end{bmatrix}$ *and* $B=\begin{bmatrix}2 & -1 & 2\\1 & 2 & 4\end{bmatrix}$

Verify that $\left(A+B\right)'=A'+B'$

Sol: $A'=\begin{bmatrix}3 & 4\\\sqrt{3} & 2\\2 & 0\end{bmatrix}$ *and* $B'=\begin{bmatrix}2 & 1\\-1 & 2\\2 & 4\end{bmatrix}$

$$A+B=\begin{bmatrix}3 & \sqrt{3} & 2\\4 & 2 & 0\end{bmatrix}+\begin{bmatrix}2 & -1 & 2\\1 & 2 & 4\end{bmatrix}=\begin{bmatrix}5 & \sqrt{3}-1 & 4\\5 & 4 & 4\end{bmatrix}$$

$$then\ \left(A+B\right)'=\begin{bmatrix}5 & 5\\\sqrt{3}-1 & 4\\4 & 4\end{bmatrix}---(1)$$

$$A'+B'=\begin{bmatrix}3 & 4\\\sqrt{3} & 2\\2 & 0\end{bmatrix}+\begin{bmatrix}2 & 1\\-1 & 2\\2 & 4\end{bmatrix}=\begin{bmatrix}5 & 5\\\sqrt{3}-1 & 4\\4 & 4\end{bmatrix}---(2)$$

From (1) and (2) $\left(A+B\right)'=A'+B'$

UNIT-4
DETERMINANTS

[Total marks: 12, Q.No − 4(1M), 14(2M), 41(5M) & 49.b(4M)]

Question No: 4 (1M)

1. Define Singular matrix.

Sol: A square matrix A is said to be singular if $|A| = 0$

2. Define non-singular matrix.

Sol: A square matrix A is said to be non-singular if $|A| \neq 0$

3. If $A = \begin{bmatrix} 3 & -2 & 0 \\ -4 & 5 & -3 \\ 2 & 4 & 6 \end{bmatrix}$ find the minor of -3.

Sol: Minor of -3 is $M_{23} = \begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix} = 18 - 0 = 18$

4. Find the cofactor of the element of a_{23} in $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$

Sol: Cofactor of a_{23} is

$A_{23} = (-1)^{2+3}(13) = -13$

5. If (i) $A = \begin{bmatrix} x & 2 \\ 8 & x \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ x & 8 \end{bmatrix}$ is a singular matrix. Find x.

Sol: (i) $|A| = \begin{vmatrix} x & 2 \\ 8 & x \end{vmatrix} = 0 \Rightarrow x^2 - 16 = 0 \Rightarrow x = \pm 4$

(ii) $|A| = \begin{vmatrix} 1 & 2 \\ x & 8 \end{vmatrix} = 0 \Rightarrow 8 - 2x = 0 \Rightarrow x = 4$

6. Find the value of x for which $\begin{vmatrix} 4 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

Sol: $\begin{vmatrix} 4 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} \Rightarrow 4 - x^2 = 3 - 8 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

7. Find the value of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

(2015-M, 2019-M, & 2019-J).

Sol: $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} \Rightarrow 3 - x^2 = 3 - 8$

$\Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$

8. Fin the value of x if $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

(2016 M, 2018-J, 2020-M)

Sol: $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix} \Rightarrow x^2 - 16 = 36 - 36$

$\Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

9. Fin the value of x if $\begin{vmatrix} x & 8 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 2 & 8 \\ 8 & 2 \end{vmatrix}$

Sol: $\begin{vmatrix} x & 8 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 2 & 8 \\ 8 & 2 \end{vmatrix} \Rightarrow x^2 - 64 = 4 - 64$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

10. Find $|3A|$, if $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$. **(2015 J)**

Sol: $3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

$\Rightarrow |3A| = 3(36 - 0) - 0 + 3(0) = 108$

11. If A is a square matrix of order 3 and $|A| = 4$ find value of $|2A|$.

Sol: We know that

$|kA| = k^n |A| \quad \therefore |2A| = 2^3 |A| = 8(4) = 32$

12. If A is a square matrix with $|A| = 6$ then find the value of $|AA^1|$.

Sol: $|A| = 6$ & $|A'| = 6$ wkt $|AB| = |A||B|$

$\therefore |AA'| = |A||A'| = 6 \times 6 = 36$

13. If A is a square matrix with $|A| = 8$ then find the value of $|AA^1|$. **(2017 M)**

Sol: $|A| = 8$ & $|A'| = 8$ wkt $|AB| = |A||B|$

$\therefore |AA'| = |A||A'| = 8 \times 8 = 64$

14. If A is square matrix of order 3, then find $|adjA|$ where $|A| = 2$

Sol: $n = 3$ & $|A| = 2$ wkt $|adjA| = |A|^{n-1} = 2^{3-1} = 4$

15. If A is a square matrix of order 3 and $A^{-1} = \frac{adjA}{10}$, then find $|3A|$.

Sol: $A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{10}(adjA) \Rightarrow |A| = 10$

We know that

$|kA| = k^n|A|$ $\therefore |3A| = 3^3|A| = 27(10) = 270$

16. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then find the value of $|2A|$ **(2014 M, 2014 J)**

Sol: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow 2A = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

$\Rightarrow |2A| = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = 16 - 24 = -8$

17. If $A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$ find $\det(adjA)$.

Sol: $|adj(A)| = |A|^{n-1} = (-14)^{2-1} = -14$

18. If A is a square matrix of order 3 & $|A| = 4$ then find $|adjA|$

Sol: $n = 3$ & $|A| = 4$ wkt $|adjA| = |A|^{n-1} = 4^{3-1} = 16$

19. If A and B are square matrices of order 3 and $A = 2B$ given that $|B| = -2$, then find $|A|$.

Sol: $A = 2B \Rightarrow |A| = |2B|$

$\Rightarrow |A| = 2^3|B| = 8(-2) = -16$ $\therefore |A| = -16$

20. What is the order of $|3I|$, where I is the identity matrix of order 3?

Sol: 3

21. If A is invertible matrix of order 2 find $|A^{-1}|$. **(2018 M)**

Sol: $|A^{-1}| = \frac{1}{|A|}$

22. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ find $|A^{-1}|$.

Sol: $|A^{-1}| = \frac{1}{|A|} = \frac{1}{4-6} = -\frac{1}{2}$

Question No: 14 (2M)

1. Find the area of triangle whose vertices are **(i)** (1, 3), (2, 5) & (7, 5) **(ii)** (2, 0), (-1, 0) & (0, 3) **(iii)** (-2, -3), (3, 2) & (-1, -8) **(2018-M, 2020-M)** **(iv)** (2,7), (1,1) & (10,8) **(2019 M)** **(v)** (3, 8), (-4, 2) & (5,1) **(2019 J)** using the determinant.

Sol: (i) Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 7 & 5 & 1 \end{vmatrix}$

Area of $\Delta = \frac{1}{2}[1(5-5) - 3(2-7) + 1(10-35)]$

$= \frac{1}{2}[0 + 15 - 25] = \frac{1}{2}|-10| = 5\,sq\ units$

(ii) Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}$

Area of $\Delta = \frac{1}{2}[2(0-3) - 0 + 1(-3-0)] = \frac{1}{2}|-9| = \frac{9}{2}sq\ units$

(iii) Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$

Area of $\Delta = \frac{1}{2}[-2(2+8) + 3(3+1) + 1(-24+2)]$

$= \frac{1}{2}[-20 + 12 - 22] = \frac{1}{2}|-30| = 15\ sq.units$

(iv) Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$

Area of $\Delta = \frac{1}{2}\left[2(1-8)-7(1-10)+1(8-10)\right]$

$= \frac{1}{2}[-14+63-2] = \frac{1}{2}|47| = \frac{47}{2}\ sq.units$

(v) Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$

Area of $\Delta = \frac{1}{2}\left[3(3-1)-8(-4-5)+1(-4-10)\right]$

$= \frac{1}{2}[3+72-14] = \frac{1}{2}|61| = \frac{61}{2}\ sq.units$

2. Find the value of 'K' using the determinant method, if the area of the triangle is

(i) 35 sq. Units with vertices (2, -6), (5, 4) & (k, 4). **(2018 J)**

(ii) 3 sq. units with vertices (1, 3), (0, 0) & (k, 0)

(iii) 4 sq. Units with vertices (-2, 0), (0, 4) & (0, K). **(2015 M, 2015 J)**

(iv) 4 sq. units with vertices (k, 0), (4, 0) & (0, 2). **(2017 M)**

Sol: (i Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = \pm 35$

Expand along R_1

$\Rightarrow \left[2(4-4)+6(5-k)+1(20-4k)\right] = \pm 70$

$30-6k+20-4k = \pm 70$

$50-10k = \pm 70$

$50-10k = +70 \ or \ 50-10k = -70$

$k = -2 \ or \ k = 12$

(ii) Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$

Expand along $R_2 \Rightarrow -1(0-3k) = \pm 6$

$\Rightarrow \ 3k = \pm 6 \ \Rightarrow k = \pm 2 \qquad k = 2 \ or \ k = -2$

(iii) Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 4$

Expand along $C_1 \Rightarrow -2(4-k) = \pm 8$

$-8+2k = +8 \ or \ -8+2k = -8$

$\Rightarrow \therefore k = +8 \ or \ k = 0$

(iv) Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \pm 4$

Expand along C_2

$\Rightarrow -2(k-4) = \pm 8$

$\Rightarrow -2(k-4) = +8 \ or \ -2(k-4) = -8$

$-2k = 0 \ or \ -2k = -16 \Rightarrow k = 0 \ or \ k = 8$

3. Find the equation of line passing through the points **(i)** (-3, -2) & (5, -7).

(ii) (1, 2) & (3, 6). **(2014 M)**

(iii) (3, 1) & (9, 3). **(2014 J)**

Sol: (i) Equation of line passing through two points is given by

$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0, \ \frac{1}{2}\begin{vmatrix} -3 & -2 & 1 \\ 5 & -7 & 1 \\ x & y & 1 \end{vmatrix} = 0$

$-3(-7-y)+2(5-x)+1(5y+7x) = 0$

$21+3y+10-2x+5y+7x = 0 \Rightarrow 5x+8y+31 = 0$

(ii) Equation of line passing through two points is given by

$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0, \ \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$

$1(6-y)-2(3-x)+1(3y-6x) = 0$

$6-y-6+2x+3y-6x = 0$

$\Rightarrow -4x+2y = 0 \Rightarrow y = 2x$

(iii) Equation of line passing through two points is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0,\ \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix}=0$$

$3(3-y)-1(9-x)+1(9y-3x)=0$

$9-3y-9+x+9y-3x=0$

$\Rightarrow\ 6y-2x=0\ \Rightarrow\ 6y=2x\ \Rightarrow 3y=x$

4. Using determinant S.T points $A(a,\ b+c),\ B(b,\ c+a)\ \&\ C(c,\ a+b)$ are collinear. **(2016 M)**

Sol: Area of $\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$=\frac{1}{2}\left[a\left((c+a)-(a+b)\right)-(b+c)(b-c)+1\left(b(a+b)-c(c+a)\right)\right]$

$=\frac{1}{2}\left[ac-ab-b^2+c^2+ba+b^2-c^2-ac\right]=\frac{1}{2}[0]=0$

Therefore, the given points are collinear

Question No: 41 (5 M)

Solve the following system of equation by matrix method

1. $x-y+z=4, 2x+y-3z=0\ \&\ x+y+z=2$ **(2014 M)**

Sol: The system of equation is written as $AX=B$

$$A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}\ \&\ B=\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$|A|=1(1+3)+1(2+3)+1(2-1)=10\neq 0$

hence A is non-singular & A^{-1} exists. Now,

$A_{11}=+\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix}=4,\ A_{12}=-\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}=-5,\ A_{13}=+\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}=1$

$A_{21}=-\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}=2,\ A_{22}=+\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0,\ A_{23}=-\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}-2$

$A_{31}=+\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix}=2,\ A_{32}=-\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}=5,\ A_{33}=+\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}=3$

Cofactor of $A=\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$ therefore

$$adjA=(cofactor\ of\ A)^T=\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Now, $A^{-1}=\frac{1}{|A|}(adjA),\ A^{-1}=\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

So $X=A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}\Rightarrow\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore x=2,\ y=-1\ \&\ z=1$

2. $x-y+2z=7, 3x+4y-5z=-5\ \&\ 2x-y+3z=12$ **(2018 M, 2014 J).**

Sol: The system of equation is written as $AX=B$

$$A=\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}\ \&\ B=\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$|A|=4\neq 0$ hence A is non-singular & A^{-1} exists.

$$Cofactor\ of\ A=\begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}$$

$$\therefore adjA=(Cofactor\ of\ A)^T=\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Now, $A^{-1}=\frac{1}{|A|}(adjA),\ A^{-1}=\frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$

So $X=A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore x = 2,\ y = 1\ \&\ z = 3$

3. $x - y + 2z = 1, 2y - 3z = 1, 3x - 2y + 4z = 2$ **(2017 M)**

Sol: *Try yourself* $Ans \Rightarrow x = 0,\ y = 5\ \&\ z = 3$

4. $2x - 3y + 5z = 11, 3x + 2y - 4z = -5, x + y - 2z = -3$ **(2016M)**

Sol: *Try yourself* $Ans \Rightarrow x = 1,\ y = 2\ \&\ z = 3$

5 $2x + 3y + 3z = 5, x - 2y + z = -4\ \&\ 3x - y - 2z = 3$ **(2015 M, 2015 J, 2020-M)**

Sol: *Try yourself* $Ans \Rightarrow x = 1,\ y = 2\ \&\ z = -1$

6. $x + y + z = 6, x - 2y + 3z = 6\ \&\ x - y + z = 2$

Sol: *Try yourself* $Ans \Rightarrow x = 1,\ y = 2\ \&\ z = 3$

7. $x + y + z = 6, x - y - z = -4\ \&\ x + 2y - 2z = -1$

Sol: *Try yourself* $Ans \Rightarrow x = 1,\ y = 2\ \&\ z = 3$

8 $4x + 3y + 2z = 60, 2x + 4y + 6z = 90\ \&\ 6x + 2y + 3z = 70$ **(2018-J)**

Sol: *Try yourself* $Ans \Rightarrow x = 5,\ y = 8\ \&\ z = 8$

9. $3x - 2y + 3z = 8, 2x + y - z = 1\ \&\ 4x - 3y + 2z = 4$ **(2019 M, 2019-J)**

Sol: *Try yourself* $Ans \Rightarrow x = 1,\ y = 2\ \&\ z = 3$

10. $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4,\ \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1\ \&\ \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

Sol: The system of equation is written as $AX = B$

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}\quad X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}\ \&\ B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$|A| = 2(120 - 45) - 3(-80 - 30) + 10(36 + 36) = 1200 \neq 0$$

hence A is non-singular & A^{-1} exists.

Now, $A_{11} = 75$ $A_{12} = 110$ $A_{13} = 72$

$A_{21} = 150$ $A_{22} = -100$ $A_{23} = 0$

$A_{31} = 75$ $A_{32} = 30$ $A_{33} = --24$

$$Cofactor\ of\ A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}$$

$$\therefore adjA = (Cofactor\ of\ A)^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|}(adjA),\ A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

So $X = A^{-1}B$

$$\begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

$\therefore x = 2,\ y = 3\ \&\ z = 5$

11. Use the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

to solve the system of equations

$x - y + 2z = 1,\ 2y - 3z = 1$ and $3x - 2y + 4z = 2$

Sol: The system of equation is written as $AX = B$

$$A=\begin{bmatrix}1&-1&2\\0&2&-3\\3&-2&4\end{bmatrix},\ X=\begin{bmatrix}x\\y\\z\end{bmatrix}\ \&\ B=\begin{bmatrix}1\\1\\2\end{bmatrix}$$

Now,

$$\begin{bmatrix}1&-1&2\\0&2&-3\\3&-2&4\end{bmatrix}\begin{bmatrix}-2&0&1\\9&2&-3\\6&1&-2\end{bmatrix}$$

$$=\begin{bmatrix}-2-9+12&0-2+2&1+3-4\\0+18-18&0+4-3&0-6+6\\-6-18+24&0-4+4&3+6-8\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

WKT $AA^{-1}=I$ therefore from above equation we can write directly

$$A^{-1}=\begin{bmatrix}-2&0&1\\9&2&-3\\6&1&-2\end{bmatrix}$$ So,

$$X=A^{-1}B\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2&0&1\\9&2&-3\\6&1&-2\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}0\\5\\3\end{bmatrix}$$

$\therefore x=0,\ y=5\ \&\ z=3$

12. The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers we get double of the second number. Represent it algebraically and find the numbers using matrix method.

Sol: Let us consider three numbers as $x, y \& z$

From the given statement $x+y+z=6$

$y+3z=11$

$x+z=2y\Rightarrow x-2y+z=0$

The system of equation is written as $AX=B$

$$A=\begin{bmatrix}1&1&1\\0&1&3\\1&-2&1\end{bmatrix},\ X=\begin{bmatrix}x\\y\\z\end{bmatrix}\ \&\ B=\begin{bmatrix}6\\11\\0\end{bmatrix}\Rightarrow|A|=9\neq0$$

hence A is non-singular & A^{-1} exists.

Now, $A_{11}=+7$ $A_{12}=3$ $A_{13}=-1$

$A_{21}=-3$ $A_{22}=0$ $A_{23}=3$

$A_{31}=+2$ $A_{32}=-3$ $A_{33}=+1$

$$cofactors\ of\ A=\begin{bmatrix}7&3&-1\\-3&0&3\\2&-3&1\end{bmatrix}$$

$$\Rightarrow adjA=(cofactors\ of\ A)'=\begin{bmatrix}7&-3&2\\3&0&-3\\-1&3&1\end{bmatrix}$$

Thus $A^{-1}=\frac{1}{|A|}adjA=\frac{1}{9}\begin{bmatrix}7&-3&2\\3&0&-3\\-1&3&1\end{bmatrix}$

So,

$$X=A^{-1}B\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{9}\begin{bmatrix}7&-3&2\\3&0&-3\\-1&3&1\end{bmatrix}\begin{bmatrix}6\\11\\0\end{bmatrix}=\frac{1}{9}\begin{bmatrix}9\\18\\27\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Therefore the number are

$\therefore x=1,\ y=2\ \&\ z=3$

13. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs.60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs.90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is Rs.70. Find the cost of each item per kg by matrix method.

Sol: Let the cost of onion, wheat and rice per kg be $Rs\,x,\ Rs\ y\ \&\ Rs\ z$ respectively. Then the given statement can be represented by a system of equations as

$4x+3y+2z=60,\ 2x+4y+6z=90\ \&\ 6x+2y+3z=70$

The system of equation is written as $AX=B$

$$A=\begin{bmatrix}4&3&2\\2&4&6\\6&2&3\end{bmatrix},\ X=\begin{bmatrix}x\\y\\z\end{bmatrix}\ \&\ B=\begin{bmatrix}60\\90\\70\end{bmatrix}\Rightarrow|A|=50\neq0$$

hence A is non-singular & A^{-1} exists.

Now, $A_{11}=0$ $A_{12}=30$ $A_{13}=-20$

$A_{21}=-5$ $A_{22}=0$ $A_{23}=10$

$A_{31}=+10$ $A_{32}=-20$ $A_{33}=+10$

$$cofactors\ of\ \ A=\begin{bmatrix}0 & 30 & -20\\ -5 & 0 & 10\\ 10 & -20 & 10\end{bmatrix}$$

$$\Rightarrow adjA=\left(cofactors\ of\ \ A\right)'=\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}$$

Thus $A^{-1}=\frac{1}{|A|}adjA=\frac{1}{50}\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}$

So,

$$X=A^{-1}B\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{50}\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}\begin{bmatrix}60\\ 90\\ 70\end{bmatrix}=\frac{1}{50}\begin{bmatrix}250\\ 400\\ 400\end{bmatrix}=\begin{bmatrix}5\\ 8\\ 8\end{bmatrix}$$

Therefore $\therefore x=5,\ y=8\ \&\ z=8$

cost of onion, wheat and rice per kg be $Rs\,5,\ Rs\ 8\ \&\ Rs\ 8$ respectively.

Question No: 49.b (4 M)

1. $\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=(a-b)(b-c)(c-a)$ **(2017 J)**

Sol: $LHS=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$

$R_2\to R_2-R_1\ \&\ R_3\to R_3-R_1$

$$=\begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix}$$

taking common $(b-a)\ \&\ (c-a)$ $From\ R_2\ \&\ R_1$

$=(b-a)(c-a)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 1 & c+a\end{vmatrix}$ expand along C_1

$=(b-a)(c-a)\left[(c-a)-(b-a)\right]$

$=(b-a)(c-a)(c-b)=(a-b)(b-c)(c-a)=RHS$

2. P. T $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$.

(2016 M)

Sol: $LHS=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}$,

$C_2\to C_2-C_1\ \&\ C_3\to C_3-C_1$

$$=\begin{vmatrix}1 & 0 & 0\\ a & b-a & c-a\\ a^3 & b^3-a^3 & c^3-a^3\end{vmatrix}$$

taking common $(b-a)\ \&\ (c-a)$ from $C_2\ \&\ C_3$

$$=(b-a)(c-a)\begin{vmatrix}1 & 0 & 0\\ a & 1 & 1\\ a^3 & b^2+ab+a^2 & c^2+ca+a^2\end{vmatrix}$$

$C_3\to C_3-C_2$

$$=(b-a)(c-a)\begin{vmatrix}1 & 0 & 0\\ a & 1 & 0\\ a^3 & b^2+ab+a^2 & c^2-b^2+ca-ab\end{vmatrix}$$

taking common $(c-b)$ $from\ C_3$

$$=(b-a)(c-a)(c-b)\begin{vmatrix}1 & 0 & 0\\ a & 1 & 0\\ a^3 & b^2+ab+a^2 & a+b+c\end{vmatrix}$$

$=(b-a)(c-a)(c-b)(a+b+c)$

$=(a-b)(b-c)(c-a)(a+b+c)$

3. P.T $\begin{vmatrix}x+4 & 2x & 2x\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}=(5x+4)(4-x)^2$

(2020-M)

Sol:

$LHS=\begin{vmatrix}x+4 & 2x & 2x\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}$, $C_1\to C_1+C_2+C_3$

$$=\begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix},$$

taking common $(5x+4)$ *from* C_1

$$=(5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix},$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$=(5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & 4-x & 2x \\ 0 & 0 & 4-x \end{vmatrix},$$

it's a upper diagonal matrix

$$=(5x-4)(1)(4-x)(4-x)=(5x-4)(4-x)^2 = RHS$$

4.Prove that $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

(2019-J)

Sol: $LHS = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}, C_1 \to C_1 + C_2 + C_3$

$$=\begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix}$$

taking common $(1+x+x^2)$ *from* C_1

$$=(1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}, R_2 \to R_2 - R_1 \ \& \ R_3 \to R_3 - R_1$$

$$=(1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-x & 1-x^2 \end{vmatrix}$$

taking common $(1-x)$ *from* R_2 & R_3

$$=(1+x+x^2)(1-x)(1-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & 1+x \end{vmatrix}$$

Expand along C_1

$$=(1+x+x^2)(1-x)(1-x)(1+x+x^2)$$

$$=(1-x^3)(1-x^3)=(1-x^3)^2 = RHS$$

5. S. T $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

(2014 J, 2018 M, 2018 J)

Sol: $LHS = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$=\begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix}$$

taking common $2(x+y+z)$ *from* C_1

$$=2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$=2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix}$$

$$=2(x+y+z)(x+y+z)(x+y+z)$$

$$=2(x+y+z)^3 = RHS$$

6. P.T $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & xz \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

(2017 M)

Sol:

$LHS = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & xz \\ z & z^2 & xy \end{vmatrix}, R_2 \to R_2 - R_1 \ \& \ R_3 \to R_3 - R_1$

$$=\begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & xz-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix}$$

taking common $(y-x)$ & $(z-x)$ *from* R_2 & R_3

$$=(y-x)(z-x)\begin{vmatrix} x & x^2 & yz \\ 1 & y+x & -z \\ 1 & z+x & -y \end{vmatrix}, R_3 \to R_3 - R_2$$

$$=(y-x)(z-x)\begin{vmatrix} x & x^2 & yz \\ 1 & y+x & -z \\ 0 & z-y & z-y \end{vmatrix}$$

taking common $(z-y)$ *from* R_3

$$=(y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & yz \\ 1 & y+x & -z \\ 0 & 1 & 1 \end{vmatrix}$$

expand along R_3

$$=(y-x)(z-x)(z-y)\left[0-1(xz-yz)+1\left(xy+x^2-x^2\right)\right]$$

$$=(y-x)(z-x)(z-y)(xy+yz+zx)=RHS$$

7. P.T $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}=(a+b+c)^3$

Sol: $LHS=\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$=\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix},$$

take $(a+b+c)$ *common from* R_1

$$=(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix},$$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

$$=(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$=(a+b+c)\left[-(a+b+c)\right]\left[-(a+b+c)\right]=(a+b+c)^3=RHS$$

8.P.T $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}=(1+a^2+b^2)^3$

Sol: $LHS=\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

$R_1 \to R_1 + bR_3$ & $R_2 \to R_2 - aR_3$

$$=\begin{vmatrix} 1+a^2+b^2 & 0 & -b\left(1+a^2+b^2\right) \\ 0 & 1+a^2+b^2 & a\left(1+a^2+b^2\right) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

take common $\left(1+a^2+b^2\right)$ *from* R_1 & R_2

$$=\left(1+a^2+b^2\right)\left(1+a^2+b^2\right)\begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$=\left(1+a^2+b^2\right)\left(1+a^2+b^2\right)\left[1\left(1-a^2-b^2+2a^2\right)-b(0-2b)\right]$$

$$=\left(1+a^2+b^2\right)\left(1+a^2+b^2\right)\left(1+a^2+b^2\right)$$

$$=\left(1+a^2+b^2\right)^3=RHS$$

9. P. T $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}=1+a^2+b^2+c^2$

(2015 M, J)

Sol: $LHS=\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

$$=\frac{abc}{abc}\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

multiply a to C_1, *b to* C_2 *& c to* C_3

$$=\frac{1}{abc}\begin{vmatrix} a\left(a^2+1\right) & ab^2 & ac^2 \\ a^2b & b\left(b^2+1\right) & bc^2 \\ ca^2 & cb^2 & c\left(c^2+1\right) \end{vmatrix}$$

take common a, b & c from R_1, R_2 & R_3

$$=\frac{abc}{abc}\begin{vmatrix}(a^2+1) & b^2 & c^2\\ a^2 & (b^2+1) & c^2\\ a^2 & b^2 & (c^2+1)\end{vmatrix},$$

$C_1 \to C_1+C_2+C_3$

$$=\begin{vmatrix}(a^2+b^2+c^2+1) & b^2 & c^2\\ (a^2+b^2+c^2+1) & (b^2+1) & c^2\\ (a^2+b^2+c^2+1) & b^2 & (c^2+1)\end{vmatrix}$$

take common $(a^2+b^2+c^2+1)$ from C_1

$$=(a^2+b^2+c^2+1)\begin{vmatrix}1 & b^2 & c^2\\ 1 & (b^2+1) & c^2\\ 1 & b^2 & (c^2+1)\end{vmatrix},$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$=(1+a^2+b^2+c^2)\begin{vmatrix}1 & b^2 & c^2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=(1+a^2+b^2+c^2)(1)=(1+a^2+b^2+c^2)=RHS$$

10. Prove that **(2019 M)**

$$\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}=ab+bc+ca+abc=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Sol: $LHS=\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}$

$$=\begin{vmatrix}a(1+1/a) & b(1/b) & c(1/c)\\ a(1/a) & b(1+1/b) & c(1/c)\\ a(1/a) & b(1/b) & a(1+1/a)\end{vmatrix}$$

take common a, b & c from C_1, C_2 & C_3

$$=abc\begin{vmatrix}(1+1/a) & (1/b) & (1/c)\\ (1/a) & (1+1/b) & (1/c)\\ (1/a) & (1/b) & (1+1/a)\end{vmatrix},$$

$C_1 \to C_1+C_2+C_3$

$$=abc\begin{vmatrix}(1+1/a+1/b+1/c) & (1/b) & (1/c)\\ (1+1/a+1/b+1/c) & (1+1/b) & (1/c)\\ (1+1/a+1/b+1/c) & (1/b) & (1+1/a)\end{vmatrix}$$

take common $(1+1/a+1/b+1/c)$ from C_1

$$=abc(1+1/a+1/b+1/c)\begin{vmatrix}1 & (1/b) & (1/c)\\ 1 & (1+1/b) & (1/c)\\ 1 & (1/b) & (1+1/a)\end{vmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$=abc(1+1/a+1/b+1/c)\begin{vmatrix}1 & (1/b) & (1/c)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=abc(1+1/a+1/b+1/c)=abc+bc+ca+ab=RHS$$

11. P.T $\begin{vmatrix}b+c & a & a\\ b & c+a & b\\ c & c & a+b\end{vmatrix}=4abc$ **(2014 M)**

Sol: $LHS=\begin{vmatrix}b+c & a & a\\ b & c+a & b\\ c & c & a+b\end{vmatrix}$, take common a, b & c from C_1, C_2 & C_3

$$=\begin{vmatrix}b+c & a & a\\ b & c+a & b\\ c & c & a+b\end{vmatrix}$$

$R_1 \to R_1-(R_2+R_3)$

$$=\begin{vmatrix}0 & -2c & -2b\\ b & c+a & b\\ c & c & a+b\end{vmatrix}$$

expanding

$$=2c((c+a)(a+b)-cb)-2b(bc-c(c+a))$$

$$=2abc+2abc=4abc=RHS$$

12. P.T $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Sol: $LHS = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

take common a, b & c from R_1, R_2 & R_3

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

take common a, b & c from C_1, C_2 & C_3

$$= (abc)(abc)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Apply $R_2 \to R_2 + R_1$ & $R_3 \to R_3 + R_1$

$$= (a^2b^2c^2)\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = a^2b^2c^2\left[(-1)(0-4)\right] = 4a^2b^2c^2$$

13. P.T $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Sol: $LHS = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \dfrac{abc}{abc}\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$,

multiply a, b & c to R_1, R_2 & R_3

$$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

take abc common from C_3

$$= \frac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

applying $C_1 \leftrightarrow C_3$ & $C_2 \leftrightarrow C_3$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^3 & c^3 \end{vmatrix} = RHS$$

UNIT-5

CONTINUITY & DIFFERENTIABILITY

$$\left[\begin{array}{l} Total\ marks: 20,\ Q.No-5(1M), 15(2M), 16(2M),\\ \quad 28(3M), 29(3M), 42(5M)\ \&\ 50.b(4M) \end{array}\right]$$

Question No: 5 (1M)

1. Find $\frac{dy}{dx}$ if **(a)** $\cos(1-x)$ **(2014-J)**

Sol: $y=\cos(1-x)$

Diff w r to x

$$\frac{dy}{dx}=-\sin(1-x)(0-1)=\sin(1-x)$$

(b) $y=log(sinx)$ **(2014-M)**

Sol: $y=\log(\sin x)$

Diff w r to x

$$\frac{dy}{dx}=\frac{1}{\sin x}.\cos x=\cot x$$

(c) $sin(x^2+1)$

Sol: $y=sin(x^2+1)$

Diff w r to x

$$\frac{dy}{dx}=\cos(x^2+1)\times 2x=2x\cos(x^2+1)$$

(d) $y=\tan(\sqrt{x})$ **(2020-M)**

Sol: $y=\tan(\sqrt{x})$

Diff w r to x

$$\frac{dy}{dx}=\sec^2(\sqrt{x})\times\frac{1}{2\sqrt{x}}=\frac{\sec^2(\sqrt{x})}{2\sqrt{x}}$$

(e) $y=cos(\sqrt{x})$ **(2015-J, 2017-M)**

Sol: $y=cos(\sqrt{x})$

Diff w r to x

$$\frac{dy}{dx}=\frac{-\sin(\sqrt{x})}{2\sqrt{x}}$$

(g) $y=\sin(x^2+5)$ **(2015-M, 2019-M)**

Sol: $y=\sin(x^2+5)$

Diff w r to x

$$\frac{dy}{dx}=\cos(x^2+5)\times 2x=2x\cos(x^2+5)$$

(h) $y=\sin(x^2)$ **(2017-J)**

Sol: $y=\sin(x^2)$

Diff w r to x

$$\frac{dy}{dx}=2x\cos(x^2)$$

(i) $y=e^{-x}$

Sol: $y=e^{-x}$

Diff w r to x

$$\frac{dy}{dx}=-e^{-x}$$

(j) $y=2\sqrt{\cot(x^2)}$

Sol: $y=2\sqrt{\cot(x^2)}$

Diff w r to x

$$\frac{dy}{dx}=\cancel{2}\frac{1}{\cancel{2}\sqrt{\cot(x^2)}}\times-\operatorname{cosec}^2(x^2)\times 2x$$

$$=\frac{-2x\operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$$

(k) $y=\sin^{-1}(x\sqrt{x}),\ 0\le x\le 1$

Sol: $y=\sin^{-1}(x\sqrt{x})=\sin^{-1}\left(x^{\frac{3}{2}}\right)$

Diff w r to x

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(x^{3/2})^2}}\times\left(\frac{3}{2}x^{\frac{1}{2}}\right)=\frac{3x^{\frac{1}{2}}}{2\sqrt{1-x^3}}$$

(l) $y=a^{\sin^{-1}x}$

Sol: $y=a^{\sin^{-1}x}$

Diff w r to x

$$\frac{dy}{dx}=a^{\sin^{-1}x}\log a\times\frac{d}{dx}(\sin^{-1}x)$$

$$=\frac{a^{\sin^{-1}x}\log a}{\sqrt{1-x^2}}$$

(m) $y = e^{\sec^2 x}$

Sol: $y = e^{\sec^2 x}$

Diff w r to x

$$\frac{dy}{dx} = e^{\sec^2 x} \times \frac{d}{dx}\left(\sec^2 x\right)$$

$$= e^{\sec^2 x} \times 2\sec x \times \sec x \tan x$$

$$= 2e^{\sec^2 x} \sec^2 x \tan x$$

(n) $y = \sin(ax+b)$ **(2018-J)**

Sol: $y = \sin(ax+b)$

Diff w r to x

$$\frac{dy}{dx} = a\cos(ax+b)$$

(o) $y = e^{3\log x}$

Sol: $y = e^{3\log x} = e^{\log x^3} = x^3$

Diff w r to x

$$\frac{dy}{dx} = 3x^2$$

(p) $y = e^{x^3}$ **(2018-M)**

Sol: $y = e^{x^3}$

Diff w r to x

$$\frac{dy}{dx} = e^{x^3} \frac{d}{dx}\left(x^3\right) = 3x^2 e^{x^3}$$

(q) $y = e^{\frac{1}{2}\log_e \cos x}$ **(2016-M)**

Sol: $y = e^{\frac{1}{2}\log_e \cos x}$

$$y = e^{\log_e \sqrt{\cos x}} = \sqrt{\cos x}$$

Diff w.r.to x

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\cos x}} \times -\sin x = \frac{-\sin x}{2\sqrt{\cos x}}$$

(r) $y = \cos(\sin x)$ **(Aug-2021)**

Sol: $y = \cos(\sin x)$

Diff w r to x

$$\frac{dy}{dx} = -\sin(\sin x) \times \cos x$$

(s) $y = (2x+1)^3$

Sol: $y = (2x+1)^3$

Diff w r to x

$$\frac{dy}{dx} = 3(2x+1)^2 \times 2 = 6(2x+1)^2$$

(t) $y = \sin(\log x),\ x > 0$

Sol: $y = \sin(\log x),\ x > 0$

Diff w r to x

$$\frac{dy}{dx} = \cos(\log x) \times \frac{1}{x} = \frac{\cos(\log x)}{x}$$

(u) $y = 2^x$

Sol: $y = 2^x$

Diff w r to x

$$\frac{dy}{dx} = 2^x \log 2$$

(v) $y = \sqrt{e^{\sqrt{x}}},\ x > 0$

Sol: $y = \sqrt{e^{\sqrt{x}}}$

Diff w r to x

$$\frac{dy}{dx} = \frac{1}{2\sqrt{e^{\sqrt{x}}}} \frac{d}{dx}\left(e^{\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \times e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{4\sqrt{x}\sqrt{e^{\sqrt{x}}}}$$

(w) $y = \cos^{-1}\left(e^x\right)$ **(2019-J)**

Sol: $y = \cos^{-1}\left(e^x\right)$

Diff w r to x

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(e^x\right)^2}} \frac{d}{dx} e^x = \frac{-e^x}{\sqrt{1-\left(e^x\right)^2}}$$

(x) $y=\sin\left(\tan^{-1}\left(e^{-x}\right)\right)$

Sol: $y=\sin\left(\tan^{-1}\left(e^{-x}\right)\right)$

Diff w r to x

$$\frac{dy}{dx}=\cos\left(\tan^{-1}\left(e^{-x}\right)\right)\frac{d}{dx}\tan^{-1}\left(e^{-x}\right)=\frac{\cos\left(\tan^{-1}\left(e^{-x}\right)\right)}{1+\left(e^{-x}\right)^2}\times -e^{-x}$$

2. Differentiate $\log(\cos e^x)$ w.r.to x.

Sol:

$$\frac{dy}{dx}=\frac{1}{\cos\left(e^x\right)}\frac{d}{dx}\cos\left(e^x\right)=\frac{-e^x\sin\left(e^x\right)}{\cos\left(e^x\right)}=-e^x\tan\left(e^x\right)$$

3. Find the derivative of $\tan(2x+3)$ w.r.to x **(2016-J)**

Sol: $\frac{dy}{dx}=\sec^2(2x+3)\frac{d}{dx}(2x+3)=2\sec^2(2x+3)$

4. If $f(x)=|\cos x|$, find $f'\left(\frac{3\pi}{4}\right)$.

Sol: when $\frac{\pi}{2}<x<\pi,\quad \cos x<0$

$\therefore |\cos x|=-\cos x.$

i.e., $f(x)=-\cos x \Rightarrow f'(x)=\sin x$

$f'\left(\frac{3\pi}{4}\right)=\sin\left(\frac{3\pi}{4}\right)=\frac{1}{\sqrt{2}}$

5. If $f(x)=|\cos x-\sin x|$ find $f'\left(\frac{\pi}{6}\right)$.

Sol: when $0<x<\frac{\pi}{4},\quad \cos x>\sin x$

so that $\cos x-\sin x>0$

i.e., $f(x)=\cos x-\sin x \Rightarrow f'(x)=-\sin x-\cos x$

$f'\left(\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{6}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}$

Question No: 15 (2M)

1. If $y+\sin y=\cos x$ *find* $\frac{dy}{dx}$ **(2014-M)**

Sol: $y+\sin y=\cos x$

Differentiate w r to x $\Rightarrow$

$\frac{dy}{dx}+\cos y\frac{dy}{dx}=-\sin x \Rightarrow \frac{dy}{dx}(1+\cos y)=-\sin x$

$\Rightarrow \frac{dy}{dx}=\frac{-\sin x}{(1+\cos y)}$

2. Find $\frac{dy}{dx}$ if $x^2+xy+y^2=100$

(2015-M, 2018-M & 2019 -J)

Sol: $x^2+xy+y^2=100 \Rightarrow$ Differentiate w r to x

$$2x+\left(x\frac{dy}{dx}+y(1)\right)+2y\frac{dy}{dx}=0$$

$\Rightarrow \frac{dy}{dx}(x+2y)=-(2x+y) \Rightarrow \frac{dy}{dx}=\frac{-(2x+y)}{(x+2y)}$

3. Find $\frac{dy}{dx}$ if $2x+3y=\sin y$ **(2017-J)**

Sol: $2x+3y=\sin y \Rightarrow$ Differentiate w r to x

$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{2}{\cos y-3}$

4. Find $\frac{dy}{dx}$ if $\sin^2 x+\cos^2 y=k$, k is constant.

Sol: Differentiate w r to x

$\Rightarrow 2\sin x\cos x+2\cos y\times -\sin y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\frac{2\sin x\cos x}{2\sin y\cos y}=\frac{\sin 2x}{\sin 2y}$

5. If $ax+by^2=\cos y$ find $\frac{dy}{dx}$ **(17-M, 2019-M)**

Sol: Differentiate w r to x

$\Rightarrow a+2by\frac{dy}{dx}=-\sin y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(2by+\sin y)=-a \Rightarrow \frac{dy}{dx}=\frac{-a}{2by+\sin y}$

6. If $\sqrt{x}+\sqrt{y}=\sqrt{10}$ S.T $\frac{dy}{dx}+\sqrt{\frac{y}{x}}=0$ **(2014-J)**

Sol: Differentiate w r to x

$$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \Rightarrow \frac{1}{\cancel{2}\sqrt{y}}\frac{dy}{dx}=-\frac{1}{\cancel{2}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}=-\sqrt{\frac{y}{x}} \qquad \therefore \frac{dy}{dx}+\sqrt{\frac{y}{x}}=0$$

7. Find, $\frac{dy}{dx}$ if $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

Sol: Diff w r to x

$$\Rightarrow \frac{2}{3}x^{\frac{2}{3}-1}+\frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx}=0 \Rightarrow \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx}=-\frac{2}{3}x^{\frac{-1}{3}}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x^{\frac{-1}{3}}}{y^{\frac{-1}{3}}} \quad \therefore \frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$$

8. Find $\frac{dy}{dx}$, if $y=x^a+a^x+a^a$ for some $a>0, x>0$

Sol: Differentiate w r to x

$$\Rightarrow \frac{dy}{dx}=ax^{a-1}+a^x\log a+0=ax^{a-1}+a^x\log a$$

9. Find $\frac{dy}{dx}$, if $y=\cos^{-1}(\sin x)$.

Sol: Differentiate w r to x

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{\sqrt{1-(\sin x)^2}}\frac{d}{dx}(\sin x)$$

$$=-\frac{\cos x}{\sqrt{1-\sin^2 x}}=-\frac{\cos x}{\sqrt{\cos^2 x}}=-1$$

10. Find $\frac{dy}{dx}$ if $y=\sec^{-1}\left[\frac{1}{2x^2-1}\right], 0<x<\frac{1}{\sqrt{2}}$

(2015-J, 2018-J)

Sol: Put $x=\cos\theta \to \theta=\cos^{-1}x$

$$y=\sec^{-1}\left[\frac{1}{2\cos^2\theta-1}\right]=\sec^{-1}\left[\frac{1}{\cos 2\theta}\right]$$

$$=\sec^{-1}[\sec 2\theta]=2\theta=2\cos^{-1}x \Rightarrow y=2\cos^{-1}x$$

Now, differentiate w r to x $\Rightarrow \frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$

11. If $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0<x<1$ find $\frac{dy}{dx}$

(2016-J)

Sol: Put $x=\tan\theta \to \theta=\tan^{-1}x$

$$y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)=2\theta$$

$$\therefore y=2\tan^{-1}x$$

Now, differentiate w r to x

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

12. If $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$ then find $\frac{dy}{dx}$

Sol: Put $x=\tan\theta \to \theta=\tan^{-1}x$

$$\Rightarrow y=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 3\theta)=3\theta \quad \therefore y=3\tan^{-1}x$$

Now, differentiate w r to x

$$\Rightarrow \frac{dy}{dx}=\frac{3}{1+x^2}$$

13. If $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ m

Sol: Put $x=\sin\theta \to \theta=\sin^{-1}x$

$$\Rightarrow y=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$=\sin^{-1}(2\sin\theta\cos\theta)=\sin^{-1}(\sin 2\theta)=2\theta$$

$$\therefore y=2\sin^{-1}x$$

Now, differentiate w r to x

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

14. Find $\frac{dy}{dx}$, if $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ then find $\frac{dy}{dx}$

Sol: Put $x=\tan\theta \to \theta=\tan^{-1}x$

$$\Rightarrow y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta$$

$$\therefore y=2\tan^{-1}x$$

Now, differentiate w r to x $\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$

15. Find the derivative of $(\cos x.\cos 2x.\cos 3x)$ w.r.to x

Sol: Let $y=(\cos x.\cos 2x.\cos 3x)$

$$\log y = \log(\cos x.\cos 2x.\cos 3x)$$
$$= \log\cos x + \log\cos 2x + \log\cos 3x$$

Differentiate w r to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}\times -\sin x + \frac{1}{\cos 2x}\times -2\sin 2x + \frac{1}{\cos 3x}\times -3\sin 3x$$
$$\frac{dy}{dx} = y(-\tan x - 2\tan 2x - 3\tan 3x)$$
$$= -\cos x\cos 2x\cos 3x(\tan x + 2\tan 2x + 3\tan 3x)$$

16. Find $\frac{dy}{dx}$, if $\sin^2 x + \cos^2 y = 1$ **(2020-M)**

Sol: Differentiate w r to x

$$\Rightarrow 2\sin x\cos x + 2\cos y \times -\sin y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{2\sin x\cos x}{2\sin y\cos y} = \frac{\sin 2x}{\sin 2y}$$

Question No: 16 (2M)

1. Find $\frac{dy}{dx}$, if $y = \log_7(\log x)$ **(2015-J)**

Sol: $y = \log_7(\log x) = \frac{\log(\log x)}{\log 7}$

Differentiate w r to x

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\log 7}\frac{d}{dx}(\log\log x)$$
$$= \frac{1}{\log 7}\times\frac{1}{\log x}\times\frac{1}{x} = \frac{1}{x\log x\log 7}$$

2. Find $\frac{dy}{dx}$, if $y = \log_5(\log x)$

Sol: $y = \log_5(\log x) = \frac{\log(\log x)}{\log 5}$

Differentiate w r to x

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\log 5}\frac{d}{dx}(\log\log x)$$
$$= \frac{1}{\log 5}\times\frac{1}{\log x}\times\frac{1}{x} = \frac{1}{x\log x\log 5}$$

3. If $y = x^x$ find $\frac{dy}{dx}$ **(14-M, 20-M, Aug-21)**

Sol: take log on both side

$\log y = \log x^x \Rightarrow \log y = x\log x$

Now, differentiate w r to

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\times\frac{1}{x} + \log x(1) \Rightarrow \frac{dy}{dx} = y(1+\log x) = x^x(1+\log x)$$

4. Find $\frac{dy}{dx}$ if $y = (\log x)^{\cos x}$

(2014-J, 2017-M, 2019-M, 2019-J)

Sol: take log on both side

$\log y = \log(\log x)^{\cos x} \Rightarrow \log y = \cos x\log(\log x)$

Now, differentiate w r to

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x\times\frac{1}{\log x}\times\frac{1}{x} + \log(\log x)\times -\sin x$$
$$\Rightarrow \frac{dy}{dx} = y\left(\frac{\cos x}{x\log x} - \sin x\log(\log x)\right)$$
$$= (\log x)^{\cos x}\left(\frac{\cos x}{x\log x} - \sin x\log(\log x)\right)$$

5. Differentiate $(\sin x)^x$ w.r.to x

Sol: take log on both side

$\log y = \log(\sin x)^x \Rightarrow \log y = x\log(\sin x)$

Now, diff w r to x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\times\frac{1}{\sin x}\times\cos x + \log(\sin x)(1)$$
$$\Rightarrow \frac{dy}{dx} = (\sin x)^x(x\cot x + \log\sin x)$$

6. Differentiate $(\sin x)^{\cos x}$ w.r.to x **(2018-J)**

Sol: take log on both side

$\log y = \log(\sin x)^{\cos x} \Rightarrow \log y = \cos x\log(\sin x)$

Now, diff w r to x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x\times\frac{1}{\sin x}\times\cos x + \log(\sin x)(-\sin x)$$
$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos x}(\cos x\cot x - \sin x\log(\sin x))$$

7. Differentiate $x^{\sin x}$, $x > 0$ w.r.to x.

Sol: take log on both side

$\log y = \log(x)^{\sin x} \Rightarrow \log y = \sin x\log(x)$

Now, diff w r to x

$$\frac{1}{y}\frac{dy}{dx} = \sin x \times \frac{1}{x} + \log x(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right)$$

8. If $y = (\sin^{-1} x)^x$ find $\frac{dy}{dx}$ **(18-M,17-J, 16-J)**

Sol: take log on both side

$$\log y = \log(\sin^{-1} x)^x \Rightarrow \log y = x\log(\sin^{-1} x)$$

Now, diff w r to x

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\sin^{-1} x} \times \frac{1}{\sqrt{1-x^2}} + \log(\sin^{-1} x)(1)$$

$$\Rightarrow \frac{dy}{dx} = (\sin^{-1} x)^x \left(\frac{x}{\sin^{-1} x\sqrt{1-x^2}} + \log(\sin^{-1} x)\right)$$

9. Differentiate $\left(x + \frac{1}{x}\right)^x$ w.r.to x **(2015-M)**

Sol: take log on both side

$$\log y = \log\left(x + \frac{1}{x}\right)^x \Rightarrow \log y = x\log\left(x + \frac{1}{x}\right)$$

Now, diff w r to x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\left(x + \frac{1}{x}\right)} \times \left(1 - \frac{1}{x^2}\right) + \log\left(x + \frac{1}{x}\right)(1)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(x + \frac{1}{x}\right)^x \left(\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right)$$

10. If $x^y = a^x$ P.T $\frac{dy}{dx} = \frac{x\log_e a - y}{x\log_e x}$ **(2016-M)**

Sol: take log on both side

$$\Rightarrow \log x^y = \log a^x \Rightarrow y\log x = x\log a$$

Now, diff w r to x

$$y \times \frac{1}{x} + \log x\frac{dy}{dx} = \log a$$

$$\Rightarrow \log x\frac{dy}{dx} = \log a - \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{x\log_e a - y}{x\log_e x}$$

11. If $xy = e^{x-y}$ Find $\frac{dy}{dx}$

Sol: $xy = e^{x-y}$

Diff w r to x

$$\Rightarrow x\frac{dy}{dx} + y = e^{x-y}\left(1 - \frac{dy}{dx}\right)$$

$$\Rightarrow x\frac{dy}{dx} - e^{x-y}\frac{dy}{dx} = e^{x-y} - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{x-y} - y}{x - e^{x-y}} = \frac{xy - y}{x - xy} = \frac{y(x-1)}{x(1-y)}$$

Question No: 28 (3M)

1. If $x = 2at^2$, $y = at^4$ then find $\frac{dy}{dx}$ **(20-M)**

Sol: $x = 2at^2$, $y = at^4 \Rightarrow \frac{dx}{dt} = 4at$ $\frac{dy}{dt} = 4at^3$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2$$

2. If $x = at^2$ & $y = 2at$ S.T $\frac{dy}{dx} = \frac{1}{t}$

Sol: $x = at^2$ & $y = 2at \Rightarrow$

$$\frac{dx}{dt} = 2at \ \& \ \frac{dy}{dt} = 2a \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

3. If $x = \sin t$, $y = \cos 2t$ then P.T $\frac{dy}{dx} = -4\sin t$

Sol: $x = \sin t$, $y = \cos 2t$

$$\Rightarrow \frac{dx}{dt} = \cos t \ \& \ \frac{dy}{dt} = -2\sin 2t$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\cos t} = \frac{-4\sin t\cos t}{\cos t} = -4\sin t$$

4. Find $\frac{dy}{dx}$ if $x = a\sec\theta$ & $y = b\tan\theta$

Sol: $x = a\sec\theta$ & $y = b\tan\theta$

$$\Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta \ \& \ \frac{dy}{d\theta} = b\sec^2\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a\sin\theta}$$

5. Find $\frac{dy}{dx}$ if $x = a(\cos\theta + \theta\sin\theta)$ & $y = a(\sin\theta - \theta\cos\theta)$. **(2014-M)**

Sol:

$$\frac{dx}{d\theta}=a\left(-\sin\theta+\left(\theta\cos\theta+\sin\theta\right)\right)=a\theta\cos\theta$$

$$\frac{dy}{d\theta}=a\left(\cos\theta-\left(-\theta\sin\theta+\cos\theta\right)\right)=a\theta\sin\theta$$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$$

6. If $x=a(\theta+\sin\theta)$ & $y=a(1-\cos\theta)$ P.T $\frac{dy}{dx}=\tan\left(\frac{\theta}{2}\right)$

(2014-J, 2019-J, Aug-2021)

Sol: $\frac{dx}{d\theta}=a(1+\cos\theta)$ & $\frac{dy}{d\theta}=a(\sin\theta)$

$$\Rightarrow\frac{dy}{dx}=\frac{a\sin\theta}{a(1+\cos\theta)}=\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}=\tan(\theta/2)$$

7. If $x=a(\theta-\sin\theta)$ & $y=a(1+\cos\theta)$ then

P.T $\frac{dy}{dx}=-\cot\left(\frac{\theta}{2}\right)$ **(2016-J, 2018-J)**

Sol: $\frac{dx}{d\theta}=a(1-\cos\theta)$ & $\frac{dy}{d\theta}=a(-\sin\theta)$

$$\Rightarrow\frac{dy}{dx}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}=-\cot(\theta/2)$$

8. If $x=a\cos^3\theta$ & $y=a\sin^3\theta$ P.T $\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$

(2016-M).

Sol:

$$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta \;\&\; \frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$$

$$\Rightarrow\frac{dy}{dx}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\tan\theta$$

Dividing given y and x

$$\Rightarrow\frac{y}{x}=\frac{a\sin^3\theta}{a\cos^3\theta}=\tan^3\theta$$

$$\Rightarrow\ \therefore\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}\Rightarrow\tan^3\theta=\frac{y}{x}\Rightarrow\tan\theta=\sqrt[3]{\frac{y}{x}}$$

9. Find $\frac{dy}{dx}$ if $x=a\left(\cos t+\log\tan\frac{t}{2}\right)$, $y=a\sin t$

(2015-J, 2017-J)

Sol: $x=a\left(\cos t+\log\tan\frac{t}{2}\right)$, $y=a\sin t$

$$\Rightarrow\frac{dx}{dt}=a\left(-\sin t+\frac{1}{\tan(t/2)}.\sec^2(t/2).\frac{1}{2}\right)$$

$$=a\left(-\sin t+\frac{1}{2\sin(t/2)\cos(t/2)}\right)$$

$$=a\left(\frac{1-\sin^2 t}{\sin t}\right)=a\frac{\cos^2 t}{\sin t}$$

$$\Rightarrow\frac{dy}{dt}=a\cos t\ \Rightarrow\frac{dy}{dx}=\frac{a\cos t}{a\cos^2 t/\sin t}=\tan t$$

10. Differentiate $\sin^2 x$ w.r.to $e^{\cos x}$. **(2017-M)**

Sol: Let $u=\sin^2 x$ & $v=e^{\cos x}$

$$\Rightarrow\frac{du}{dx}=2\sin x\cos x\quad \frac{dv}{dx}=-\sin xe^{\cos x}$$

$$\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{2\sin x\cos x}{-\sin xe^{\cos x}}=-\frac{2\cos x}{e^{\cos x}}$$

11. If $x=\sqrt{a^{\sin^{-1}t}}$, $y=\sqrt{a^{\cos^{-1}t}}$ S.T $\frac{dy}{dx}=\frac{-y}{x}$

(2015-M)

Sol: Multiply given x and y

$$\Rightarrow xy=\sqrt{a^{\sin^{-1}t}}\times\sqrt{a^{\cos^{-1}t}}=\sqrt{a^{\sin^{-1}t}a^{\cos^{-1}t}}$$

$$=\sqrt{a^{\sin^{-1}t+\cos^{-1}t}}=\sqrt{a^{\pi/2}}$$

$$xy=\sqrt{a^{\pi/2}}\ \Rightarrow x\frac{dy}{dt}+y=0\Rightarrow\frac{dy}{dx}=-\frac{y}{x}$$

$\left(wkt\ \sin^{-1}t+\cos^{-1}t=\pi/2\ \&\ \sqrt{a^{\pi/2}}\ is\ cons\tan t\right)$

12. Find $\frac{dy}{dx}$, if $x^y=y^x$

Sol: $x^y=y^x$ taking log on both side

$$y\log x=x\log y$$

Now diff w r to x

$$\Rightarrow y.\frac{1}{x}+\log x\frac{dy}{dx}=x\frac{1}{y}\frac{dy}{dx}+\log y(1)$$

$$\Rightarrow\frac{dy}{dx}\left(\log x-\frac{x}{y}\right)=\log y-\frac{y}{x}\Rightarrow\therefore\frac{dy}{dx}=\frac{y(x\log y-y)}{x(y\log x-x)}$$

13. Find $\frac{dy}{dx}$ if $x^y + y^x = 1$

Sol: Let $u = x^y$ & $v = y^x \Rightarrow u + v = 1 \Rightarrow$

$$diff\ w\ r\ to\ x \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0 -------(1)$$

$$\log u = y\log x \Rightarrow \frac{1}{u}\frac{du}{dx} = y\frac{1}{x} + \log x\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$$

$$\log v = x\log y$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = x\frac{1}{y}\frac{dy}{dx} + \log y \Rightarrow \frac{dv}{dx} = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$

Eqn (1) becomes

$$x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$$

$$\Rightarrow \frac{dy}{dx}\left[x^y\log x + x.y^{x-1}\right] = -\left[y.x^{y-1} + y^x\log y\right]$$

$$\Rightarrow \frac{dy}{dx} = -\left[\frac{y.x^{y-1} + y^x\log y}{x^y\log x + x.y^{x-1}}\right]$$

14. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ w.r.to x

Sol: $u = x^{\sin x}$ taking log $\Rightarrow$

$$\log u = \sin x\log x \Rightarrow \frac{1}{u}\frac{du}{dx} = \sin x\frac{1}{x} + \cos x\log x$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x\log x\right)$$

$$v = (\sin x)^{\cos x} \Rightarrow \log v = \cos x\log(\sin x)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{\cos x}{\sin x}.\cos x - \sin x\log(\sin x)$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}(\cot x.\cos x - \sin x.\log(\sin x))$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x\log x\right) + (\sin x)^{\cos x}(\cot x.\cos x - \sin x.\log(\sin x))$$

15. Differentiate $x^x - 2^{\sin x}$ w.r.to x

Sol: $u = x^x$ taking log on both side

$$\Rightarrow \log u = x\log x \Rightarrow \frac{1}{u}\frac{du}{dx} = x.\frac{1}{x} + \log x$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

$$v = 2^{\sin x} \Rightarrow \frac{dv}{dx} = 2^{\sin x}\log 2.\cos x$$

$$\therefore y = u - v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$$= x^x(1 + \log x) - 2^{\sin x}\log 2.\cos x$$

16. Find $\frac{dy}{dx}$, if $(x+3)^2.(x+4)^3.(x+5)^4$

Sol: Let $y = (x+3)^2.(x+4)^3.(x+5)^4$

taking log on both side

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$

$$(\because \log abc = \log a + \log b + \log c)$$

$$\log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Now diff w r to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{(x+3)} + \frac{3}{(x+4)} + \frac{4}{(x+5)}$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2.(x+4)^3.(x+5)^4\left(\frac{2}{(x+3)} + \frac{3}{(x+4)} + \frac{4}{(x+5)}\right)$$

Question No: 29 (3M)

1. Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8,\ x \in [-4, 2]$.

(2015-M, 2017-M, 2019-M)

Sol: $f(x) = x^2 + 2x - 8$ is a polynomial function **i.** $f(x)$ is continuous in $[-4, 2]$

ii. $f(x)$ is differentiable in $(-4, 2)$

Now, $f'(x) = 2x + 2$

$$f(a) = f(-4) = (-4)^2 + 2(-4) - 8 = 0$$

$$f(b) = f(2) = (2)^2 + 2(2) - 8 = 0$$

hence $f(a) = f(b)$

There exists at least one value

$c \in (-4, 2)$ *such that* $f'(c) = 0$

$2c+2=0 \Rightarrow c=-1\in(-4,2)$.

hence Rolle's theorem is verified.

2. Verify Rolle's Theorem for the function $y=x^2+2,\ x\in[-2,2]$. **(2014-M)**

Sol: Let $y=x^2+2$ is a polynomial function

i. $f(x)$ is continuous in $[-2,2]$

ii. $f(x)$ is differentiable in $(-2,2)$

Now, $f'(x)=2x$

$f(a)=f(-2)=(-2)^2+2=6$

$f(b)=f(2)=(2)^2+2=6$

$hence\ f(a)=f(b)$

There exists at least one value $c\in(-2,2)$ *such that* $f'(c)=0$

$2c=0 \Rightarrow c=0\in(-2,2)$.

Hence Rolle's theorem is verified.

3. Verify M.V.T for $f(x)=x^2$ in the interval [-2, 2]. **(2017-J, 2018-M)**

Sol: Let $f(x)=x^2$ is a polynomial function

i. $f(x)$ is continuous in $[-2,2]$

ii. $f(x)$ is differentiable in $(-2,2)$

Now, $f'(x)=2x$

$f(a)=f(-2)=(-2)^2=4$

$f(b)=f(2)=(2)^2=4$

There exists at least one value

$c\in(-2,2)$ *such that* $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$2c=\dfrac{4-4}{2-(-2)} \Rightarrow 2c=0 \Rightarrow c=0\in(-2,2)$.

Hence MVT theorem is verified.

4. Verify M.V.T for $f(x)=x^2$ in the interval [2, 4]. **(2016-J)**

Sol: Let $f(x)=x^2$ is a polynomial function

i. $f(x)$ is continuous in $[2,4]$

ii. $f(x)$ is differentiable in $(2,4)$

Now, $f'(x)=2x$

$f(a)=f(2)=(2)^2=4$

$f(b)=f(4)=(4)^2=16$

There exists at least one value

$c\in(2,4)$ *such that* $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$2c=\dfrac{16-4}{4-2} \Rightarrow 2c=6 \Rightarrow c=3\in(2,4)$.

Hence MVT theorem is verified.

5. Verify M.V.T for the function $f(x)=x^2-4x-3$ in the interval [1, 4]. **(14-J, 15-J 19-J, 2020-M)**

Sol: Let $f(x)=x^2-4x-3$ is a polynomial function

i. $f(x)$ is continuous in $[1,4]$

ii. $f(x)$ is differentiable in $(1,4)$

Now, $f'(x)=2x-4$

$f(a)=f(1)=(1)^2-4(1)-3=-6$

$f(b)=f(4)=(4)^2-4(4)-3=-3$

There exists at least one value

$c\in(1,4)$ *such that* $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$2c-4=\dfrac{-3-(-6)}{4-1} \Rightarrow 2c-4=1 \Rightarrow c=\dfrac{5}{2}=2.5\in(1,4)$

Hence MVT theorem is verified.

6. Verify mean value theorem for the function $f(x)=x^3-5x^2-3x$ in the interval [1, 3]. **(2016-M)**

Sol: Let $f(x)=x^3-5x^2-3x$ is a polynomial function.

i. $f(x)$ is continuous in $[1,3]$

ii. $f(x)$ is differentiable in $(1,3)$

Now, $f'(x)=3x^2-10x-3$

$$f(a)=f(1)=(1)^3-5(1)^2-3(1)=-7$$
$$f(b)=f(3)=(3)^3-5(3)^2-3(3)=-27$$

There exists at least one value

$$c\in(1,3)\ such\ that\ f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$3c^2-10c-3=\frac{-27-(-7)}{3-1}\Rightarrow 3c^2-10c-3=-10$$
$$\Rightarrow 3c^2-10c+7=0$$
$$(c-1)(3c-7)=0\Rightarrow c=1\quad c=7/3$$
$$here,\ c=1\notin(1,3)\ \therefore c=7/3=2.33\in(1,3)$$

Hence MVT theorem is verified.

Question No: 42 (5M)

1. If $y=Ae^{mx}+Be^{nx}$ then prove that

$$\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=0$$ **(2015-M, 2018-J)**

Sol: $y=Ae^{mx}+Be^{nx}$ differentiate w r to x

$$\frac{dy}{dx}=Ame^{mx}+Bne^{nx},$$

differentiate again w r to x

$$\frac{d^2y}{dx^2}=Am^2e^{mx}+Bn^2e^{nx}$$

$$LHS=\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny$$
$$=Am^2e^{mx}+Bn^2e^{nx}-(m+n)\left(Ame^{mx}+Bne^{nx}\right)$$
$$+mn\left(Ae^{mx}+Be^{nx}\right)$$
$$=Am^2e^{mx}+Bn^2e^{nx}-Am^2e^{mx}-Bmne^{nx}$$
$$-Amne^{mx}-Bn^2e^{nx}+Amne^{mx}+Bmne^{nx}$$
$$=0=RHS$$

2. If $y=3e^{2x}+2e^{3x}$ Prove that

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$ **(2014-M)**

Sol: $y=3e^{2x}+2e^{3x}$, differentiate w r to x

$$\frac{dy}{dx}=6e^{2x}+6e^{3x},$$

differentiate again w r to x

$$\frac{d^2y}{dx^2}=12e^{2x}+18e^{3x}$$

$$LHS=\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y$$
$$=12e^{2x}+18e^{3x}-5\left(6e^{2x}+6e^{3x}\right)+6\left(3e^{2x}+2e^{3x}\right)$$
$$=12e^{2x}+18e^{3x}-30e^{2x}-30e^{3x}+18e^{2x}+12e^{3x}$$
$$=0=RHS$$

3. If $y=500e^{7x}+600e^{-7x}$ show that $y_2=49y$.

Sol: $y=500e^{7x}+600e^{-7x}$ differentiate w r to x

$$\frac{dy}{dx}=500(7)e^{7x}+600(-7)e^{-7x}$$

differentiate again w r to x

$$\frac{d^2y}{dx^2}=500(7)(7)e^{7x}+600(-7)(-7)e^{-7x}$$
$$\frac{d^2y}{dx^2}=(49)500e^{7x}+(49)600e^{-7x}$$
$$\frac{d^2y}{dx^2}=49\left(500e^{7x}+600e^{-7x}\right)=49y=RHS$$

4. If $y=3cos(logx)+4sin(logx)$ Show that

$x^2y_2+xy_1+y=0$ **(2014-J, 16-J, 17-J, 19-J)**

Sol: $y=3cos(logx)+4sin(logx)$,

differentiate w r to x

$$\frac{dy}{dx}=-3\sin(logx).\frac{1}{x}+4\cos(logx).\frac{1}{x}$$
$$x\frac{dy}{dx}=-3\sin(\log x)+4\cos(\log x)$$

differentiate again w r to x

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}(1)=-3\cos(\log x).\frac{1}{x}-4\sin(\log x).\frac{1}{x}$$

$$x\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)=-(3cos(logx)+4sin(logx))$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y \quad \Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

5. If $y=5\cos(\log x)+7\sin(\log x)$ show that

$$x^2y_2+xy_1+y=0$$

Sol: $y=5cos(logx)+7sin(logx)$,

differentiate w r to x

$$\frac{dy}{dx}=-5\sin(logx).\frac{1}{x}+7\cos(logx).\frac{1}{x}$$

$$x\frac{dy}{dx}=-5\sin(\log x)+7\cos(\log x),$$

differentiate again w r to x

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}(1)=-5\cos(\log x).\frac{1}{x}-7\sin(\log x).\frac{1}{x}$$

$$x\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)=-(5cos(logx)+7sin(logx))$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y \quad \Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

6. If $y=\sin^{-1}x$ Show that

$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$ **(16-M, 19-M, Aug-21)**

Sol: $y=\sin^{-1}x$, differentiate w r to x

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \sqrt{1-x^2}\frac{dy}{dx}=1,$$

differentiate again w r to x

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}+\frac{dy}{dx}.\frac{1}{\cancel{2}\sqrt{1-x^2}}.(-\cancel{2}x)=0$$

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}-\frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}=0$$

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$$

7. If $y=(\sin^{-1}x)^2$ Show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$$

Sol: $y=(\sin^{-1}x)^2$, differentiate w r to x

$$\frac{dy}{dx}=2\sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$$

$$\left(\sqrt{1-x^2}\right)\frac{dy}{dx}=2\sin^{-1}x,$$

differentiate again w r to x

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}+\frac{dy}{dx}.\frac{1}{\cancel{2}\sqrt{1-x^2}}.(-\cancel{2}x)=2.\frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}-\frac{x}{\sqrt{1-x^2}}\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$$

8. If $y=(\tan^{-1}x)^2$ S.T $(1+x^2)^2y_2+2x(1+x^2)y_1=2$

(2015-J, 17-M, 18-M, 2020-M)

Sol: $y=(\tan^{-1}x)^2$, differentiate w r to x

$$\frac{dy}{dx}=2\tan^{-1}x.\frac{1}{1+x^2}$$

$$(1+x^2)\frac{dy}{dx}=2\tan^{-1}x,$$

differentiate again w r to x

$$(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=2.\frac{1}{1+x^2}$$

$$(1+x^2)\left((1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}\right)=2$$

$$(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$$

9. If $e^y(x+1)=1$ Prove that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

Sol: $e^y(x+1)=1\Rightarrow e^y=\frac{1}{(x+1)}----(1)$

differentiate w r to x

$$e^y \frac{dy}{dx} = -\frac{1}{(x+1)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right]$$

$$\frac{1}{(x+1)}\frac{dy}{dx} = -\frac{1}{(x+1)^2} \Rightarrow \frac{dy}{dx} = -\frac{1}{(x+1)}$$

$$\frac{dy}{dx} = -e^y ----(2),$$

differentiate again w r to x

$$\frac{d^2y}{dx^2} = -e^y \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left(-\frac{dy}{dx}\right)\frac{dy}{dx} \quad \text{(from (2))}$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

10. If $y = e^{a\cos^{-1}x}, -1 < x < 1$, S.T

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Sol: $y = e^{a\cos^{-1}x}$, differentiate w r to x

$$\frac{dy}{dx} = e^{a\cos^{-1}x}.a\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\frac{dy}{dx} = -ae^{a\cos^{-1}x},$$

differentiate again w r to x

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{1}{2\sqrt{1-x^2}}(-2x) = -a^2e^{a\cos^{-1}x}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

11. If $y = \cos^{-1}x$ then find $\frac{d^2y}{dx^2}$ in-terms of y alone.

Sol: $y = \cos^{-1}x \Rightarrow x = \cos y$,

differentiate w r to y

$$\frac{dx}{dy} = -\sin y \qquad \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} ---(1)$$

$\frac{dy}{dx} = -\operatorname{cosec} y$, now differentiate w r to x

$$\frac{d^2y}{dx^2} = -\left(-\operatorname{cosec} y \cot y\right)\frac{dy}{dx} = \operatorname{cosec} y \cot y\left(-\operatorname{cosec} y\right)$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 y \cot y \quad \Rightarrow \frac{d^2y}{dx^2} + \operatorname{cosec}^2 y \cot y = 0$$

12. If

$x = a(\cos t + t\sin t)$ & $y = a(\sin t - t\cos t)$, *find* $\frac{d^2y}{dx^2}$.

Sol: $x = a(\cos t + t\sin t)$ & $y = a(\sin t - t\cos t)$,

differentiate w r to t

$$\frac{dx}{dt} = a\left(-\sin t + \left(t\cos t + \sin t\right)\right)$$

$$\frac{dy}{dt} = a\left(\cos t - \left(-t\sin t + \cos t\right)\right)$$

$$\frac{dx}{dt} = at\cos t \;\&\; \frac{dy}{dt} = at\sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\Rightarrow \frac{dy}{dx} = \tan t$$

now differentiate w r to x

$$\frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx} = \sec^2 t.\frac{1}{at\cos t} \Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$$

13. If $y = 5\cos x - 3\sin x$ then prove that

$$\frac{d^2y}{dx^2} + y = 0$$

Sol: $y = 5\cos x - 3\sin x$, differentiate w r to x

$$\frac{dy}{dx} = -5\sin x - 3\cos x,$$

differentiate again w r to x

$$\frac{d^2y}{dx^2} = -5\cos x + 3\sin x = -(5\cos x - 3\sin x)$$

$$\frac{d^2y}{dx^2} = -y \qquad \Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Question No: 50.b (4M)

1. Find the value of k so that the function $f(x) = \begin{cases} kx+1, & if\ x \le 5 \\ 3x-5, & if\ x > 5 \end{cases}$ at $x = 5$ is a continuous function. **(2015-M, 2019-M)**

Sol: $LHL = \lim_{x\to 5^-} f(x) = \lim_{x\to 5}(kx+1) = 5k+1$

$RHL = \lim_{x\to 5^+} f(x) = \lim_{x\to 5}(3x-5) = 3(5)-5 = 10$

$f(x) = kx+1\ \ at\ \ x = 5\ \therefore f(5) = 5k+1$

Given function is continuous at $x = 5$

$\therefore LHL = RHL = f(5)$

$5k+1 = 10 = 5k+1 \Rightarrow 5k+1 = 10 \Rightarrow 5k = 9 \Rightarrow k = 9/5$

2. Find the value of k if

$f(x) = \begin{cases} kx^2, & if\ x \le 2 \\ 3, & if\ x > 2 \end{cases}$

is continuous at $x = 2$ **(2016-J, 2018-J)**

Sol: $LHL = \lim_{x\to 2^-} f(x) = \lim_{x\to 2}(kx^2) = k(2)^2 = 4k$

$RHL = \lim_{x\to 2^+} f(x) = \lim_{x\to 5^+}(3) = 3$

$f(x) = kx^2\ \ at\ \ x = 2\ \therefore f(2) = 4k$

Given function is continuous at $x = 2$

$\therefore LHL = RHL = f(2)$

$4k = 3 = 4k \Rightarrow 4k = 3 \Rightarrow k = 3/4$

3. Find the value of K, if $f(x) = \begin{cases} Kx+1, & if\ x \le \pi \\ \cos x, & if\ x > \pi \end{cases}$ is continuous at $x = \pi$

(2020-M, Aug-2021)

Sol: $LHL = \lim_{x\to \pi^-} f(x) = \lim_{x\to \pi}(kx+1) = \pi k+1$

$RHL = \lim_{x\to \pi^+} f(x) = \lim_{x\to \pi}\cos x = \cos\pi = -1$

$f(x) = kx+1\ \ at\ \ x = \pi\ \therefore f(\pi) = \pi k+1$

Given function is continuous at $x = \pi$

$\therefore LHL = RHL = f(\pi)$

$\pi k+1 = -1 = \pi k+1 \Rightarrow \pi k+1 = -1$

$\Rightarrow \pi k = -2 \Rightarrow k = -2/\pi$

4. Determine the value of k if

$f(x) = \begin{cases} x^2\sin\frac{1}{x} & if\ x \ne 0 \\ k, & if\ x = 0 \end{cases}$ is continuous at $x = 0$

Sol: Let Now,

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right) = \left(0^2\sin\frac{1}{0}\right) = 0$

and $f(0) = k$

$\lim_{x\to 0} f(x) = f(0)$

$0 = k \Rightarrow k = 0$

5. Determine the value of k if

$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & if\ x \ne \frac{\pi}{2} \\ 3 & if\ x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

(2014-M, 14-J, 17-M, 19-J)

Sol:

$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{x\to\frac{\pi}{2}} \frac{k\sin(\pi/2 - x)}{2(\pi/2 - x)}$

$= \frac{k}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad \left(where\ \theta = \frac{\pi}{2} - x\ if\ x \to \frac{\pi}{2}\ then\ \theta \to 0\right)$

$= \frac{k}{2}(1) = \frac{k}{2}$

Given that function is continuous at

$x = \pi/2 \qquad \& \quad f\left(\frac{\pi}{2}\right) = 3$

$\lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6$

6. Find the value of k if

$f(x) = \begin{cases} \frac{1-\cos 2x}{1-\cos x}, & x \ne 0 \\ k & x = 0 \end{cases}$

is continuous at $x = 0$ **(2016-M)**

Sol: $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1-\cos 2x}{1-\cos x}\right)$ & $f(0)=k$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{2\sin^2\left(\frac{x}{2}\right)} = \lim_{x\to 0}\frac{\sin^2 x}{\sin^2\left(\frac{x}{2}\right)} = \lim_{x\to 0}\frac{\left(2\sin(x/2)\cos(x/2)\right)^2}{\sin^2\left(\frac{x}{2}\right)}$$

$$= 4\lim_{x\to 0}\cos^2(x/2) = 4(1) = 4$$

Given that function is continuous at $x=0$

$$\therefore \lim_{x\to 0} f(x) = f(0) \Rightarrow 4 = k \Rightarrow k = 4$$

7. Find the values of a & b such that $f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$ is a continuous function. **(2015-J)**

Sol: Now, at $x = 2$

$$LHL = \lim_{x\to 2^-} f(x) = \lim_{x\to 2}(5) = 5$$

$$RHL = \lim_{x\to 2^+} f(x) = \lim_{x\to 2}(ax+b) = 2a+b$$

$f(x) = 5$ at $x=2$ $\therefore f(2) = 5$

Given function is continuous at $x=2$

$\therefore LHL = RHL = f(2)$

$2a+b = 5 ------(1)$

Now, at $x = 10$

$$LHL = \lim_{x\to 10^-} f(x) = \lim_{x\to 10}(ax+b) = 10a+b$$

$$RHL = \lim_{x\to 10^+} f(x) = \lim_{x\to 10}(21) = 21$$

$f(x) = 21$ at $x = 10$ $\therefore f(10) = 21$

Given function is continuous at $x=10$

$\therefore LHL = RHL = f(10)$

$10a+b = 21 ------(2)$

Solving (1) and (2) we get $a = 2$ & $b = 1$

8. Find the relationship between a & b so that the function f is defined by $f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$ is continuous at $x=3$. **(2015-J, 2018-M)**

Sol: $LHL = \lim_{x\to 3^-} f(x) = \lim_{x\to 3}(ax+1) = 3a+1$

$$RHL = \lim_{x\to 3^+} f(x) = \lim_{x\to 3}(bx+3) = 3b+3$$

$f(x) = ax+1$ at $x=3$ $\therefore f(3) = 3a+1$

Given that function is continuous at $x=3$

Then we have $LHL = RHL = f(3)$

$3a+1 = 3b+3$

$3a = 3b+2 \Rightarrow a = b + \frac{2}{3}$

9. For what value of is the function defined by $f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$ is continuous at $x=0$. **(2016-J)**

Sol:

$$LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0}\lambda(x^2-2x) = \lambda(0^2 - 2(0)) = 0$$

$$RHL = \lim_{x\to 0+} f(x) = \lim_{x\to 0}(4x+1) = 4(0)+1 = 1$$

$f(x) = \lambda(x^2-2x)$ at $x=0$ $\therefore f(0) = 0$

$LHL \ne RHL$ $\therefore$ for any value of λ the function is not continuous.

10. Discuss the continuity of the function $f(x) = \begin{cases} |x|+3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \ge 3 \end{cases}$

Sol: Now, at $x = -3$

$$LHL = \lim_{x\to(-3)^-} f(x) = \lim_{x\to(-3)}(|x|+3) = |-3|+3 = 6$$

$$RHL = \lim_{x\to(-3)^+} f(x) = \lim_{x\to(-3)}(-2x) = 6$$

$f(x) = (|x|+3)$ at $x=-3$ $\therefore f(-3) = |-3|+3 = 6$

$\therefore LHL = RHL = f(-3)$

$\therefore$ Given function is continuous at $x=-3$

Now, at $x=3$

$$LHL=\lim_{x\to(3)^-} f(x)=\lim_{x\to(3)}(-2x)=-6$$

$$RHL=\lim_{x\to(3)^+} f(x)=\lim_{x\to(3)}(6x+2)=6(3)+2=20$$

$$f(x)=(6x+2)\ at\ x=3 \qquad \therefore f(3)=6(3)+2=20$$

$$\therefore LHL\neq RHL$$

Given function is not continuous at $x=3$

- If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, Prove that $\dfrac{dy}{dx}=\dfrac{\cos^2(a+y)}{\sin a}$ **(Aug-2021, 3M)**

Sol: $\cos y = x\cos(a+y)$

$$x=\frac{\cos y}{\cos(a+y)}$$

Differentiate with respect to y

$$\frac{dx}{dy}=\frac{\cos(a+y)\frac{d}{dy}\cos y-\cos y\frac{d}{dy}\cos(a+y)}{\cos^2(a+y)}$$

$$\frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)-\cos y(-\sin(a+y))}{\cos^2(a+y)}$$

$$\frac{dx}{dy}=\frac{\sin(a+y)\cos y-\cos(a+y)\sin y}{\cos^2(a+y)}$$

$$\frac{dx}{dy}=\frac{\sin(a+y-y)}{\cos^2(a+y)}=\frac{\sin a}{\cos^2(a+y)}$$

$$\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

- If $y=x.\cos x$, *find* $\dfrac{d^2y}{dx^2}$. **(Aug-21, 2M)**

Sol: $y=x.\cos x$

Differentiate w. r.to x

$$\frac{dy}{dx}=x(-\sin x)+\cos x(1)$$

$$\frac{dy}{dx}=-x\sin x+\cos x$$

Differentiate again w. r. to x

$$\frac{d^2y}{dx^2}=-\left[x(\cos x)+\sin x(1)\right]-\sin x$$

$$\frac{d^2y}{dx^2}=-x\cos x-2\sin x$$

UNIT-6 APPLICATION OF DERIVATIVES

$\left[Total\ marks:10,\ Q.No-17(2M),\ 30(3M)\ \&\ 43(5M)\right]$

Question No: 17 (2M)

1. Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 2% **(2014-M, 2019-M)**

Sol: $V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$ and

$\Delta x = 0.02x \Rightarrow \Delta x = 2\%\ of\ x$

Change in volume $\Delta V = \frac{dv}{dx}.\Delta x$

$\Rightarrow \Delta V = (3x^2).(0.02x) = 0.06x^3\ m^3$

2. Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 3% **(2017-M)**

Sol: $V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$ and $\Delta x = 0.03x \Rightarrow \Delta x = 3\%\ of\ x$

Change in volume $\Delta V = \frac{dv}{dx}.\Delta x$

$\Rightarrow \Delta V = (3x^2).(0.03x) = 0.09x^3\ m^3$

3. Approximate $\sqrt{36.6}$ by using differential. **(2014-J, 2020-M)**

Sol: Consider $f(x) = \sqrt{x} \quad \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

Let $x = 36$ & $\Delta x = 0.6$

Now, $f(x+\Delta x) = f(x) + \Delta x.f'(x)$

$$\sqrt{x+\Delta x} = \sqrt{x} + \Delta x.\frac{1}{2\sqrt{x}}$$

$$\sqrt{36.6} = \sqrt{36} + \frac{0.6}{2\sqrt{36}}$$

$$\sqrt{36.6} = 6 + 0.05$$

$$\sqrt{36.6} = 6.05$$

4. Approximate $\sqrt{0.6}$ by using differential.

Sol: Consider $f(x) = \sqrt{x} \quad \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

Let $x = 0.64$ & $\Delta x = -0.04$

Now, $f(x+\Delta x) = f(x) + \Delta x.f'(x)$

$$\sqrt{x+\Delta x} = \sqrt{x} + \Delta x.\frac{1}{2\sqrt{x}}$$

$$\sqrt{0.6} = \sqrt{0.64} - \frac{0.04}{2\sqrt{0.64}}$$

$$\sqrt{0.6} = 0.8 - 0.025$$

$$\sqrt{0.6} = 0.775$$

5. Using differentials, find the approximate value of $\sqrt{49.5}$. **(2015-J)**

Sol: Consider $f(x) = \sqrt{x} \quad \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

Let $x = 49$ & $\Delta x = 0.5$

Now, $f(x+\Delta x) = f(x) + \Delta x.f'(x)$

$$\sqrt{x+\Delta x} = \sqrt{x} + \Delta x.\frac{1}{2\sqrt{x}}$$

$$\sqrt{49.5} = \sqrt{49} + \frac{0.5}{2\sqrt{49}}$$

$\sqrt{49.5} = 7 + 0.036 \qquad \therefore \sqrt{49.5} = 7.036$

6. Using differentials, Find the approximate value of $(25)^{\frac{1}{3}}$. **(2016-M, J)**

Sol: Consider $f(x) = (x)^{\frac{1}{3}} \quad \Rightarrow f'(x) = \frac{1}{3x^{2/3}}$

Let $x = 27$ & $\Delta x = -2$

Now, $f(x+\Delta x) = f(x) + \Delta x.f'(x)$

$$(x+\Delta x)^{1/3} = (x)^{1/3} + \Delta x.\frac{1}{3x^{2/3}}$$

$$(25)^{\frac{1}{3}} = (27)^{\frac{1}{3}} - 2.\frac{1}{3(27)^{2/3}}$$

$$(25)^{\frac{1}{3}} = 3 - \frac{2}{3\times 9}$$

$(25)^{\frac{1}{3}} = 3 - 0.074 \qquad \therefore (25)^{\frac{1}{3}} = 2.926$

7. If the radius of a sphere is measured as 7m with an error of 0.02m, then find the approximate error in calculating its volume. **(2018-J)**

Sol: Let r be the radius of sphere and Δr be the error in measuring radius

Given $r = 7m$ & $\Delta r = 0.02m$

Volume of sphere is $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dv}{dr} = 4\pi r^2$

Approximate error in calculating volume

$\Delta V = \frac{dV}{dr}.\Delta r$

$\Delta V = \left(4\pi r^2\right).\Delta r$

$\Delta V = \left(4\pi (7)^2\right).(0.02)$ $\Rightarrow \Delta V = 3.92\pi m^3$

8. If the radius of a sphere is measured as 9cm with an error of 0.03cm, then find the approximate error in calculating its volume.

Sol: Let r be the radius of sphere and Δr be the error in measuring radius

Given $r = 9cm$ & $\Delta r = 0.03cm$

Volume of sphere is $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dv}{dr} = 4\pi r^2$

Approximate error in calculating volume

$\Delta V = \frac{dV}{dr}.\Delta r$

$\Delta V = \left(4\pi r^2\right).\Delta r$

$\Delta V = \left(4\pi (9)^2\right).(0.03)$ $\Rightarrow \Delta V = 9.72\pi cm^3$

9. Find the points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are parallel to x-axis.

Sol: $\frac{x^2}{4} + \frac{y^2}{25} = 1$ differentiate w r to x**(2017-J)**

$\frac{2x}{4} + \frac{2y}{25}\frac{dy}{dx} = 0$ $\Rightarrow \frac{dy}{dx} = -\frac{25x}{4y}$

If the tangents are parallel to the x-axis, then the slope is equal to zero

$\Rightarrow \frac{dy}{dx} = 0$ $\Rightarrow -\frac{25x}{4y} = 0$ $\Rightarrow x = 0$

If $x = 0$, substituting value of x in given equation we get $y = \pm 5$

Therefore, the points on the curve are $(0,+5)$ & $(0,-5)$

10. Find the points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are parallel to x-axis. **(2017-J)**

Sol: $\frac{x^2}{9} + \frac{y^2}{16} = 1$ differentiate w r to x

$\frac{2x}{9} + \frac{2y}{16}\frac{dy}{dx} = 0$ $\Rightarrow \frac{dy}{dx} = -\frac{16x}{9y}$

If the tangents are parallel to the x-axis, then the slope is equal to zero

$\Rightarrow \frac{dy}{dx} = 0$ $\Rightarrow -\frac{19x}{9y} = 0,$ $\Rightarrow \boxed{x = 0}$

If $x = 0$, substituting value of x in given equation we get $\boxed{y = \pm 4}$

Therefore, the points on the curve are $(0,+4)$ & $(0,-4)$

11. Find the point, at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope 2/3.

Sol: $y = \sqrt{4x-3} - 1$ $\Rightarrow \frac{dy}{dx} = \frac{4}{2\sqrt{4x-3}}$ and

slope of tangent $\frac{dy}{dx} = \frac{2}{3}$

$\therefore \frac{4}{2\sqrt{4x-3}} = \frac{2}{3}$ $\Rightarrow \sqrt{4x-3} = 3$

square on both side

$4x - 3 = 9$ $\Rightarrow 4x = 12$ $\Rightarrow \boxed{x = 3}$

Put $x = 3$ i

$y = \sqrt{4x-3} - 1 = \sqrt{4(3)-3} - 1 = 2$ $\Rightarrow \boxed{y = 2}$

Therefore, the required point is $(3,2)$

12. Find the slope of tangent to the curve

$y = x^3 - x \ at \ x = 2$ **(2018-M)**

Sol: $y = x^3 - x \ at \ x = 2$, differentiate w r to x

$$\frac{dy}{dx} = 3x^2 - 1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = 3(2)^2 - 1 = 11$$

The slope of the tangent to the curve is 11

13. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2 \ at \ x = 10$ **(2015-M)**

Sol: $y = \frac{x-1}{x-2} \Rightarrow \frac{dy}{dx} = \frac{(x-2)(1)-(x-1)(1)}{(x-2)^2} = -\frac{1}{(x-2)^2}$

$\left(\frac{dy}{dx}\right)_{x=10} = -\frac{1}{(10-2)^2} = -\frac{1}{64}$ thus, the slope of the tangent to the curve is $-\frac{1}{64}$

14. Find the slope of normal to the curve

$y = 2x^2 + 3\sin x \ \ at \ \ x = 0$

Sol: $y = 2x^2 + 3\sin x$, differentiate w r to x

$$\frac{dy}{dx} = 4x + 3\cos x \Rightarrow \left(\frac{dy}{dx}\right)_{x=0} = 4(0) + 3\cos(0) = 3$$

Slope of the normal to the curve is

$$= -\frac{1}{(dy/dx)} = -\frac{1}{3}$$

15. Find the slope of the normal to the curve

$x = 1 - a\sin\theta \ \& \ y = b\cos^2\theta \quad at \quad \theta = \frac{\pi}{2}$

Sol: $x = 1 - a\sin\theta$ $\qquad$ $y = b\cos^2\theta$

$\frac{dx}{d\theta} = -a\cos\theta$ $\qquad$ $\frac{dy}{d\theta} = -2b\cos\theta\sin\theta$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = \frac{2b}{a}\sin\left(\frac{\pi}{2}\right) = \frac{2b}{a}$$

Therefore, the slope of the normal is

$$= -\frac{1}{(dy/dx)} = -\frac{a}{2b}$$

Question No: 30 (3M)

1. Find the interval in which the function f given by $f(x) = x^2 - 4x + 6$ is

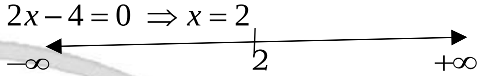
(i) Strictly increasing (ii) Strictly decreasing

(2014-M, 2020-M)

Sol: $f(x) = x^2 - 4x + 6 \quad \Rightarrow f'(x) = 2x - 4$

To find the intervals, we have to consider, $f'(x) = 0$

$2x - 4 = 0 \Rightarrow x = 2$



The point $x = 2$ divides the real line into two disjoint intervals $(-\infty, 2) \ \& \ (2, +\infty)$

For all x in $(-\infty, 2)$, $f'(x) < 0$. Therefore $f(x)$ is strictly decreasing in $(-\infty, 2)$

For all x in $(2, \infty)$, $f'(x) > 0$. Therefore $f(x)$ is strictly increasing in $(2, \infty)$

2. Find the interval in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is strictly increasing. **(2019-M)**

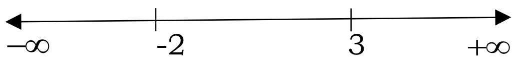
Sol: $f(x) = 2x^3 - 3x^2 - 36x + 7$

$\Rightarrow f'(x) = 6x^2 - 6x - 36$

To find the intervals, we have to consider, $f'(x) = 0$

$6x^2 - 6x - 36 = 0 \quad \Rightarrow x^2 - x - 6 = 0$

$(x+2)(x-3) = 0 \quad \Rightarrow x = -2 \ \& \ x = 3$



The point $x = -2, 3$ divides the real line into three disjoint intervals $(-\infty, -2), (-2, 3) \ \& \ (3, +\infty)$

For all x in $(-\infty, -2) \& (3, \infty)$, $f'(x) > 0$.

$\therefore f(x)$ is strictly increasing in $(-\infty, -2) \& (3, \infty)$

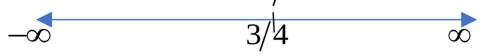
For all x in $(-2, 3)$, $f'(x) < 0$. Therefore $f(x)$ is strictly decreasing in $(-2, 3)$

3. Find the interval in which the function f given by $f(x)=2x^2-3x$ is strictly increasing. **(2016-J, 2019-J)**

Sol: $f(x)=2x^2-3x \qquad \Rightarrow f'(x)=4x-3$

To find the intervals, we have to consider, $f'(x)=0$

$4x-3=0 \qquad \Rightarrow x=3/4$



The point $x=\dfrac{3}{4}$ divides the real line into two disjoint intervals $\left(-\infty,\dfrac{3}{4}\right)$ & $\left(\dfrac{3}{4},+\infty\right)$

For all x in $\left(\dfrac{3}{4},\infty\right)$, $f'(x)>0$. Therefore $f(x)$ is strictly increasing in $\left(\dfrac{3}{4},\infty\right)$

For all x in $\left(-\infty,\dfrac{3}{4}\right)$, $f'(x)<0$. Therefore $f(x)$ is strictly decreasing in $\left(-\infty,\dfrac{3}{4}\right)$

4. Prove that the function f given by $f(x)=x^2-x+1$ is neither strictly increasing nor strictly decreasing on $(-1,1)$

Sol: $f(x)=x^2-x+1 \qquad f'(x)=2x-1$

Now, $f'(x)=0 \qquad \Rightarrow x=\dfrac{1}{2}$

The point $x=\dfrac{1}{2}$ divides the interval $(-1,1)$ into two disjoint intervals $\left(-1,\dfrac{1}{2}\right)$ & $\left(\dfrac{1}{2},1\right)$

Now, in interval $\left(-1,\dfrac{1}{2}\right)$, $f'(x)<0$.

$\therefore f(x)$ is strictly decreasing in $\left(-1,\dfrac{1}{2}\right)$

Now, in interval $\left(\dfrac{1}{2},1\right)$, $f'(x)>0$.

$\therefore f(x)$ is strictly increasing in $\left(\dfrac{1}{2},1\right)$

Hence $f(x)$ is neither strictly increasing nor strictly decreasing on $(-1,1)$.

5. Find two numbers whose product is 100 and whose sum is minimum. **(2016-M)**

Sol: Let x and y be two positive numbers then $xy=100$

$y=\dfrac{100}{x}$ and whose sum is

$S(x)=x+y \qquad \therefore S(x)=x+\dfrac{100}{x}$

$S'(x)=1-\dfrac{100}{x^2}, \qquad S''(x)=\dfrac{200}{x^3}>0$

therefore, whose sum is minimum

For maximum or minimum

$S'(x)=0, \qquad \Rightarrow 1-\dfrac{100}{x^2}=0$

$x^2=100 \Rightarrow \boxed{x=\pm 10}$ and

$y=\dfrac{100}{x}=\dfrac{100}{\pm 10}=\pm 10 \qquad \Rightarrow \boxed{y=\pm 10}$

Therefore, the required numbers are $x=10$ & $y=10$

6. Find the two positive numbers whose sum is 15 and sum of whose square is minimum. **(2015-M, 2017-J, 2018-J)**

Sol: Let x and y be two positive numbers then $x+y=15 \qquad \Rightarrow y=15-x$

Sum of squares is $S(x)=x^2+y^2$

$S(x)=x^2+(15-x)^2$

$S'(x)=2x-2(15-x)$

$S''(x)=2-2(1)=4>0$,

$\therefore$ sum of whose squares is minimum.

For maximum or minimum

$S'(x)=0, \Rightarrow 2x-2(15-x)=0$

$4x-30=0 \Rightarrow 4x=30 \Rightarrow x=\frac{15}{2}$ and

$y=15-x=15-\frac{15}{2}=\frac{15}{2} \Rightarrow y=\frac{15}{2}$

Therefore, the requires numbers are $x=\frac{15}{2}$ & $y=\frac{15}{2}$.

7. Find the two positive numbers whose sum is 16 and sum of whose cubes is minimum.

Sol: Let x and y be two positive numbers then $x+y=16 \quad \Rightarrow y=16-x$

Sum of cubes is $S(x)=x^3+y^3$

$$S(x)=x^3+(16-x)^3$$

$$S'(x)=3x^2-3(16-x)^2$$

$$S''(x)=6x+6(16-x)=96>0,$$

$\therefore$ sum of whose cubes is minimum.

For maximum or minimum

$S'(x)=0, \Rightarrow 3x^2-3(16-x)^2=0$

$\Rightarrow 3x^2-3(256+x^2-32x)=0$

$x^2-(256+x^2-32x)=0$

$\Rightarrow x^2-256+x^2-32x=0$

$\Rightarrow 32x=256 \Rightarrow \boxed{x=8}$ *and* $\boxed{y=8}$

$\therefore$ the required numbers are $8 \& 8$

8. Find two numbers whose sum is 24 and whose product is as large as possible. **(2015-J, 2018-M)**

Sol: Let x and y be two positive numbers then $x+y=24 \Rightarrow y=24-x$

Product is $P(x)=xy=x(24-x)$

$P(x)=24x-x^2 \qquad \Rightarrow P'(x)=24-2x$

$P''(x)=-2<0.$

Therefore, product is maximum.

For maximum or minimum

$P'(x)=0 \Rightarrow 24-2x=0 \Rightarrow \boxed{x=12}$ & $\boxed{y=12}$

Therefore, the required numbers are 12 & 12.

9. Find two positive numbers x & y such that $x+y=60$ & xy^3 is maximum. **(2014-J, 2017-M)**

Sol: $x+y=60 \qquad x=60-y$

$$P(y)=xy^3=(60-y)y^3$$

$$P(y)=60y^3-y^4$$

$$P'(y)=180y^2-4y^3$$

$$p''(y)=360y-12y^2$$

For maximum or minimum $P'(y)=0$

$180y^2-4y^3=0 \Rightarrow 4y^2(45-y)=0$

$\Rightarrow \boxed{y=0}$ *or* $\boxed{y=45}$

At $y=0, P''(y)=0$ therefore the function is neither maximum nor minimum.

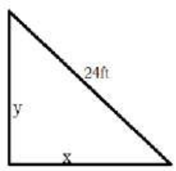
At $y=45, P''(y)=-1800<0$ therefore xy^3 is maximum when $y=45$

Therefore, two numbers are $\boxed{x=15}$ & $\boxed{y=45}$

Question No: 43 (5M)

1. A ladder 24 feet long is leans against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 3 feet/s. Find the rate at which the top of the ladder is moving downwards, if the foot is 8 ft from the wall. **(2014-M,16-M)**

Sol: The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 3 feet/s is $\frac{dx}{dt}=+3\,ft/s$



$\frac{dy}{dt}=?$ when $x=8\,ft$

Now, $x^2+y^2=(24)^2$ we get $y=16\sqrt{2}\,ft$

We have $x^2+y^2=(24)^2$

$$\Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

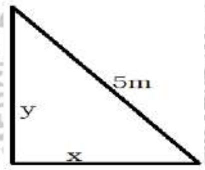
$$\Rightarrow 2(8)(3)+2\left(16\sqrt{2}\right)\frac{dy}{dt}=0$$

$$\Rightarrow \frac{dy}{dt}=-\frac{48}{32\sqrt{2}}=-\frac{3}{2\sqrt{2}}\ ft/s=-1.06\,ft/s$$

Therefore, height on the wall decreasing at rate of 1.06 ft./s

2. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall? **(2016-J)**

Sol: The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. i.e.,



$$\frac{dx}{dt}=+2cm/s=+0.02m/\sec$$

$$\frac{dy}{dt}=?\quad \text{when } x=4m$$

Now, $x^2+y^2=5^2$, we get $y=3m$ when $x=4m$

We have $x^2+y^2=5^2 \quad \Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

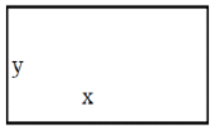
$$\Rightarrow x\frac{dx}{dt}+y\frac{dy}{dt}=0 \Rightarrow 4(0.02)+3\frac{dy}{dt}=0$$

$$\frac{dy}{dt}=-\frac{0.08}{3}\ m/s$$

Therefore, height on the wall decreasing at rate of $\frac{0.08}{3}\ m/\sec=\frac{8}{3}cm/s$.

3. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x=8cm$ and $y=6cm$ find the rates of change of **(a)** the perimeter and **(b)** the area of the rectangle. **(2014-J, 2017-M)**

Sol: The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute.



$$\frac{dx}{dt}=-5cm/\min \quad \& \quad \frac{dy}{dt}=4cm/\min$$

When $x=8cm$ and $y=6cm$

(a) the perimeter of rectangle $P=2(x+y)$

$$\frac{dP}{dt}=2\left(\frac{dx}{dt}+\frac{dy}{dt}\right) \quad \Rightarrow \frac{dP}{dt}=2(-5+4)=-2cm/\min$$

Perimeter of rectangle is decreasing at the rate of 2cm/min

(b) The area of rectangle $A=xy$

$$\frac{dA}{dt}=x\frac{dy}{dt}+y\frac{dx}{dt}$$

$$\Rightarrow \frac{dA}{dt}=(8)(4)+(6)(-5)=+2cm^2/\min$$

The area of the rectangle is increasing at the rate of 2cm²/min.

4. The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2 cm/minute. When $x=10cm$ and $y=6cm$, find the rates of change of **(a)** the perimeter and **(b)** the area of the rectangle. **(2019-M)**

Sol: The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2 cm/minute

$$\frac{dx}{dt}=-3cm/\min \quad \& \quad \frac{dy}{dt}=+2cm/\min$$

When $x=10cm$ and $y=6cm$

(a) the perimeter of rectangle $P=2(x+y)$

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \Rightarrow \frac{dP}{dt} = 2(-3+2) = -2cm/\min$$

Perimeter of rectangle is decreasing at the rate of 2cm/min

(b) The area of rectangle $A = xy$

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\Rightarrow \frac{dA}{dt} = (10)(2) + (6)(-3) = +2cm^2/\min$$

The area of the rectangle is increasing at the rate of 2cm²/min.

5. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate. **(2015-M, 2018-J)**

Sol: The y-coordinate is changing 8 times as fast as the x-coordinate. i.e., $\frac{dy}{dt} = 8\frac{dx}{dt}$

Equation of curve $6y = x^3 + 2$

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

$$6\left(8\frac{dx}{dt}\right) = 3x^2\frac{dx}{dt}$$

$$48 = 3x^2 \quad \Rightarrow x^2 = 16,\ x = \pm 4$$

When $x = 4$ we get $y = 11$

therefore, the point is $(4,11)$

When $x = -4$ we get $y = -\frac{31}{3}$

therefore, the point is $\left(-4, -\frac{31}{3}\right)$

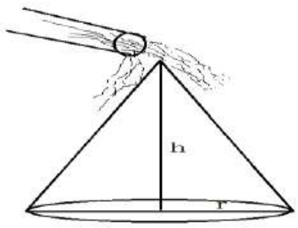
6. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm? **(2015-J, 2017-J, 2018-M, 2019-J, 2020-M)**

Sol: Let h be the height, r be radius and V be the volume of sand cone at any time t

$$\frac{dV}{dt} = 12cm^3/\sec \ \&\ h = \frac{1}{6}r \Rightarrow r = 6h$$

Volume of cone

$$V = \frac{1}{3}\pi r^2 h$$



$$V = \frac{1}{3}\pi(6h)^2 h, \qquad (\because r = 6h)$$

$$V = 12\pi h^3$$

$$\frac{dV}{dt} = 36\pi h^2\frac{dh}{dt}$$

$$12 = 36\pi(4)^2\frac{dh}{dt} \qquad (\because h = 4)$$

$$\frac{dh}{dt} = \frac{12}{36\pi(4)^2} = \frac{1}{48\pi}cm/\sec$$

The height of the sand cone increasing at the rate of $\frac{1}{48\pi}cm/\sec$.

7. A man of height 2 meters walks at uniform speed of 5km/hr, away from a lamp post which is 6 meters high. Find the rate at which the length of his shadow increases.

Sol: Let AB be the lamp post

MN be the man at a particular point at time t

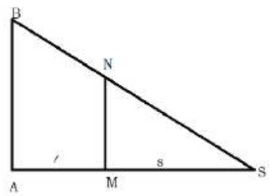
$AM = l,\ MS = s$ is shadow of man

$$MN = 2, AB = 6\ \frac{dl}{dt} = 5km/hr$$

From figure,

$\Delta MSN \approx \Delta ASB$

We have $\frac{MS}{AS} = \frac{MN}{AB}$



$$\frac{s}{l+s} = \frac{2}{6}$$

$$l + s = 3s \quad \Rightarrow l = 2s$$

$diff\ \ w\ r\ to\ t$

$$\frac{dl}{dt} = 2\frac{ds}{dt}$$

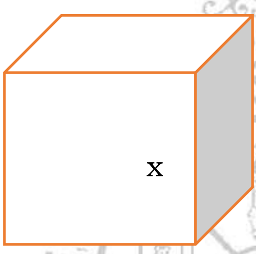
$$5 = 2\frac{ds}{dt} \quad \Rightarrow \frac{ds}{dt} = \frac{5}{2} km/hr$$

Therefore, length of the shadow is increasing at rate of $\frac{5}{2} km/hr$

8. The volume of the cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 cm?

Sol: The volume of the cube is increasing at a rate of $9cm^3/\sec$. i.e., $\frac{dV}{dt} = 9cm^3/\sec$

$\frac{ds}{dt} = ?$ when $x = 10cm$



Volume of the cube $V = x^3$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$9 = 3x^2\frac{dx}{dt} \quad \Rightarrow \frac{dx}{dt} = \frac{3}{x^2}$$

Surface area of cube $S = 6x^2$

$$\frac{ds}{dt} = 12x\frac{dx}{dt}$$

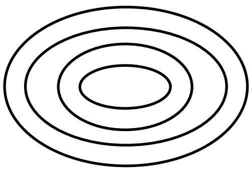
$$\frac{ds}{dt} = 12x.\frac{3}{x^2}$$

$$\frac{ds}{dt} = \frac{36}{x} \quad \Rightarrow at\ x = 10, \ \frac{ds}{dt} = \frac{36}{10} = 3.6cm^2/\sec$$

Therefore, surface area of cube is increasing at the rate of $3.6cm^2/\sec$

9. A stone is dropped into a quit lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10cm, how fast is the enclosed area increasing?

Sol: Let r be the radius and A be the area of circular waves



$\frac{dr}{dt} = 4cm/\sec \quad r = 10cm$

Area of circle, $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \quad \Rightarrow \frac{dA}{dt} = 2\pi(10)(4)$$

$$\Rightarrow \frac{dA}{dt} = 80\pi cm^2/\sec$$

Hence, enclosed area of circular waves is increasing at the rate of $80\pi cm^2/\sec$

10. A bubble, which always remains spherical has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Sol: Diameter $d = \frac{3}{2}(2x+1)$,

Radius $r = \frac{3}{4}(2x+1) \quad (\because r = d/2)$

Volume of the sphere is $V = \frac{4}{3}\pi r^3$

$$V = \frac{4}{3}\pi\left(\frac{3}{4}(2x+1)\right)^3 = \frac{4}{3}\pi\frac{27}{64}(2x+1)^3$$

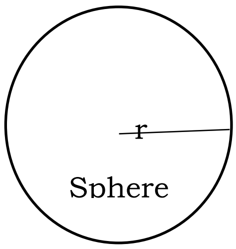
$$V = \frac{9\pi}{16}(2x+1)^3 \Rightarrow \frac{dV}{dx} = \frac{9\pi}{16}\times 3(2x+1)^2\times 2$$

$$\frac{dV}{dx} = \frac{27\pi}{8}(2x+1)^2$$

11. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetre of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm?

Sol: $\frac{dV}{dt} = 900cm^3/\sec \quad \& \ r = 15cm$

Volume of the sphere is $V = \frac{4}{3}\pi r^3$



$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$900 = 4\pi r^2\frac{dr}{dt}$$

$$900 = 4\pi(15)^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{900}{900\pi} = \frac{1}{\pi}cm/s$$

UNIT-7
INTEGRALS

$\left[\begin{array}{l} Total\ marks: 22,\ Q.No-6(1M),\ 18\&19(2M\ each) \\ 31\&32(3M\ each)\ 44(5M)\ \&\ 50.a(6M) \end{array}\right]$

Question No: 6 (1M)

1. Evaluate $\int(\sin x+\cos x)dx$. **(2014-M)**

Sol: $\int(\sin x+\cos x)dx=-\cos x+\sin x+c$

2. Find $\int\cos 3x\,dx$. **(2017-J)**

Sol: $\int\cos 3x\,dx=\frac{\sin 3x}{3}+c$

3. Find $\int\left(2x^2+e^x\right)dx$. **(2016-J, 2020-M)**

Sol: $\int\left(2x^2+e^x\right)dx=2\frac{x^3}{3}+e^x+c$

4. Evaluate $\int\left(2x-3\cos x+e^x\right)dx$. **(14-J, Aug-21)**

Sol: $\int\left(2x-3\cos x+e^x\right)dx=x^2-3\sin x+e^x+c$

5. Find $\int\operatorname{cosec}x(\operatorname{cosec}x+\cot x)dx$ **(2016-M)**

Sol: $\int\operatorname{cosec}x(\operatorname{cosec}x+\cot x)dx$

$=\int\operatorname{cosec}^2x\,dx+\int\operatorname{cosec}x\cot x\,dx=-\cot x-\operatorname{cosec}x+c$

6. Evaluate $\int\sec x(\sec x+\tan x)dx$. **(2015-J)**

Sol: $\int\sec x(\sec x+\tan x)dx=\tan x+\sec x+c$

7. Find $\int\operatorname{cosec}x(\operatorname{cosec}x-\cot x)dx$

Sol:

$\int\operatorname{cosec}x(\operatorname{cosec}x-\cot x)dx=-\cot x+\operatorname{cosec}x+c$

8. Evaluate: $\int\sec^2(7-4x)dx$ **(2019-J)**

Sol: $\int\sec^2(7-4x)dx=-\frac{1}{4}\tan x(7-4x)+c$

9. Simplify $\int\frac{\sin^3x+\cos^3x}{\sin^2x.\cos^2x}dx$

Sol:

$\int\frac{\sin^3x+\cos^3x}{\sin^2x.\cos^2x}dx=\int\tan x\sec x\,dx+\int\cot x\operatorname{cosec}x\,dx$

$=\sec x-\operatorname{cosec}x+c$

10. Evaluate $\int\tan^2 2x\,dx$

Sol:

$\int\tan^2 2x\,dx=\int\left(\sec^2 2x-1\right)dx=\frac{1}{2}\tan 2x-x+c$

11. Evaluate $\int\frac{\sec^2x}{\operatorname{cosec}^2x}dx$

Sol: $\int\frac{\sec^2x}{\operatorname{cosec}^2x}dx=\int\frac{\sin^2x}{\cos^2x}dx=\int\tan^2x\,dx$

$=\int\left(\sec^2x-1\right)dx=\tan x-x+c$

12.Find $\int(ax+b)^2\,dx$

Sol: $\int(ax+b)^2\,dx=\frac{(ax+b)^3}{3a}+c$

13. Find $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$ **(2017-M)**

Sol: $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=\frac{2}{3}x^{3/2}+2\sqrt{x}+c$

14. Find $\int(1-x)\sqrt{x}\,dx$ **(2019-M)**

Sol: $\int(1-x)\sqrt{x}\,dx=\int\sqrt{x}\,dx-\int x\sqrt{x}\,dx$

$=\int x^{1/2}dx-\int x^{3/2}dx=\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}+c$

15. Find $\int\frac{dx}{\sqrt{1-x}}$

Sol: $\int\frac{dx}{\sqrt{1-x^2}}=\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x+C$

16. Find $\int\frac{(1-x)}{\sqrt{x}}dx$

Sol:

$$\int \frac{(1-x)}{\sqrt{x}}\,dx = \int\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)dx = 2\sqrt{x}-\frac{2}{3}x^{3/2}+C$$

17. Evaluate: $\int \frac{3\cos x+4}{\sin^2 x}\,dx$

Sol:

$$\int \frac{3\cos x+4}{\sin^2 x}\,dx = \int 3\cot x \operatorname{cosec} x\,dx + \int 4\operatorname{cosec}^2 x\,dx$$

$$= -3\operatorname{cosec} x - 4\cot x + c$$

18. Evaluate: $\int e^{3x+2}\,dx$

Sol: $\int e^{3x+2}\,dx = \frac{e^{3x+2}}{3}+c$

19. Find $\int \frac{3\sin x}{\cos^2 x}\,dx$

Sol: $\int \frac{3\sin x}{\cos^2 x}\,dx = \int 3\tan x \sec x\,dx = 3\sec x + c$

20. Find $\int \frac{x^3-1}{x^2}\,dx$ **(2018-M)**

Sol: $\int \frac{x^3-1}{x^2}\,dx = \int x\,dx - \int \frac{1}{x^2}\,dx = \frac{x^2}{2}+\frac{1}{x}+c$ **s**

21. Evaluate $\int e^x\left(\frac{x-1}{x^2}\right)dx$ **(2015-M)**

Sol: $\int e^x\left(\frac{x-1}{x^2}\right)dx = \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = e^x\left(\frac{1}{x}\right)+c$

22. Evaluate: $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx.$

Sol: $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx. = e^x\tan^{-1}x+c$

23. Evaluate: $\int e^x \sec x(1+\tan x)\,dx$

Sol: $\int e^x \sec x(1+\tan x)\,dx = \int e^x(\sec x+\sec x\tan x)\,dx$

$$= e^x\sec x + c$$

24. Find $\int e^x(\sin x+\cos x)\,dx$

Sol: $\int e^x(\sin x+\cos x)\,dx = e^x\sin x + c$

25. Find the anti-derivative of $\log_e x$ w.r.to x

Sol: $\int \log_e x\,dx = x\log_e x - x + c$

26. Find the anti-derivative of $\cot^2 x$ w.r.to x

Sol: $\int \cot^2 x\,dx = \int(\operatorname{cosec}^2 - 1)\,dx = -\cot x - x + c$

27. Find the anti-derivative of 3^x w. r.to x

Sol: $\int 3^x\,dx = \frac{3^x}{\log 3}+c$

28. Find the anti-derivative of $\sin 2x - 4e^{3x}$ w. r.to x

Sol: $\int 3^x\,dx = -\frac{\cos 2x}{2}-\frac{4e^{3x}}{3}+C$

29. Evaluate $\int \sqrt{ax+b}\,dx$

Sol: $\int \sqrt{ax+b}\,dx = \frac{2(ax+b)^{\frac{3}{2}}}{3a}+C$

<u>Question No: 18 & 19 (2M each)</u>

1. Evaluate $\int \frac{\sin^2 x}{1+\cos x}\,dx$ **(2014-M)**

Sol:

$$\int \frac{\sin^2 x}{1+\cos x}\,dx = \int \frac{(\sin x)^2}{2\cos^2(x/2)}\,dx = \int \frac{(2\sin(x/2)\cos(x/2))^2}{2\cos^2(x/2)}$$

$$= \int 2\sin^2(x/2)\,dx = \int(1-\cos x)\,dx = x - \sin x + c$$

Or

$$\int \frac{\sin^2 x}{1+\cos x}\,dx = \int \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)}\,dx = x - \sin x + C$$

2. Integrate $\sin x.\sin(\cos x)$ w.r.to x **(2014-J)**

Sol: $\int \sin x.\sin(\cos x)\,dx = \int \sin t(-dt)$

$(put\ \cos x = t,\ -dt = \sin x\,dx)$

$$= \cos t + c = \cos(\cos x) + C$$

3. Evaluate $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx$ **(2015-M)**

Sol: $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx$

$$= \int \frac{\left(2\cos^2 x - 1\right) - \left(2\cos^2\alpha - 1\right)}{\cos x - \cos\alpha}dx$$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2\alpha + 1}{\cos x - \cos\alpha}dx$$

$$= \int \frac{2\cos^2 x - 2\cos^2\alpha}{\cos x - \cos\alpha}dx$$

$$= 2\int \frac{(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{(\cos x - \cos\alpha)}dx$$

$$= 2\int (\cos x + \cos\alpha)dx = 2(\sin x + x\cos\alpha) + c$$

4. Evaluate $\int \frac{1}{x - \sqrt{x}}dx$ **(2015-M)**

Sol: $\int \frac{1}{x - \sqrt{x}}dx = \int \left(\frac{1}{\sqrt{x}\left(\sqrt{x} - 1\right)}\right)dx$

$$\left(put\ \sqrt{x} - 1 = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt \Rightarrow \frac{dx}{\sqrt{x}} = 2dt \right)$$

$$= \int \frac{1}{t}(2dt) = 2\log|t| + c = 2\log\left|\sqrt{x} - 1\right| + c$$

5. Find $\int x^2 \log x\, dx$ **(2016-J)**

Sol: $\int x^2 \log x\, dx = \int \log x . x^2\, dx$

$$= \log x \int x^2 dx - \int \left[\int x^2 . \frac{d}{dx}\log x\right]dx$$

$$= \log x\left(\frac{x^3}{3}\right) - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx$$

$$= \log x\left(\frac{x^3}{3}\right) - \frac{1}{3}\int x^2 dx = \log x\left(\frac{x^3}{3}\right) - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$$

6. Evaluate $\int \log\sin x . \cot x\, dx$ **(2020-M)**

Sol: $\int \log\sin x . \cot x\, dx$

$$\left(\therefore put\ \log\sin x = t \Rightarrow \frac{1}{\sin x}\cos x\, dx = dt \Rightarrow \cot x\, dx = dt \right)$$

$$= \int t\, dt = \frac{t^2}{2} + c = \frac{1}{2}(\log\sin x)^2 + c$$

7. Evaluate $\int \frac{x^2}{1 - x^6}dx$ **(2015-J)**

Sol: $\int \frac{x^2}{1 - x^6}dx = \int \frac{x^2}{1 - \left(x^3\right)^2}dx = \frac{1}{3}\int \frac{1}{1 - t^2}dt$

$$\left(put\ x^3 = t \Rightarrow 3x^2\, dx = dt \Rightarrow x^2 dx = dt/3 \right)$$

$$\left(wkt \int \frac{1}{a^2 - x^2}dx = \frac{1}{2a}\log\left|\frac{1 + t}{1 - t}\right| + c \right)$$

$$= \frac{1}{3}\left[\frac{1}{2(1)}\log\left|\frac{1 + t}{1 - t}\right|\right] + c = \frac{1}{6}\log\left|\frac{1 + x^3}{1 - x^3}\right| + c$$

8. Find $\int \frac{1}{\sin x \cos^3 x}dx$ **(2016-M)**

Sol: $\int \frac{1}{\sin x \cos^3 x}dx = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}dx$

$$= \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x}dx$$

$$\left(put\ \tan x = t \Rightarrow \sec^2 x\, dx = dt \right)$$

$$= \int t\, dt + \int \frac{1}{t}dt = \frac{t^2}{2} + \log t + c = \frac{\tan^2 x}{2} + \log|\tan x| + c$$

9. Integrate $\frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}$ w.r.to x**(2017-M)**

Sol: $\int \frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}dx = \int t^4(2dt)$

$$\left(put\ \tan\sqrt{x} = t \Rightarrow \frac{\sec^2\sqrt{x}}{2\sqrt{x}}dx = dt \quad \Rightarrow \frac{\sec^2\sqrt{x}}{\sqrt{x}}dx = 2dt \right)$$

$$= 2\frac{t^5}{5} + c = \frac{2}{5}\left(\tan\sqrt{x}\right)^5 + c \quad \left(\sec^2\sqrt{x} . \frac{1}{2\sqrt{x}} = \frac{dt}{dx} \right)$$

10. Evaluate $\int \frac{\sqrt{\tan x}}{\sin x . \cos x}dx$ **(2017-J)**

Sol: $\int \frac{\sqrt{\tan x}}{\sin x . \cos x}dx = \int \frac{\sqrt{\tan x}}{\sin x . \cos x . \frac{\cos x}{\cos x}}dx$

$$= \int \frac{\sqrt{\tan x}}{\tan^2 x . \cos^2 x}dx = \int \frac{\sqrt{\tan x} . \sec^2 x}{\tan^2 x}dx$$

$$\left(put\ \tan x = t,\ \sec^2 x\, dx = dt \right)$$

$$=\int\frac{\sqrt{t}}{t}dt=\int\frac{1}{\sqrt{t}}dt=2\sqrt{t}+c=2\sqrt{\tan x}+c$$

11. Evaluate $\int\sin 3x.\cos 4x\,dx$

Sol: $\int\sin 3x.\cos 4x\,dx=\int\cos 4x.\sin 3x\,dx$

$$\left(use\ \cos x\sin y=\frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]\right)$$

$$=\frac{1}{2}\int[\sin 7x-\sin x]dx$$

$$=\frac{1}{2}\left[-\frac{1}{7}\cos 7x+\cos x\right]+c=-\frac{1}{14}\cos 7x+\frac{1}{2}\cos x+c$$

12. Evaluate: $\int\sin 2x.\cos 3x\,dx$ **(19-M, Aug-21)**

Sol: $\int\sin 2x.\cos 3x\,dx=\frac{1}{2}\int[\sin 5x-\sin x]dx$

$$=\frac{1}{2}\left[-\frac{\cos 5x}{5}+\cos x\right]+c=-\frac{\cos 5x}{10}+\frac{\cos x}{2}+C$$

13. Evaluate $\int\frac{2-3\sin x}{\cos^2 x}dx$

Sol: $\int\frac{2-3\sin x}{\cos^2 x}dx=\int\frac{2}{\cos^2 x}dx-\int\frac{3\sin x}{\cos^2 x}dx$

$$=2\int\sec^2 x dx-3\int\tan x\sec x dx$$

$$=2\tan x-3\sec x+c$$

14. Integrate $\frac{e^{\tan^{-1}x}}{1+x^2}$ w.r.to x **(2018-M)**

Sol: $\int\frac{e^{\tan^{-1}x}}{1+x^2}dx=\int e^t dt=e^t+c=e^{\tan^{-1}x}+c$

$$\left(put\ \tan^{-1}x=t\Rightarrow\frac{1}{1+x^2}dx=dt\right)$$

15. Evaluate: $\int\frac{1}{\cos^2 x(1-\tan x)^2}dx$ **(2019-M)**

Sol: $\int\frac{1}{\cos^2 x(1-\tan x)^2}dx=\int\frac{\sec^2 x}{(1-\tan x)^2}dx$

$$\left(put\ 1-\tan x=t\Rightarrow\sec^2 x dx=-dt\right)$$

$$=\int\frac{-dt}{t^2}=-\int\frac{1}{t^2}dt=-\left(-\frac{1}{t}\right)+c=\frac{1}{1-\tan x}+c$$

16. Evaluate: $\int\frac{\sec x}{\sec x+\tan x}dx$

Sol:

$$\int\frac{\sec x}{\sec x+\tan x}dx=\int\frac{\sec x}{\sec x+\tan x}\times\frac{\sec x-\tan x}{\sec x-\tan x}dx$$

$$=\int\frac{\sec^2 x-\sec x\tan x}{\sec^2 x-\tan^2 x}dx\quad(\because\sec^2 x-\tan^2 x=1)$$

$$=\int(\sec^2 x-\sec x\tan x)dx=\tan x-\sec x+c$$

17. Evaluate: $\int\frac{\tan x}{\sec x+\cos x}dx$

Sol:

$$\int\frac{\tan x}{\sec x+\tan x}dx=\int\frac{\tan x}{\sec x+\tan x}\times\frac{\sec x-\tan x}{\sec x-\tan x}dx$$

$$=\int\frac{\tan x\sec x-\tan^2 x}{\sec^2 x-\tan^2 x}dx$$

$$=\int\left(\sec x\tan x-(\sec^2 x-1)\right)dx=\sec x-\tan x+x+c$$

$$(\because 1+\tan^2 x=\sec^2 x)$$

18. Evaluate $\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx$

Sol: $\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx$

$$put\ \sqrt{x}=t\Rightarrow\frac{1}{2\sqrt{x}}dx=dt\Rightarrow\frac{1}{\sqrt{x}}dx=2dt$$

$$=2\int\sin t\,dt=-2\cos t+c=-2\cos(\sqrt{x})+c$$

19. Evaluate: $\int\frac{\sin x}{\sin(x-a)}dx$

Sol: $\int\frac{\sin x}{\sin(x-a)}dx=\int\frac{\sin(t+a)}{\sin t}dt$

$$put\ x-a=t\Rightarrow dx=dt$$

$$(wkt\ \sin(x-y)=\sin x\cos y-\cos x\sin y)$$

$$=\int\frac{\sin t\cos a-\cos t\sin a}{\sin t}dt$$

$$=\cos a\int 1dx-\sin a\int\cot t\,dt=\cos a.t-\sin a.\log(\sin t)+c$$

$$=\cos a.(x-a)-\sin a.\log\sin(x-a)+c$$

20. Evaluate: $\int\frac{1}{1-\cos x}dx$

Sol:

$$\int\frac{1}{1-\cos x}dx=\int\frac{1}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}dx=\int\frac{1+\cos x}{1-\cos^2 x}dx$$

$$= \int \frac{1+\cos x}{\sin^2 x}\,dx = \int \left(\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x\right) dx$$

$$= -\cot x - \operatorname{cosec} x + c$$

21. Evaluate: $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx$

Sol:

$$\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx = \int \tan^{-1}\left(\frac{2\sin x\cos x}{2\cos^2 x}\right) dx$$

$$= \int \tan^{-1}(\tan x)\,dx \quad = \int x\,dx = \frac{x^2}{2} + c$$

22. Evaluate: $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\,dx$

Sol: $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\,dx$

$$\left(put\ \ x^{10} + 10^x = t \Rightarrow \left(10x^9 + 10^x \log_e 10\right) dx = dt\right)$$

$$= \int \frac{1}{t}\,dt = \log|t| + c = \log\left|x^{10} + 10^x\right| + c$$

23. Find $\int \frac{\left(x^4 - x\right)^{\frac{1}{4}}}{x^5}\,dx$ **(2019-J)**

Sol:

$$\int \frac{\left(x^4 - x\right)^{\frac{1}{4}}}{x^5}\,dx = \int \frac{\left(x^4\right)^{\frac{1}{4}}\left(1 - \frac{1}{x^3}\right)^{\frac{1}{4}}}{x^5}\,dx = \int \frac{(x)\left(1 - \frac{1}{x^3}\right)^{\frac{1}{4}}}{x^5}\,dx$$

$$= \int \frac{\left(1 - \frac{1}{x^3}\right)^{\frac{1}{4}}}{x^4}\,dx = \int t^{\frac{1}{4}} . \frac{1}{3}\,dt$$

$$\left(put\ \left(1 - \frac{1}{x^3}\right) = t \Rightarrow dt = \frac{3}{x^4}\,dx \Rightarrow \frac{1}{3}\,dt = \frac{1}{x^4}\,dx\right)$$

$$= \frac{1}{3}\left(\frac{4}{5} t^{\frac{5}{4}}\right) + c = \frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{\frac{5}{4}} + c$$

24. Find the anti-derivative of F given by $f(x) = 4x^3 - 6x$, where $F(0) = 3$.

Sol: Given

$$F(x) = \int f(x)\,dx = \int \left(4x^3 - 6x\right) dx = x^4 - 3x^2 + c$$

But $F(0) = 3 \Rightarrow 3 = 0 - 0 + c \Rightarrow c = 3$

$$F(x) = x^4 - 3x^2 + 3$$

25. Evaluate $\int_1^e \frac{1}{x}\,dx$ **(2014-M)**

Sol: $\int_1^e \frac{1}{x}\,dx = [\log x]_1^e = \log e - \log 1 = 1 - 0 = 1$

26. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ **(2014-J)**

Sol:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

27. Evaluate $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ **(2015-J)**

Sol: $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx = e^x\left(\frac{1}{x}\right) + c$

28. Evaluate $\int_0^\pi \left(sin^2\left(\frac{x}{2}\right) - \cos^2\left(\frac{x}{2}\right)\right) dx$ **(2016-M)**

Sol: $\int_0^\pi \left(sin^2\left(\frac{x}{2}\right) - \cos^2\left(\frac{x}{2}\right)\right) dx$

$$= -\int_0^\pi \cos x\,dx = -[\sin x]_0^\pi = -\sin\pi + \sin 0 = 0$$

29. Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ **(2016-J)**

Sol:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1} x\right]_0^1 = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2}$$

30. Evaluate $\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}\,dx$ **(2017-M)**

Sol:

$$\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}\,dx = \int_0^{\frac{2}{3}} \frac{1}{2^2 + (3x)^2}\,dx = \frac{1}{3}.\frac{1}{2}\left[\tan^{-1}\left(\frac{3x}{2}\right)\right]_0^{\frac{2}{3}}$$

$$=\frac{1}{6}\left[\tan^{-1}1-\tan^{-1}0\right]=\frac{1}{6}.\frac{\pi}{4}=\frac{\pi}{24}$$

31. Evaluate $\int \frac{x-3}{(x-1)^3}e^x\ dx$ **(2017-J)**

Sol: $\int \frac{x-3}{(x-1)^3}e^x\ dx=\int \frac{(x-1)-2}{(x-1)^3}dx$

$$=\int\left(\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right)e^x\,dx=e^x\frac{1}{(x-1)^2}+c$$

32. Evaluate $\int_2^3 \frac{x}{x^2+1}dx$ **(2018-M)**

Sol: $\int_2^3 \frac{x}{x^2+1}dx=\int_2^3\frac{1}{t}.\frac{1}{2}dt\quad\left[put\ x^2+1=t,\ xdx=\frac{1}{2}dt\right]$

$$=\frac{1}{2}[\log t]_2^3=\frac{1}{2}\left[\log\left(x^2+1\right)\right]_2^3$$

$$=\frac{1}{2}(\log 10-\log 5)=\frac{1}{2}\log\left(\frac{10}{5}\right)=\frac{1}{2}\log 2=\log\sqrt{2}$$

33. Evaluate $\int\frac{e^{2x}-1}{e^{2x}+1}dx$

Sol: $\int\frac{e^{2x}-1}{e^{2x}+1}dx$ divide both numerator and denominator by e^x

$$=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx\quad\left[put\ e^x+e^{-x}=t\Rightarrow\left(e^x-e^{-x}\right)dx=dt\right]$$

$$=\int\frac{1}{t}dt=\log t+c=\log\left(e^x+e^{-x}\right)+c$$

34. Evaluate: $\int\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx$

Sol: $\int\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx$

$$\left[put\ e^{2x}+e^{-2x}=t\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt\right]$$

$$=\frac{1}{2}\int\frac{1}{t}dt=\frac{1}{2}\log t+c=\frac{1}{2}\log\left(e^{2x}+e^{-2x}\right)+c$$

35. Evaluate $\int\frac{2x\tan^{-1}x^2}{\left(1+x^4\right)}dx$

Sol: $\int\frac{2x\tan^{-1}x^2}{\left(1+\left(x^2\right)^2\right)}dx$

$$\left[put\ \tan^{-1}x^2=t\Rightarrow\frac{1}{\left(1+\left(x^2\right)^2\right)}2x\,dx=dt\right]$$

$$=\int t\,dt=\frac{t^2}{2}+c=\frac{\left(\tan^{-1}x^2\right)^2}{2}+c$$

36. Evaluate $\int\sin(ax+b)\cos(ax+b)dx$

Sol: $\int\sin(ax+b)\cos(ax+b)dx$

$$\left[put\ \sin(ax+b)=t\Rightarrow\cos(ax+b)dx=dt/a\right]$$

$$=\int t\frac{dt}{a}=\frac{t^2}{2a}+c=\frac{\left[\sin(ax+b)\right]^2}{2a}+C$$

37. Evaluate $\int_0^{\frac{\pi}{2}}\cos 2x\,dx$

Sol: $\int_0^{\frac{\pi}{2}}\cos 2x\,dx=\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$$=\frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right)-\sin 2(0)\right]=\frac{1}{2}(0-0)=0$$

38. Integrate $x\sec^2 x$ with r to x (**19-J, 20-M)**

Sol: $\int x\sec^2x\,dx=x\int\sec^2x\,dx-\int\left[\int\sec^2x.\frac{d}{dx}x\right]dx$

$$=x.\tan x-\int\tan x(1)dx=x\tan x-\left(-\log|\cos x|\right)+c$$

$$=x\tan x+\log|\cos x|+C$$

Question No: 31 & 32 (3M Each)

1. Find $\int\sin 3x\cos 4x\,dx$ **(2014-J)**

Sol: $\int\sin 3x\cos 4x\,dx=\int\frac{1}{2}[\sin 7x-\sin x]dx$

$$=\frac{1}{2}\left[\frac{-\cos 7x}{7}+\cos x\right]+c$$

2. Evaluate $\int \frac{1}{1+\tan x}dx$

Sol: $\int \frac{1}{1+\tan x}dx=\int \frac{1}{1+\frac{\sin x}{\cos x}}dx=\int \frac{\cos x}{\cos x+\sin x}dx$

$$=\frac{1}{2}\int \frac{(\cos+\sin x)+(\cos x-\sin x)}{\cos x+\sin x}dx$$

$$=\frac{1}{2}\int 1dx+\frac{1}{2}\int \frac{\cos x-\sin x}{\cos x+\sin x}dx$$

$\left(put\ \cos x+\sin x=t\Rightarrow(-\sin x+\cos x)dx=dt\right)$

$$=\frac{1}{2}(x)+\frac{1}{2}\int\frac{1}{t}dt=\frac{x}{2}+\frac{1}{2}\log t+c=\frac{x}{2}+\frac{1}{2}\log(\cos x+\sin x)+c$$

3. Evaluate $\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$ **(2017-J)**

Sol: $I=\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$\left[put\ \cos^{-1}x=t\Rightarrow\frac{1}{\sqrt{1-x^2}}dx=-dt\right]$ $[x=\cos t]$

$$=\int \cos t.t(-dt)=-\left[t\int\cos t\,dt-\int\left[\int\cos t\frac{d}{dt}(t)\right]dt\right]$$

$$=-\left[t(\sin t)-\int(\sin t)(1)\right]dt$$

$$=-t\sin t-\cos t+c=-\cos^{-1}(x)\sin\left(\cos^{-1}(x)\right)-x+C$$

4. Evaluate $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Sol: $I=\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

$\left[put\ \sin^{-1}x=t\Rightarrow\frac{1}{\sqrt{1-x^2}}dx=dt\right]$ $[x=\sin t]$

$$=\int \sin t.t\,dt=t\int\sin t\,dt-\int\left[\int\sin t\frac{d}{dt}(t)\right]dt$$

$$=t(-\cos t)-\int(-\cos t(1))$$

$$=-t\cos t+\sin t+c=-\sin^{-1}(x)\cos\left(\sin^{-1}(x)\right)+x+c$$

5. Evaluate $\int \frac{(1+\log x)^2}{x}dx$

Sol: $I=\int \frac{(1+\log x)^2}{x}dx$

$\left[put\,1+\log x=t\Rightarrow\frac{1}{x}dx=dt\right]$

$$I=\int t^2\,dt=\frac{t^3}{3}+c=\frac{(1+\log x)^3}{3}+c$$

6. Evaluate $\int \frac{1}{x+x\log x}dx$

Sol: $I=\int \frac{1}{x+x\log x}dx=\int\frac{1}{x(1+\log x)}dx$

$\left[put\ 1+\log x=t\Rightarrow\frac{1}{x}dx=dt\right]$

$$=\int\frac{1}{t}dt=\log t+c=\log(1+\log x)+c$$

7. Evaluate $\int \frac{1}{x(x^2+1)}dx$

Sol: *Let* $\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$

$1=A(x^2+1)+(Bx+C)x$

on solving we get $A=1$, $B=-1$ & $C=0$

$$I=\int\frac{1}{x(x^2+1)}dx=\int\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx$$

$$=\log|x|-\frac{1}{2}\log|x^2+1|+c$$

8. Evaluate $\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$

Sol: $I=\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$

$=\int \frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)^2}dx=\int \frac{\cos x-\sin x}{\cos x+\sin x}dx$

$\left[put\ (\sin x+\cos x)=t\Rightarrow(\cos x-\sin x)dx=dt\right]$

$=\int \frac{1}{t}dt=\log|t|+c=\log|(\sin x+\cos x)|+c$

9. Evaluate $\int \frac{x}{(x+1)(x+2)}dx$ **(2015, 2016, 2017-J, 2018-M & 2019-J, MQP-21, Aug-21)**

Sol: Let $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$x=A(x+2)+B(x+1)-----(1)$

$Put\ x=-1\ we\ get\ A=-1\ \&\ put\ x=-2\ we\ get\ B=2$

$\therefore I=\int\left[\frac{-1}{(x+1)}+\frac{2}{(x+2)}\right]dx$

$=-\log|x+1|+2\log|x+2|+c=\log\left|\frac{(x+2)^2}{(x+1)}\right|+c$

10. Find $\int \frac{x}{(x-1)(x-2)}dx$ **(2016-M)**

Sol: Let $\frac{x}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

$x=A(x-2)+B(x-1)-----(1)$

$Put\ x=1\ we\ get\ A=-1\ \&\ put\ x=2\ we\ get\ B=2$

$\therefore I=\int\left[\frac{-1}{(x-1)}+\frac{2}{(x-2)}\right]dx$

$=-\log|x-1|+2\log|x-2|+c=\log\left|\frac{(x-2)^2}{(x-1)}\right|+c$

11. Integrate $\frac{2x}{(x+1)(x+2)}$ w.r.to x **(2016-M)**

Sol: Let $\frac{2x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$2x=A(x+2)+B(x+1)$

$Put\ x=-1\Rightarrow A=-2\ \ \&\ \ Put\ x=-2\Rightarrow B=4$

$I=\int\frac{-2}{(x+1)}dx+\int\frac{4}{(x+2)}dx=-2\log(x+1)+4\log(x+2)+C$

12. Evaluate $\int \frac{2x}{x^2+3x+2}dx$ **(2017-M)**

Sol:

$\frac{2x}{x^2+3x+2}=\frac{2x}{(x+2)(x+1)}=\frac{A}{(x+2)}+\frac{B}{(x+1)}$

$2x=A(x+1)+B(x+2)$

$Put\ x=-1\Rightarrow B=-2\ \ \&\ Put\ x=-2\Rightarrow A=4$

$I=\int\frac{4}{(x+2)}dx-\int\frac{2}{(x+1)}dx=4\log(x+2)-2\log(x+1)+c$

13. Evaluate: $\int\frac{e^x}{(1+e^x)(2+e^x)}dx$

Sol: $\int\frac{e^x}{(1+e^x)(2+e^x)}dx$ (put $e^x=t\Rightarrow e^xdx=dt$

$I=\int\frac{e^x}{(1+e^x)(2+e^x)}dx=\int\frac{1}{(1+t)(2+t)}dt$

$I=\int\left(\frac{1}{1+t}-\frac{1}{2+t}\right)dt=\log|1+t|-\log|2+t|+c$

$I=\log\left|\frac{1+t}{2+t}\right|+c$

14. Evaluate: $\int\frac{1}{x^2-9}dx$

Sol: $I=\int\frac{1}{x^2-9}dx$

$=\int\frac{1}{x^2-3^2}dx=\frac{1}{2(3)}\log\left|\frac{x-3}{x+3}\right|+c=\frac{1}{6}\log\left|\frac{x-3}{x+3}\right|+c$

15. Evaluate $\int \tan^{-1} x\,dx$ **(2014-M)**

Sol: $I = \int \tan^{-1} x\,dx = \int \tan^{-1} x.1dx$

$$I = (\tan^{-1} x)(x) - \int (x)\frac{1}{1+x^2}\,dx$$

$$I = x\tan^{-1} x - \int \frac{x}{1+x^2}dx$$

$$(\text{put}\,1+x^2 = t \Rightarrow x\,dx = dt/2)$$

$$I = x\tan^{-1} x - \int \frac{1}{t}\frac{dt}{2}$$

$$I = x\tan^{-1} x - \frac{1}{2}\log t + C$$

$$I = x\tan^{-1} x - \frac{1}{2}\log(1+x^2) + C$$

16. Integrate $x^2 e^x$ with respect to x **(14-J)**

Sol: $I = \int x^2 e^x\,dx = x^2\int e^x\,dx - \int\left[\int e^x \frac{d}{dx}x^2\right]dx$

$$I = x^2 e^x - 2\int xe^x dx$$

$$I = x^2 e^x - 2\left[xe^x - e^x\right] + c = x^2 e^x - 2xe^x + 2e^x + c$$

17. Evaluate $\int x\tan^{-1} x\,dx$ **(2015-M)**

Sol: $I = \int x\tan^{-1} x\,dx = \int \tan^{-1} x.x\,dx$

$$= \tan^{-1} x\int x\,dx - \int\left[\int x\frac{d}{dx}\tan^{-1} x\right]dx$$

$$= \tan^{-1} x.\frac{x^2}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int\frac{1+x^2-1}{1+x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left[\int 1\,dx - \int\frac{1}{1+x^2}dx\right]$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left[x - \tan^{-1} x\right] + c$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + c$$

18. Evaluate $\int e^x \sin x\,dx$ **(2017, 18-M)**

Sol: $I = \int \sin x\,e^x\,dx ----(1)$

$$I = \sin x\int e^x\,dx - \int\left[\int e^x\frac{d}{dx}\sin x\right]dx$$

$$= e^x\sin x - \int\cos x.e^x\,dx$$

$$= e^x\sin x - \left\{\cos x\int e^x\,dx - \int\left[\int e^x\frac{d}{dx}\cos x\right]dx\right\}$$

$$= e^x\sin x - e^x\cos x + \int -\sin x.e^x\,dx$$

$$= e^x\sin x - e^x\cos x - I \qquad (from\ (1))$$

$$2I = e^x\sin x - e^x\cos x$$

$$I = \frac{e^x}{2}(\sin x - \cos x) + C$$

19. Find $\int x.\log x\,dx$ **(2019-M)**

Sol: $\int x.\log x\,dx = \int \log x.x\,dx$

$$= \log x\int x\,dx - \int\left[\int x\frac{d}{dx}\log x\right]dx$$

$$= \log x.\frac{x^2}{2} - \int\frac{x^2}{2}.\frac{1}{x}dx$$

$$= \log x.\frac{x^2}{2} - \frac{1}{2}\int x\,dx = \log x.\frac{x^2}{2} - \frac{x^2}{4} + c$$

20. Evaluate: $\int x\sin^{-1} x\,dx$

Sol: $I = \int x\sin^{-1} x\,dx = \int \sin^{-1} x.x\,dx$

$$= \sin^{-1} x\int x\,dx - \int\left[\int x\frac{d}{dx}(\sin^{-1} x)\right]dx$$

$$= \sin^{-1} x\left(\frac{x^2}{2}\right) - \int\frac{x^2}{2}.\frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int\frac{(1-x^2)-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int\sqrt{1-x^2}\,dx - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] - \frac{1}{2}\sin^{-1} x + c$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + c$$

21. Find $\int \frac{(x^2+1)e^x}{(x+1)^2}dx$ **(2014-M, Aug-21)**

Sol: $\int \frac{(x^2+1)e^x}{(x+1)^2}dx = \int e^x \frac{(x^2-1+2)}{(x+1)^2}\,dx$

$= \int e^x \left[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}\right]dx$

$= \int e^x \left[\frac{x-1}{(x+1)} + \frac{2}{(x+1)^2}\right]dx$

$\left(wkt \int e^x\left(f(x)+f'(x)\right)dx = e^x.f(x)+C\right)$

$= e^x\left(\frac{x-1}{x+1}\right)+C$

22. Find $\int e^x\left[\frac{1+sinx}{1+cosx}\right]dx$ **(2016-J)**

Sol:

$\int e^x\left[\frac{1+sinx}{1+cosx}\right]dx = \int e^x\left[\frac{1}{1+\cos x}+\frac{\sin x}{1+\cos x}\right]dx$

$= \int e^x\left[\frac{1}{2\cos^2(x/2)} + \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right]dx$

$= \int e^x\left(\frac{1}{2}\sec^2(x/2)+\tan(x/2)\right)dx$

$= e^x.\tan(x/2)+C$

23. Evaluate $\int \frac{x-3}{(x-1)^3}e^x\;dx$ **(2020-M)**

Sol:

$\int \frac{x-3}{(x-1)^3}e^x\;dx = \int e^x\left[\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3}\right]dx$

$= \int e^x\left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right]dx = e^x\frac{1}{(x-1)^2}+C$

24. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$ **(2019-M)**

Sol: $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{1}{1+t^2}dt$

Put $t=\cos x,\ dt=-\sin x\,dx \quad -dt=\sin x\,dx$

If $x=0$ *then* $t=1$*and* $x=\pi/2$ *then* $t=0$

$= -\left[\tan^{-1}t\right]_1^0$

$= -\left[\tan^{-1}(0)-\tan^{-1}(1)\right] = -\left[0-\frac{\pi}{4}\right] = \frac{\pi}{4}$

25. Evaluate $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$

Sol: $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \int_0^2 \sqrt{t}\,dt$

Put $t = x^5+1 \;\Rightarrow\; dt = 5x^4\,dx$

$\left(If\ x=1\ then\ t=0,\ x=1\ then\ t=2\right)$

$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4}{3}\sqrt{2}$

26. Evaluate $\int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}}dx$

Sol:

$\int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}}dx = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} \times \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}\,dx$

$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{(1+x)-x}dx = \int_0^1\left(\sqrt{1+x}+\sqrt{x}\right)dx$

$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{3}\left[(1+1)^{\frac{3}{2}}-(1+0)^{\frac{3}{2}}\right] + \frac{2}{3}\left[1^{\frac{3}{2}}-0\right]$

$= \frac{2}{3}\left[2\sqrt{2}-1+1\right] = \frac{4\sqrt{2}}{3}$

27. Evaluate $\int \frac{\sin^3 x+\cos^3 x}{\sin^2 x.\cos^2 x}dx$

Sol: $\int \frac{\sin^3 x+\cos^3 x}{\sin^2 x.\cos^2 x}dx$

$= \int \left(\frac{\sin^3 x}{\sin^2 x.\cos^2 x}+\frac{\cos^3 x}{\sin^2 x.\cos^2 x}\right)dx$

$= \int (\sec x.\tan x + \cos ecx.\cot x)dx = \sec x - \cos ecx + C$

28. Evaluate $\int \frac{dx}{(x+1)(x+2)}$

Sol: Let $\frac{1}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$1 = A(x+2) + B(x+1) - - - - - (1)$

Put $x=-1$ we get $A=1$ & put $x=-2$ we get $B=-1$

$$I = \int \left[\frac{1}{(x+1)} - \frac{1}{(x+2)} \right] dx$$

$$I = \log|x+1| - \log|x+2| + C = \log\left|\frac{(x+1)}{(x+2)}\right| + C$$

29. $\int_0^2 e^x dx$ as a limit of sum. **(2015-M & J)**

Sol: $I = \int_0^2 e^x dx = \int_0^2 f(x)dx$ *where* $f(x) = e^x$

$$I = \lim_{h\to 0} h \begin{Bmatrix} f(0) + f(0+h) + f(0+2h) + ... \\ .... + f(0+(n-1)h) \end{Bmatrix}$$

$$= \lim_{h\to 0} h\left\{e^0 + e^h + e^{2h} + ... + e^{(n-1)h}\right\}$$

$$= \lim_{h\to 0} h \frac{e^{(n-1)h} - 1}{e^h - 1} = \lim_{h\to 0} \frac{e^{nh-h} - 1}{\frac{e^h - 1}{h}} = e^2 - 1$$

$$\left(\because nh = b - a = 2 - 0 = 2 \ \& \ \lim_{h\to 0} \frac{e^h - 1}{n} = 1 \right)$$

30. $\int_0^2 (x^2+1)dx$ as a limit of sum **(2019-J)**

Sol: $\int_0^2 (x^2+1)dx = \int_0^2 f(x)dx$

where $f(x) = x^2 + 1$, $nh = b - a = 2 - 0 = 2$

$$= \lim_{h\to 0} h \begin{Bmatrix} f(0) + f(0+h) + f(0+2h) + ... \\ ...... + f(0+(n-1)h) \end{Bmatrix}$$

$$I = \lim_{h\to 0} h\left\{1 + h^2 + 1 + 2h^2 + 1 + ... + (n-1)^2 h^2 + 1\right\}$$

$$= \lim_{h\to 0} h\left\{ [1+1+....+1(n\,times)] + \begin{bmatrix} h^2 + 2h^2 + ... \\ .... + (n-1)^2 h^2 \end{bmatrix} \right\}$$

$$= \lim_{h\to 0} h\left\{ n + h^2\left(1^2 + 2^2 + 3^2 + ... + (n-1)^2\right) \right\}$$

$$= \lim_{h\to 0} \left\{ nh + h^3 \frac{n(n-1)(2n-1)}{6} \right\}$$

$$\left(1^2 + 2^2 + 3^2 + ... + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \right)$$

$$= \lim_{h\to 0} \left\{ nh + \frac{nh(nh-h)(2nh-h)}{6} \right\}$$

$$= \lim_{h\to 0} \left\{ 2 + \frac{2(2-h)(2(2)-h)}{6} \right\}$$

$$= \left\{ 2 + \frac{2(2-0)(2(2)-0)}{6} \right\} = 2 + \frac{16}{6} = \frac{14}{3}$$

31. Evaluate: $\int \tan^4 x\,dx$.

Sol: $I = \int \tan^4 x\,dx = \int \tan^2 x \ \tan^2 x \ dx$

$$= \int \tan^2 x(\sec^2 x - 1)\,dx = \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$= \int t^2\,dt - \int (\sec^2 x - 1)dx$$

put $\tan x = t$ in first integral

$\Rightarrow \sec^2 x\,dx = dt$

$$= \frac{t^3}{3} - \tan x + x + C = \frac{\tan^3 x}{3} - \tan x + x + C$$

32. Evaluate: $\int_0^{\frac{\pi}{4}} \sin 2x dx$

Sol: $I = \int_0^{\frac{\pi}{4}} \sin 2x dx = \left(\frac{-\cos 2x}{2} \right)_0^{\frac{\pi}{4}}$

$$= \frac{1}{2}\left(-\cos 2\frac{\pi}{4} + \cos 0 \right) = \frac{1}{2}\left(-\cos\frac{\pi}{2} + 1 \right) = \frac{1}{2}(0+1) = \frac{1}{2}$$

Question No: 44 (5M)

1. Find $\int \frac{1}{x^2 - a^2} dx$ and hence evaluate

(a) $\int \frac{1}{3x^2 + 13x - 10} dx$ **(2014-M)** **(b)** $\int \frac{1}{x^2 - 16} dx$

Sol: $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \int \frac{2a}{x^2 - a^2}$

$$=\frac{1}{2a}\int\frac{(x+a)-(x-a)}{(x+a)(x-a)}dx=\frac{1}{2a}\int\left[\frac{1}{(x-a)}-\frac{1}{(x+a)}\right]dx$$

$$=\frac{1}{2a}\left[\log|x-a|-\log|x+a|\right]+C=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$$

$$\therefore\ \int\frac{1}{x^2-a^2}dx==\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$$

(a) $\int\frac{1}{3x^2+13x-10}dx=\frac{1}{3}\int\frac{1}{x^2+\frac{13}{3}x-\frac{10}{3}}dx$

to make perfect square, add and substract half of the square of co-efficient of the x.

i.e., $\left(\frac{13}{6}\right)^2$

$$=\frac{1}{3}\int\frac{1}{x^2+\frac{13}{3}x-\frac{10}{3}+\left(\frac{13}{6}\right)^2-\left(\frac{13}{6}\right)^2}dx$$

$$=\frac{1}{3}\int\frac{1}{\left(x^2+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2}dx$$

$$=\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{6}{17}\log\left(\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right)+C$$

$$=\frac{1}{17}\log\left|\frac{6x-4}{6x+30}\right|+C=\frac{1}{17}\log\left|\frac{3x-2}{3x+15}\right|+C$$

(b) $\int\frac{1}{x^2-16}dx=\int\frac{1}{x^2-4^2}dx$ **(2019-M)**

$$=\frac{1}{2(4)}\log\left|\frac{x-4}{x+4}\right|+C=\frac{1}{8}\log\left|\frac{x-4}{x+4}\right|+C$$

2. Find the integral of $\frac{1}{x^2+a^2}$ w.r.to x and evaluate **(a)** $\int\frac{1}{3+2x+x^2}dx$ **(2016-M)**

(b) $\int\frac{1}{x^2+2x+2}dx$ **(2020-M)** **(c)** $\int\frac{1}{x^2-6x+13}dx$ **(2018-M)**

Sol: $\int\frac{1}{x^2+a^2}dx$ $Put\ x=a\tan\theta\Rightarrow dx=a\sec^2\theta\,d\theta$

$$I=\int\frac{1}{x^2+a^2}dx=\int\frac{1}{a^2\tan^2\theta+a^2}a\sec^2\theta\,d\theta$$

$$=\int\frac{a\sec^2\theta}{a^2\left(\tan^2\theta+1\right)}d\theta=\frac{1}{a}\int d\theta=\frac{1}{a}\theta+C$$

$$=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$\therefore\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

(a) $\int\frac{1}{3+2x+x^2}dx=\int\frac{1}{x^2+2x+3}dx=\int\frac{1}{x^2+2x+1+2}dx$

$$=\int\frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2}dx=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C$$

(b) $\int\frac{1}{x^2+2x+2}dx=\int\frac{1}{x^2+2x+1+1}dx$

$$=\int\frac{1}{(x+1)^2+1}dx=\tan^{-1}(x+1)+C$$

(c) $\int\frac{1}{x^2-6x+13}dx=\int\frac{1}{x^2-(2)(3)(x)+9-9+13}dx$

$$=\int\frac{1}{(x-3)^2+2^2}dx=\frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+C$$

3. Find the integral of $\frac{1}{\sqrt{x^2+a^2}}$ with respect to x & hence evaluate $\int\frac{1}{\sqrt{x^2+7}}dx$

(2014-J)

Sol: $I=\int\frac{1}{\sqrt{x^2+a^2}}dx=\int\frac{1}{\sqrt{a^2\tan^2\theta+a^2}}\times a\sec^2\theta\,d\theta$

$$\left[Put\ x=a\tan\theta,\ dx=a\sec^2\theta\,d\theta\right]$$

$$=\int\frac{a\sec^2\theta}{\sqrt{a^2\left(\tan^2\theta+1\right)}}d\theta=\int\frac{a\sec^2\theta}{a\sec\theta}d\theta=\int\sec\theta\,d\theta$$

$$=\log|\sec\theta+\tan\theta|=\log\left|\frac{x}{a}+\frac{\sqrt{x^2+a^2}}{a}\right|$$

$$\left[\tan\theta=\frac{x}{a}\ then\ \sec\theta=\sqrt{\left(\frac{x}{a}\right)^2+1}=\frac{\sqrt{x^2+a^2}}{a}\right]$$

$$=\log\left|x+\sqrt{x^2+a^2}\right|-\log a+k$$

$= \log\left|x+\sqrt{x^2+a^2}\right|+C \qquad [-\log a + k = C]$

$\therefore \int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left|x+\sqrt{x^2+a^2}\right|+C$

Now,

$I = \int \frac{1}{\sqrt{x^2+7}}dx = \int \frac{1}{\sqrt{x^2+\left(\sqrt{7}\right)^2}}dx = \log\left|x+\sqrt{x^2+7}\right|+C$

4. Find the integral of $\frac{1}{\sqrt{x^2-a^2}}$ w.r.to x & hence evaluate **(a)** $\int \frac{1}{\sqrt{x^2-25}}dx$

(b) $\int \frac{1}{\sqrt{x^2+6x-7}}dx$ **(2015-M)**

Sol:

$I = \int \frac{1}{\sqrt{x^2-a^2}}dx \qquad Put\ x = a\sec\theta,\ dx = a\sec\theta\tan\theta d\theta$

$\therefore I = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}d\theta$

$= \int \frac{a\sec\theta\tan\theta}{a\tan\theta}d\theta = \int \sec\theta\, d\theta$

$= \log|\sec\theta + \tan\theta| + C_1$

$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right| + C_1$

$\left(\because x = a\sec\theta, \sec\theta = \frac{x}{a},\ \tan\theta = \sqrt{\frac{x^2}{a^2}-1}\right)$

$= \log\left|x+\sqrt{x^2-a^2}\right| + C_1 - \log|a|$

$= \log\left|x+\sqrt{x^2-a^2}\right| + C$

$\therefore \int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right|+C$

(a) $I = \int \frac{1}{\sqrt{x^2-25}}dx = \int \frac{1}{\sqrt{x^2-5^2}}dx = \log\left|x+\sqrt{x^2-25}\right|+C$

(b) $I = \int \frac{1}{\sqrt{x^2+6x-7}}dx = \int \frac{1}{\sqrt{x^2+6x+3^2-3^2-7}}dx$

$= \int \frac{1}{\sqrt{(x+3)^2-4^2}}dx = \log\left|x+3+\sqrt{x^2+6x-7}\right|+C$

5. Find the integral of $\frac{1}{\sqrt{a^2-x^2}}$ w.r.to x & hence evaluate $\int \frac{1}{\sqrt{7-6x-x^2}}dx$ **(Aug-21)**

Sol: $I = \int \frac{1}{\sqrt{a^2-x^2}}dx$

$Put\ x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta \quad \left[\theta = \sin^{-1}\left(\frac{x}{a}\right)\right]$

$= \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}d\theta = \int \frac{a\cos\theta}{\sqrt{a^2\left(1-\sin^2\theta\right)}}d\theta$

$== \int \frac{a\cos\theta}{a\cos\theta}d\theta = \int 1 d\theta = \theta + C = \sin^{-1}\left(\frac{x}{a}\right)+C$

$\therefore \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+C$

$I = \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{-\left(x^2+6x-7\right)}}dx$

$= \int \frac{1}{\sqrt{-\left(x^2+2.3.x-7+3^2-3^2\right)}}dx$

$= \int \frac{1}{\sqrt{-\left(x^2+2.3.x+3^2-16\right)}}dx = \int \frac{1}{\sqrt{-\left((x+3)^2-4^2\right)}}dx$

$= \int \frac{1}{\sqrt{4^2-(x+3)^2}}dx = \sin^{-1}\left(\frac{x+3}{4}\right)+C$

6. Find the integral of $\sqrt{x^2+a^2}$ with respect to x.and hence evaluate

(a) $\int \sqrt{x^2+4x+6}\,dx$ **(2015-J)**

(b) $\int \sqrt{x^2+2x+5}\,dx$ **(2019-J)**

(c) $\int \sqrt{1+x^2}\,dx$ **(2017-J)**

Sol: $I = \int \sqrt{x^2+a^2}\,dx ----------(1)$

$$I=\int\sqrt{x^2+a^2}\,.1dx$$

$$=\sqrt{x^2+a^2}\int 1dx-\int\left[\int 1\frac{d}{dx}\sqrt{x^2+a^2}\right]dx$$

$$=\sqrt{x^2+a^2}.x-\int x\times\frac{2x}{2\sqrt{x^2+a^2}}dx$$

$$=\sqrt{x^2+a^2}.x-\int\frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$=\sqrt{x^2+a^2}.x-\int\frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

$$=\sqrt{x^2+a^2}.x-\int\frac{x^2+a^2}{\sqrt{x^2+a^2}}dx+\int\frac{a^2}{\sqrt{x^2+a^2}}dx$$

$$=\sqrt{x^2+a^2}.x-\int\sqrt{x^2+a^2}\,dx+a^2\int\frac{1}{\sqrt{x^2+a^2}}dx$$

$$I=x.\sqrt{x^2+a^2}-I+a^2\log\left|x+\sqrt{x^2+a^2}\right|$$

$$2I=x\sqrt{x^2+a^2}+a^2\log\left|x+\sqrt{x^2+a^2}\right|$$

$$I=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

$$\therefore\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

(a) $\int\sqrt{x^2+4x+6}\,dx=\int\sqrt{x^2+2.2.x+2^2+2}\,dx$

$$=\int\sqrt{(x+2)^2+\left(\sqrt{2}\right)^2}\,dx$$

$$=\frac{x+2}{2}\sqrt{x^2+4x+6}+\frac{\left(\sqrt{2}\right)^2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$

$$=\frac{x+2}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$

(b) $\int\sqrt{x^2+2x+5}\,dx=\int\sqrt{x^2+2.1.x+1^2+4}\,dx$

$$=\int\sqrt{(x+1)^2+2^2}\,dx$$

$$=\frac{(x+1)}{2}\sqrt{x^2+2x+5}+2\log\left|(x+1)+\sqrt{x^2+2x+5}\right|+C$$

(c) $\int\sqrt{1+x^2}\,dx$

$$=\int\sqrt{x^2+1}\,dx=\frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\log\left|x+\sqrt{x^2+1}\right|+C$$

7. Find the integral of $\sqrt{a^2-x^2}$ with respect to x.and hence evaluate

(a) $\int\sqrt{5-x^2+2x}\,dx$ **(2016-J)**

(b) $\sqrt{1-4x-x^2}\;dx$

Sol: $I=\int\sqrt{a^2-x^2}\,dx----(1)$

$$I=\int\sqrt{a^2-x^2}\,.1dx=\sqrt{a^2-x^2}\int 1dx-\int\left[\int 1\times\frac{d}{dx}\sqrt{a^2-x^2}\right]dx$$

$$=\sqrt{a^2-x^2}.x-\int x\times\frac{-2x}{2\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{-x^2}{\sqrt{a^2-x^2}}dx=x\sqrt{a^2-x^2}-\int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx+\int\frac{a^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{1}{\sqrt{a^2-x^2}}dx$$

$$I=x\sqrt{a^2-x^2}-I+a^2\sin^{-1}\left(\frac{x}{a}\right)+C$$

$$2I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)+C$$

$$I=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$$

$$\therefore\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$$

(a)

$$I=\int\sqrt{5-x^2+2x}\,dx=\int\sqrt{5-\left(x^2-2x+1-1\right)}\,dx$$

$$=\int\sqrt{5-(x-1)^2+1}\,dx=\int\sqrt{6-(x-1)^2}\,dx$$

$$=\int\sqrt{\left(\sqrt{6}\right)^2-(x-1)^2}\,dx=\frac{(x-1)}{2}\sqrt{5-x^2+2x}+\frac{\left(\sqrt{6}\right)^2}{2}\sin^{-1}\left(\frac{x-1}{\sqrt{6}}\right)+C$$

$$=\frac{(x-1)}{2}\sqrt{5-x^2+2x}+3\sin^{-1}\left(\frac{x-1}{\sqrt{6}}\right)+C$$

(b $I=\int\sqrt{1-4x-x^2}\;dx=\int\sqrt{1+4-\left(x^2+4x+4\right)}\,dx$

$$= \int \sqrt{5-\left(x^2+4x+4\right)}\, dx = \int \sqrt{5-\left(x+2\right)^2}\, dx$$

$$= \frac{(x+2)}{2}\sqrt{1-4x-x^2} + \frac{\left(\sqrt{5}\right)^2}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$$

$$= \frac{(x+2)}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$$

8. Find the integral of $\sqrt{x^2-a^2}$ w.r.to x & hence evaluate $\int \sqrt{x^2-8x+7}\, dx$ **(2017-M)**

Sol: $I = \int \sqrt{x^2-a^2}\, dx ------(1)$

$$= \int \sqrt{x^2-a^2}.1dx = \sqrt{x^2-a^2}\int 1dx - \int\left[\int 1 \frac{d}{dx}\sqrt{x^2-a^2}\right]dx$$

$$= \sqrt{x^2-a^2}.x - \int x.\frac{2x}{2\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2}{\sqrt{x^2-a^2}}dx - \int \frac{a^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\, dx - a^2\int \frac{1}{\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - I - a^2\log\left|x+\sqrt{x^2-a^2}\right|$$

$$2I = x\sqrt{x^2-a^2} - a^2\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\therefore \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$I = \int \sqrt{x^2-8x+7}\, dx = \int \sqrt{x^2-8x+4^2-4^2+7}\, dx$$

$$= \int \sqrt{(x-4)^2-9}\, dx = \int \sqrt{(x-4)^2-3^2}\, dx$$

$$= \frac{x-4}{2}\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right| + C$$

Question No: 50.a (6M)

1. Prove that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ and hence evaluate

(a) $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{tanx}}dx$ **(14-J, 15-J, 19-J, Aug-21)**

(b) $\int_1^3 \frac{\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}dx$

Sol: Let $Put\ x=a+b-t$

$$x = a+b-t \qquad \Rightarrow dx = -dt$$

$$If\ x=a\ then\ t=b\ \ \&\ x=b\ then\ t=a$$

Now,

$$\int_a^b f(x)dx = -\int_b^a f(a+b-t)dt$$

$$\left(wkt \int_a^b f(x)dx = -\int_b^a f(x)dx \ \& \int_a^b f(x)dx = \int_a^b f(t)dt\right)$$

$$= \int_a^b f(a+b-t)dt = \int_a^b f(a+b-x)dx$$

$$\therefore \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

(a) $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{tanx}}dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx --(1)\left(\frac{\pi}{6}+\frac{\pi}{3}=\frac{\pi}{2}\right)$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx ----(2)$$

Adding (1) and (2)

$$I + I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \int_{\pi/6}^{\pi/3} 1dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$$I = \frac{\pi}{12} \qquad \therefore I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \frac{\pi}{12}$$

(b) $I=\int_1^3 \frac{\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}dx------(1)$

$$I=\int_1^3 \frac{\sqrt{4-x}}{\sqrt{4-(4-x)}+\sqrt{4-x}}dx=\int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}dx...(2)$$

Adding (1) and (2)

$$I+I=\int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}dx+\int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}dx$$

$$2I=\int_1^3 \frac{\sqrt{x}+\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}dx=\int_1^3 1dx=[x]_1^3=3-1=2$$

$$2I=2 \Rightarrow I=1 \qquad \therefore \int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}dx=1$$

2. Prove that $\int_0^a f(x)dx=\int_0^a f(a-x)dx$ and

hence evaluate **(a)** $\int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}}dx$ **(2016-J)**

(b) $\int_0^{\frac{\pi}{4}} \log(1+tanx)dx$ **(2014-M)**

(c) $\int_0^{\frac{\pi}{2}} (2\log\sin x-\log\sin 2x)dx$ **(2017-M)**

(d) $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ **(2018-M)**

(e) $\int_0^{\frac{\pi}{2}} \log(\sin x)dx$

(f) $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x+\cos^5 x}dx$ **(2019-M)**

Sol: Let $t=a-x \qquad \Rightarrow dx=-dt$

If $x=a$ *then* $t=0$ & $x=0$ *then* $t=a$

Now, $\int_0^a f(x)dx=-\int_a^0 f(a-t)dt$

$$\left(wkt\int_a^b f(x)dx=-\int_b^a f(x)dx \ \& \int_a^b f(x)dx=\int_a^b f(t)dt\right)$$

$$=\int_0^a f(a-t)dt$$

$$=\int_0^a f(a-x)dx \qquad \therefore \int_0^a f(x)dx=\int_0^a f(a-x)dx$$

(a) $I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{sinx}}{\sqrt{sinx}+\sqrt{cosx}}dx------(1)$

$$I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx----(2)$$

Adding (1) and (2)

$$I+I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx+\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$2I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx=\int_0^{\frac{\pi}{2}} 1dx=[x]_0^{\pi/2}=\frac{\pi}{2}-0=\frac{\pi}{2}$$

$$I=\frac{\pi}{4} \qquad \therefore I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx=\frac{\pi}{4}$$

(b) $I=\int_0^{\frac{\pi}{4}} \log(1+tanx)dx----(1)$

$$I=\int_0^{\frac{\pi}{4}} \log\left(1+tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$\left[WKT \Rightarrow tan\left(\frac{\pi}{4}-x\right)=\frac{1-\tan x}{1+\tan x}\right]$$

$$I=\int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx=\int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx--(2)$$

adding $(1)\&(2)$

$$I+I=\int_0^{\frac{\pi}{4}} \log(1+tanx)dx+\int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$2I=\int_0^{\frac{\pi}{4}} \log\left[(1+tanx)\times\left(\frac{2}{1+\tan x}\right)\right]dx=\int_0^{\frac{\pi}{4}} \log 2\,dx$$

$$2I=\log 2\int_0^{\frac{\pi}{4}} 1dx=\log 2[x]_0^{\pi/4}$$

$$2I=\log 2[\pi/4-0]=\pi/4\log 2$$

$$I=\frac{\pi}{8}\log 2$$

(c) $I = \int_0^{\frac{\pi}{2}} \left(2\log\sin x - \log\sin 2x\right)dx$

$I = \int_0^{\frac{\pi}{2}} \left(\log\sin^2 x - \log 2\sin x\cos x\right)dx$

$\left[\because \log m^n = n\log m \text{ \& } \sin 2x = 2\sin x\cos x\right]$

$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin^2 x}{2\sin x\cos x}\right)dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan x}{2}\right)dx$

$I = \int_0^{\frac{\pi}{2}} \left(\log\tan x - \log 2\right)dx$

$I = \int_0^{\frac{\pi}{2}} \log\tan x\,dx - \int_0^{\frac{\pi}{2}} \log 2\,dx = \int_0^{\frac{\pi}{2}} \log\tan x\,dx - \log 2\left[x\right]_0^{\pi/2}$

$I = \int_0^{\frac{\pi}{2}} \log\tan x\,dx - \frac{\pi}{2}\log 2 ----(1)$

Now let take $I_1 = \int_0^{\frac{\pi}{2}} \log\tan x\,dx -----(2)$

$I_1 = \int_0^{\frac{\pi}{2}} \log\tan\left(\frac{\pi}{2} - x\right)dx = \int_0^{\frac{\pi}{2}} \log\cot x\,dx ----(3)$

Adding $(2) \& (3)$

$I_1 + I_1 = \int_0^{\frac{\pi}{2}} \log\tan x\,dx + \int_0^{\frac{\pi}{2}} \log\cot x\,dx = \int_0^{\frac{\pi}{2}} \log\left(\tan x.\cot x\right)dx$

$2I_1 = \int_0^{\frac{\pi}{2}} \log 1\,dx = 0 \qquad \left[\because \log 1 = 0\right]$

$I_1 = 0$

Therefore equation (1) becomes

$I = -\frac{\pi}{2}\log 2$

(d) $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}dx ----(1)$

$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-(a-x)} + \sqrt{a-x}}dx = \int_0^a \frac{\sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}}dx ...(2)$

Adding (1) and (2)

$I + I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}}dx$

$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}}dx = \int_0^a 1\,dx = \left[x\right]_0^a = a - 0 = a$

$2I = a \Rightarrow I = \frac{a}{2} \qquad \therefore \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}dx = \frac{a}{2}$

(e) $I = \int_0^{\frac{\pi}{2}} \log\left(\sin x\right)dx ----(1)$

$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right)dx = I = \int_0^{\frac{\pi}{2}} \log\cos x\,dx --(2)$

$(1) + (2)$ *gives*

$2I = \int_0^{\frac{\pi}{2}} \left(\log\sin x + \log\cos x\right)dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx$

$2I = \int_0^{\frac{\pi}{2}} \log\left(\sin 2x\right)dx - \int_0^{\frac{\pi}{2}} \log 2\,dx$

$2I = \int_0^{\frac{\pi}{2}} \log\left(\sin 2x\right)dx - \frac{\pi}{2}\log 2 ---(3)$

Consider $\int_0^{\frac{\pi}{2}} \log\left(\sin 2x\right)dx$

Put

$2x = t \Rightarrow dx = \frac{1}{2}dt \qquad$ *when* $x = 0, t = 0$ & $x = \frac{\pi}{2}, t = \pi$

$\therefore \int_0^{\frac{\pi}{2}} \log\left(\sin 2x\right)dx = \int_0^{\pi} \log\sin t\,\frac{1}{2}dt = \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log\sin t\,dt$

$= \int_0^{\frac{\pi}{2}} \log\sin t\,dt = \int_0^{\frac{\pi}{2}} \log\sin x\,dx = I$

$\therefore$ equation (3) reduces to

$2I = I - \frac{\pi}{2}\log 2 \qquad \Rightarrow I = -\frac{\pi}{2}\log 2$

(f) $I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x}dx ----(1)$

$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)}dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx --- (2)$$

$(1)+(2)$ *gives*

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

$$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

3. Prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

and hence evaluate **(a)** $\int_{-1}^{2} |x^3 - x| dx$ **(b)** $\int_{-5}^{5} |x+2| dx$

Sol: Let $\int_a^b f(x)dx = F(b) - F(a) ----- (1)$

$$\int_a^c f(x)dx = F(c) - F(a) ----- (2)$$

$$\int_c^b f(x)dx = F(b) - F(c) ----- (3)$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a) = \int_a^b f(x)dx$$

$$\therefore \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(a) $I = \int_{-1}^{2} |x^3 - x| dx$, Let

$x^3 - x = 0$ *if* $x = -1, 0, +1$

therefore, split the given interval into $[-1,0],[0,1]$ & $[1,2]$

on

$[-1,0], x^3 - x \ge 0 \quad [0,1], x^3 - x \le 0 \quad [1,2], x^3 - x \ge 0$

$$I = \int_{-1}^{2} |x^3 - x| dx = \int_{-1}^{0} (x^3 - x) dx + \int_0^1 -(x^3 - x) dx + \int_1^2 (x^3 - x) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^2$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) + (4-2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{4} + 2 + \frac{1}{4} = \frac{11}{4}$$

(b)

$$I = \int_{-5}^{5} |x+2| dx = \left[\frac{(x+2)|x+2|}{2}\right]_{-5}^{5} \quad \left[\because \int |x| dx = \frac{x|x|}{2}\right]$$

$$= \frac{1}{2}\left[(5+2)|5+2| - (-5+2)|-5+2|\right] = \frac{1}{2}(49+9) = 29$$

4. P. T

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if\ (2a-x) = f(x) \\ 0 & if\ (2a-x) = -f(x) \end{cases}$$ and

hence evaluate **(a)** $\int_0^{2\pi} cos^5 x\ dx$ **(2016-M)**

(b) $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x} dx$

Sol: We have

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx -- (1)$$

Now, if $f(2a-x) = f(x)$ *then* (1) *becomes*

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

And if $f(2a-x) = -f(x)$ *then* (1) *becomes*

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$$

$$\therefore \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if\ (2a-x) = f(x) \\ 0 & if\ (2a-x) = -f(x) \end{cases}$$

(a)

$$I = \int_0^{2\pi} cos^5 x\ dx == 2\int_0^{\pi} cos^5 x\ dx = 2\int_0^{\pi} cos^5(\pi - x)\ dx$$

$$= 2\int_0^{\pi} (-\cos x)^5\ dx = -2\int_0^{\pi} cos^5 x\ dx = 0$$

(b) $I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x} dx$

$$I=\int_0^{\pi}\frac{(\pi-x)}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}dx$$

$$I=\pi\int_0^{\pi}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx-\int_0^{\pi}\frac{x}{a^2\cos^2x+b^2\sin^2x}dx$$

$$I=\pi\int_0^{\pi}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx-I$$

$$2I=\pi\int_0^{\pi}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx$$

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx=2.\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx$$

$$I=\pi\left[\int_0^{\frac{\pi}{4}}\frac{1}{a^2\cos^2x+b^2\sin^2x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx\right]$$

$$I=\pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2x\,dx}{a^2+b^2\tan^2x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\operatorname{cosec}^2x}{a^2\cot^2x+b^2}dx\right]$$

$$I=\pi\left[\int\frac{1}{a^2+b^2t^2}dt-\int_1^0\frac{1}{a^2u^2+b^2}du\right]$$

$(put\ \tan x=t\ \&\ \cot x=u)$

$$I=\frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^1-\frac{\pi}{ab}\left[\tan^{-1}\frac{au}{b}\right]_1^0$$

$$I=\frac{\pi}{ab}\left[\tan^{-1}\frac{b}{a}+\tan^{-1}\frac{a}{b}\right]=\frac{\pi^2}{2ab}$$

5. P.T

$$\int_{-a}^{a}f(x)dx=\begin{cases}2\int_0^a f(x)dx & if\ (x)is\ even\ function\\ 0 & if\ (x)is\ odd\ function\end{cases}$$

hence evaluate **(a)** $\int_{-1}^{1}sin^5xcos^4xdx$ **(15-M, 20-M)**

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7x.\cos^4x\,dx$ **(c)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5x+1\right)dx$ **(17-J)**

Sol: $\int_{-a}^{a}f(x)dx=\int_{-a}^{0}f(x)dx+\int_0^a f(x)dx--(1)$

In the first integral, consider
$t=-x \Rightarrow x=-t$

$dx=-dt \qquad if\ \ x=0\to t=0,\ x=-a\to t=a$

Equation (1) becomes

$$\int_{-a}^{a}f(x)dx=-\int_a^0 f(-t)dt+\int_0^a f(x)dx$$

$$=\int_0^a f(-t)dt+\int_0^a f(x)dx$$

$$\int_{-a}^{a}f(x)dx=\int_0^a f(-x)dx+\int_0^a f(x)dx-----(2)$$

Now, if f is even function i.e., $f(-x)=f(x)$
Then equation (2) becomes

$$\int_{-a}^{a}f(x)dx=\int_0^a f(x)dx+\int_0^a f(x)dx=2\int_0^a f(x)dx$$

Now, if f is odd function i.e.,
$f(-x)=-f(x)$
Then equation (2) becomes

$$\int_{-a}^{a}f(x)dx=-\int_0^a f(x)dx+\int_0^a f(x)dx=0$$

$$\therefore\int_{-a}^{a}f(x)dx=\begin{cases}2\int_0^a f(x)dx & if\ (x)is\ even\ function\\ 0 & if\ (x)is\ odd\ function\end{cases}$$

(a) $I=\int_{-1}^{1}sin^5xcos^4x\,dx$

$f(x)=\sin^5x\cos^4x$ then

$f(-x)=\sin^5(-x)\cos^4(-x)=-\sin^5x\cos^4x=-f(x)$

Therefore, f is odd function.

$$\therefore\int_{-1}^{1}sin^5xcos^4x\,dx=0$$

(b) $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7x.\cos^4x\,dx$

$f(x)=\sin^7x\cos^4x$ then

$f(-x)=\sin^7(-x)\cos^4(-x)=-\sin^7x\cos^4x=-f(x)$

Therefore, f is odd function.

$$\therefore\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7x.\cos^4x\,dx=0$$

(c) $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5x+1\right)dx$

Here x^3, $x\cos x$, $\tan^5x$ are odd functions

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 + x\cos x + \tan^5 x + 1\right) dx$$

$$= 0 + 0 + 0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

* Find $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx$ **(Aug-21, 2M)**

Sol: $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx = \int \frac{e^{\log x^5} - e^{\log x^4}}{e^{\log x^3} - e^{\log x^2}} dx$

$$= \int \frac{x^5 - x^4}{x^3 - x^2} dx = \int \frac{x^4(x-1)}{x^2(x-1)} dx = \int x^2\,dx$$

$$= \frac{x^3}{3} + C$$

* Evaluate $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx.$ **(Aug-21, 3M)**

Sol: $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx.$

put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

when $x = 0 \Rightarrow t = 0$ & $x = 1 \Rightarrow t = \pi/4$

$$= \int_0^{\frac{\pi}{4}} t\,dt = \left(\frac{t^2}{2}\right)_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}$$

UNIT-8

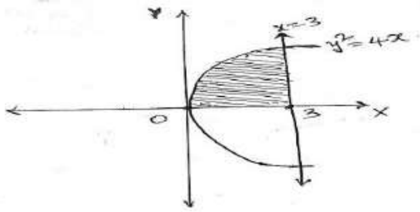
APPLICATIONS OF INTEGRALS

[Total marks : 8, Q.no : 33(3M),45(5M)]

Question No: 33 (3M)

1. Find the area of the region bounded by the curve $y^2 = 4x$ & line $x = 3$.**(15-M, 19-M)**

Sol: Required area



$$A = 2\int_a^b |y|\,dx, \quad y^2 = 4x, \ |y| = 2\sqrt{x}$$

$$= 2\int_0^3 2\sqrt{x}\,dx = 4\int_0^3 \sqrt{x}\,dx$$

$$= 4.\frac{2}{3}\left[x^{3/2}\right]_0^3 = \frac{8}{3}\left[3^{3/2} - 0^{3/2}\right]$$

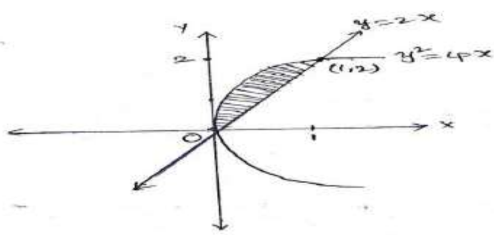
$$= \frac{8}{3}\left[3\sqrt{3} - 0\right] = 8\sqrt{3}\ sq\,units$$

2. Find the area lying between curve $y^2 = 4x$ & line $y = 2x$. **(2015-J, 2016-M)**

Sol: Given $y^2 = 4x$ & $y = 2x$

$(2x)^2 = 4x \quad \Rightarrow 4x^2 = 4x$

then $x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0$ *and* $x = 1$



Area between the curves (A)

$$= \int_a^b |y_2 - y_1|\,dx = \int_0^1 \sqrt{4x}\,dx - \int_0^1 2x\,dx$$

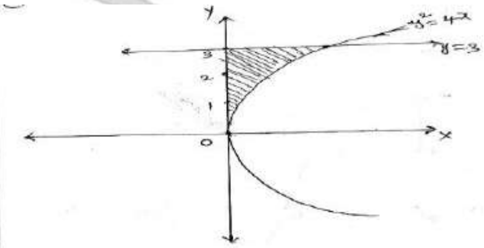
$$= 2\left[\int_0^1 \sqrt{x}\,dx - \int_0^1 x\,dx\right] = 2\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[\left(\frac{2}{3} - \frac{1}{2}\right) - 0\right] = \frac{1}{3}\,sq\,units$$

3. Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$. **(2017-M)**

Sol: Required area

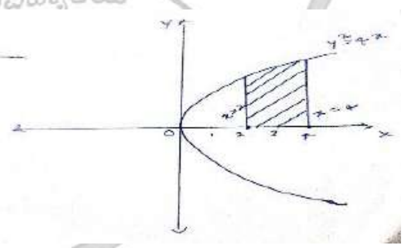
$$= \int_a^b |x|\,dy = \int_0^3 \frac{y^2}{4}\,dy = \frac{1}{4}\int_0^3 y^2\,dy$$



$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3 = \frac{1}{12}\left[3^3 - 0\right] = \frac{27}{12} = \frac{9}{4}\,sq.units$$

4. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ *and* $x - axis$ in the first quadrant. **(2016, 2019-J)**

Sol: Required area



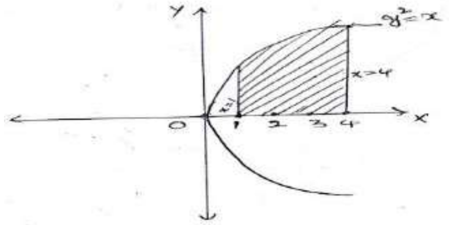
$$A = \int_a^b |y|\,dx = \int_2^4 3\sqrt{x}\,dx \quad (y^2 = 9x,\ |y| = 3\sqrt{x})$$

$$= 3\int_2^4 \sqrt{x}\,dx = 3\left[\frac{2}{3}x^{\frac{3}{2}}\right]_2^4 = 3\times\frac{2}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$

$$= 2\left(8 - 2\sqrt{2}\right) = 16 - 4\sqrt{2}\ sq.units$$

5. Determine the area of the region bounded by $y^2 = x$ and lines $x = 1$ and $x = 4$ and the $x - axis$ in the first quadrant. **(2014-J)**

Sol: Required area



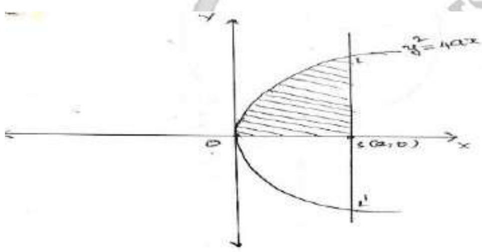
$A=\int_a^b |y|\,dx=\int_1^4 \sqrt{x}\,dx \quad (y^2=x,\ |y|=\sqrt{x}\,)$

$=\frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^4=\frac{2}{3}\left[4^{\frac{3}{2}}-1^{\frac{3}{2}}\right]=\frac{2}{3}(8-1)=\frac{14}{3}\,sq.units$

6. Find the area of the parabola $y^2=4ax$ bounded by its latus rectum. **(2018-J)**

Sol: Equation of the latus rectum is $x=a$

The required area of the parabola $y^2=4ax$ bounded by $x=a$



$A=2\int_0^a y\,dx=2\int_0^a 2\sqrt{ax}\,dx$

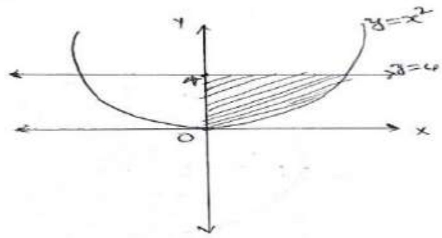
$=4\sqrt{a}\int_0^a \sqrt{x}\,dx=4\sqrt{a}\left(\frac{2}{3}x^{\frac{3}{2}}\right)_0^a$

$=4\sqrt{a}\times\frac{2}{3}\left(a^{\frac{3}{2}}-0\right)=\frac{8}{3}\sqrt{a}\left(a\sqrt{a}-0\right)$

$=\frac{8}{3}a^2\ sq.units$

7. Find the area of the region bounded by the curve $y=x^2$ and the line $y=4$ **(2014-M, 2018-M)**

Sol: The required area of the region bounded by the curve $y=x^2$ and line $y=4$

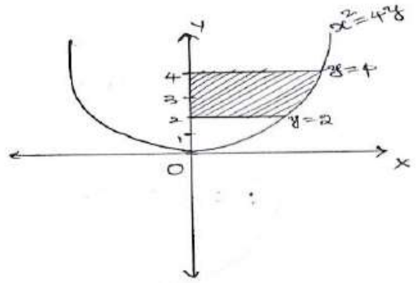


$A=2\int_0^4 x\,dy=2\int_0^4 \sqrt{y}\,dy=2\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^4$

$=\frac{4}{3}\left[4^{\frac{3}{2}}-0\right]=\frac{32}{3}\,sq.units$

8. Find the area of the region bounded by the curves $x^2=4y, y=2, y=4$ and the y-axis in the first quadrant. **(2020-M)**

Sol: $A=\int_2^4 x\,dy=\int_2^4 2\sqrt{y}\,dy$

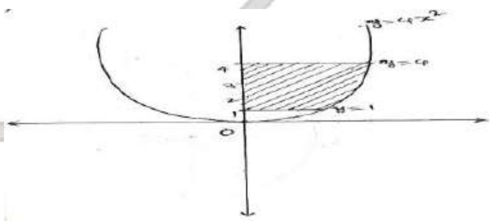


$=2\int_2^4 \sqrt{y}\,dy=2\left[\frac{2}{3}y^{\frac{3}{2}}\right]_2^4=\frac{4}{3}\left(4^{\frac{3}{2}}-2^{\frac{3}{2}}\right)$

$=\frac{4}{3}\left(8-2\sqrt{2}\right)sq.units$

9. Find the area of the region lying in the first quadrant and bounded by $y=4x^2, x=0, y=1$ & $y=4$.

Sol: $A=\int_1^4 x\,dy=\int_1^4 \frac{1}{2}\sqrt{y}\,dy$



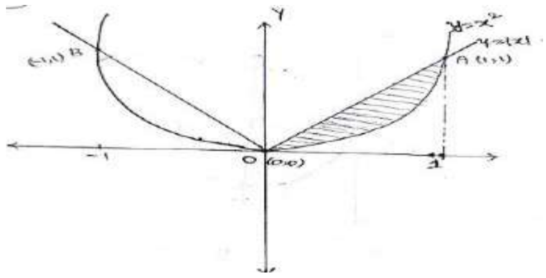
$=\frac{1}{2}\int_1^4 \sqrt{y}\,dy=\frac{1}{2}\left(\frac{2}{3}y^{\frac{3}{2}}\right)_1^4=\frac{1}{3}\left(4^{\frac{3}{2}}-1^{\frac{3}{2}}\right)=\frac{7}{3}\,sq.units$

10. Find the area of the region bounded by the parabola $y=x^2$ & $y=|x|$

Sol: $y=x^2$ & $y=|x|=\pm x$

$\Rightarrow \pm x=x^2 \ \Rightarrow x^2\pm x=0 \qquad \Rightarrow x\left(x\pm1\right)=0$

we get $x=0$ & $x=\pm1$



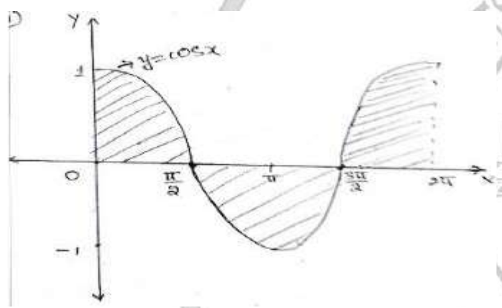
Area between the curve

$$A = 2\int_a^b (y_2 - y_1)\,dx = 2\int (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}\,sq.units$$

11. Find the area bounded by the curve $y = \cos x$ between $x = 0$ & $x = 2\pi$ **(2017-J)**

Sol: $A = \int_a^b |y|\,dx = \int_0^{2\pi} \cos x\,dx$



$$= \int_0^{\frac{\pi}{2}} \cos x\,dx + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx\right| + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\,dx$$

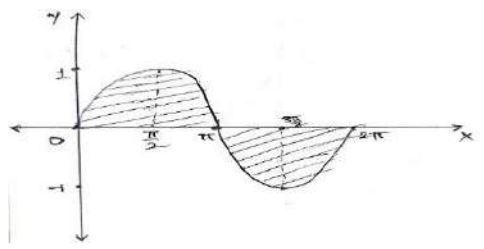
$$= [\sin x]_0^{\pi/2} + \left|[\sin x]_{\pi/2}^{3\pi/2}\right| + [\sin x]_{3\pi/2}^{2\pi}$$

$$= (\sin \pi/2 - \sin 0) + |\sin 3\pi/2 - \sin \pi/2| + (\sin 2\pi - \sin 3\pi/2)$$

$$= (1-0) + |-1-1| + (0-(-1)) = 1+2+1 = 4\,sq.units$$

12. Find The area bounded by the curve $y = \sin x$ between $x = 0$ & $x = 2\pi$

Sol: $A = \int_a^b |y|\,dx = \int_0^{2\pi} \sin x\,dx$



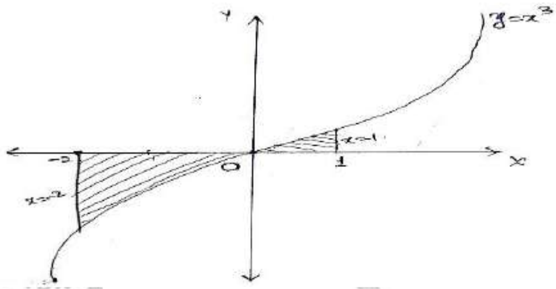
$$= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx = [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi}$$

$$= -[\cos \pi - \cos 0] + [\cos 2\pi - \cos \pi]$$

$$= -[-1-1] + [1+1] = 4\,sq.units$$

13. Find The area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ & $x = 1$

Sol: $A = \int_{-2}^{1} |x^3|\,dx = \int_{-2}^{0} |x^3|\,dx + \int_0^1 x^3\,dx$



$$= \int_{-2}^{0} -x^3\,dx + \int_0^1 x^3\,dx = -\left[\frac{x^4}{4}\right]_{-2}^{0} + \left[\frac{x^4}{4}\right]_0^1$$

$$= -\left(0 - \frac{(-2)^4}{4}\right) + \left(\frac{1}{4} - 0\right) = 4 + \frac{1}{4} = \frac{17}{4}\,sq.units$$

Question No: 45 (5M)

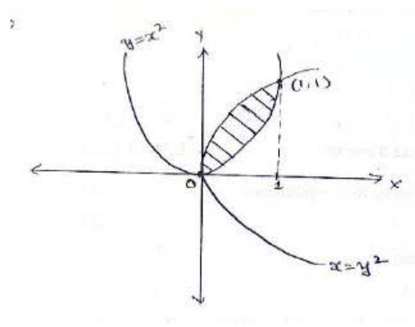
1. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$ **(2014-M)**

Sol: $y = x^2 ------(1)$ $\qquad y^2 = x ---(2)$

Solving (1) and (2)

$$(x^2)^2 = x \quad \Rightarrow x^4 - x = 0 \quad \Rightarrow x(x^3 - 1) = 0$$

We get $x = 0, 1$



Required area

$$A=\int_{0}^{1}(y_2-y_1)\,dx=\int_0^1\left(\sqrt{x}-x^2\right)dx$$

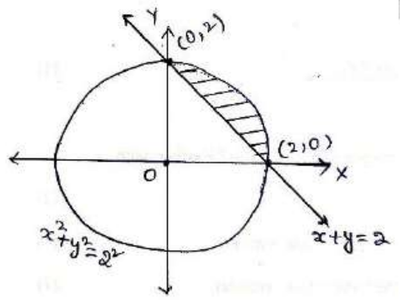
$$=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{x^3}{3}\right]_0^1=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}\,sq.units$$

2. Find the smaller area enclosed by the circle $x^2+y^2=4$ and the line $x+y=2$

(2017-J, 2019-M)

Sol: From the figure required area is

$$A=\int_a^b(y_2-y_1)\,dx$$



$$A=\int_0^2\left[\sqrt{4-x^2}-(2-x)\right]dx$$

$$=\int_0^2\left[\sqrt{4-x^2}-2+x\right]dx$$

$$=\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}-2x+\frac{x^2}{2}\right]_0^2$$

$$=0+2\sin^{-1}1-4+2=2\frac{\pi}{2}-2=\pi-2\,sq.units$$

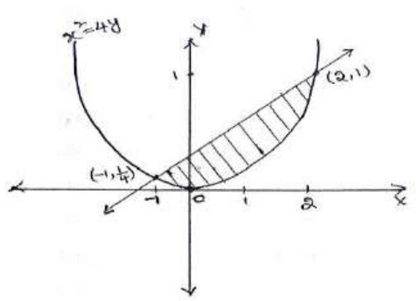
3. Find the area bounded by the curves $x^2=4y$ & the line $x=4y-2$

Sol: Given equations are

$x^2=4y-----(1)$ & $x=4y-2----(2)$

Solving (1) and (2) we get

$x^2=x+2 \quad \Rightarrow x=2,-1$



From the figure required area is

$$A=\int_a^b(y_2-y_1)\,dx=\int_{-1}^{2}\left(\frac{x+2}{4}-\frac{x^2}{4}\right)dx$$

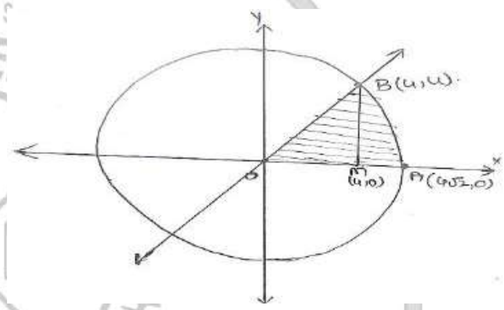
$$=\frac{1}{4}\int_{-1}^{2}\left[x+2-x^2\right]dx=\frac{1}{4}\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)_{-1}^{2}$$

$$=\frac{1}{4}\left[\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)\right]=\frac{1}{4}\left(5-\frac{1}{2}\right)$$

$$=\frac{9}{8}\,sq.units$$

4. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y=x$ & the circle $x^2+y^2=32$ **(2019-J)**

Sol: Equation of circle



$x^2+y^2=32-----(1)$

Equation of line $y=x------(2)$

On solving (1) and (2) we get $x=\pm 4, y=\pm 4$

$\therefore B(4,4)$ is the point of intersection

Draw BM $\perp^r$ to x-axis.

Required area

$=Area\ of\ OMB+Area\ of\ MBAM$

$$=\int_0^4 x\,dx+\int_4^{4\sqrt{2}}\sqrt{32-x^2}\,dx$$

$$=\left(\frac{x^2}{2}\right)_0^4+\left(\frac{x}{2}\sqrt{32-x^2}+\frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right)_4^{4\sqrt{2}}$$

$$=(8-0)+\left(0+16\sin^{-1}(1)-\frac{4}{2}\sqrt{16}-16\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$=8+8\pi-8-16\left(\frac{\pi}{4}\right)=4\pi\,sq.units$$

5. Using integration, find the area enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.

Sol: The given equations are

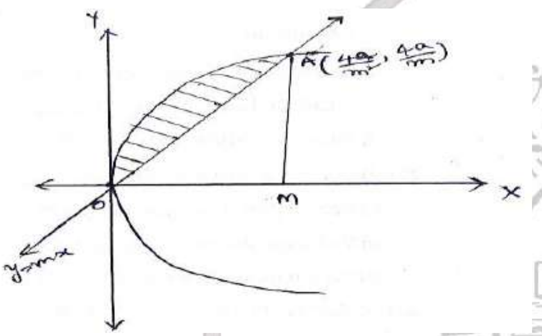
$y^2 = 4ax ----(1)$ & $y = mx ----(2)$

Solving (1) and (2) we get

$x = 0,\ x = \frac{4a}{m^2}$ $\quad \therefore y = 0,\ y = \frac{4a}{m}$

$\therefore$ Point of intersection are

$O(0,0)$ & $A\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$



Required area

$$A = \int_0^{4a/m^2} 2\sqrt{a}\sqrt{x}\,dx - \int_0^{4a/m^2} mx\,dx$$

$$= \left[2\sqrt{a}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - m\frac{x^2}{2}\right]_0^{\frac{4a}{m^2}} = \frac{4\sqrt{a}}{3}\frac{8a\sqrt{a}}{m^3} - \frac{m}{2}\left(\frac{16a^2}{m^4}\right)$$

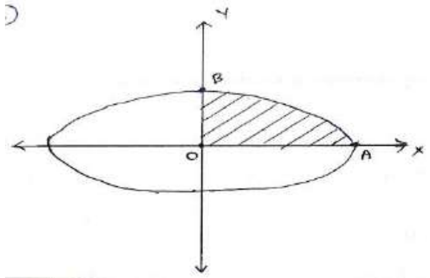
$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{m^3}\ sq.units$$

6. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by method of integration & hence find the area of the ellipse

(i) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ **(2016-J, 2018-J)**

(ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Sol:



Required area $= 4(area\ of\ AOB)$

$$= \int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left(0 + \frac{a^2}{2}.\frac{\pi}{2} - 0 - 0\right) = \pi ab\ sq.units$$

(i) $\frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow \frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$

Comparing with standard equation of ellipse we get $a = 4, b = 3$

Required area

$$A = 4\int_0^a y\,dx = 4\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,dx$$

$$= 3\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= 3\left(0 + 8\sin^{-1}(1)\right) = 3\left(8.\frac{\pi}{2}\right) = 12\pi\ sq.units$$

(ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$

Comparing with standard equation of ellipse we get $a = 2, b = 3$

Required area $A = 4\int_0^a y\,dx = 4\int_0^2 \frac{3}{2}\sqrt{2 - x^2}\,dx$

$$= 6\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 6\left(0 + 2\sin^{-1}(1)\right) = 6\left(2.\frac{\pi}{2}\right) = 6\pi\ sq.units$$

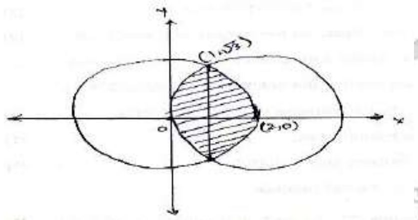
7. Find the area of the region enclosed between the two circles: $x^2+y^2=4$ and $(x-2)^2+y^2=4$. **(2014-J)**

Sol: Given circles are $x^2+y^2=4---(1)$

and $(x-2)^2+y^2=4--(2)$

Solve (1) and (2) we get

$x^2=(x-2)^2 \quad \Rightarrow x=1$



Required area

$$A=2\left[\int_0^1\sqrt{4-(x-2)^2}\,dx+\int_1^2\sqrt{4-x^2}\,dx\right]$$

$$=2\left[\begin{array}{l}\left\{\dfrac{x-2}{2}\sqrt{4-(x-2)^2}+\dfrac{4}{2}\sin^{-1}\dfrac{x-2}{2}\right\}_0^1\\+\left\{\dfrac{x}{2}\sqrt{4-x^2}+\dfrac{4}{2}\sin^{-1}\dfrac{x}{2}\right\}_1^2\end{array}\right]$$

$$=2\left[\begin{array}{l}\left\{-\dfrac{1}{2}\sqrt{3}+2\left(-\dfrac{\pi}{6}\right)-0-2\left(-\dfrac{\pi}{2}\right)\right\}\\+\left\{0+2\left(\dfrac{\pi}{2}\right)-\dfrac{1}{2}\sqrt{3}-2\left(\dfrac{\pi}{6}\right)\right\}\end{array}\right]$$

$$=-\sqrt{3}-\frac{2\pi}{3}+2\pi+2\pi-\sqrt{3}-\frac{2\pi}{3}$$

$$=\frac{8\pi}{3}-2\sqrt{3}\ sq.units$$

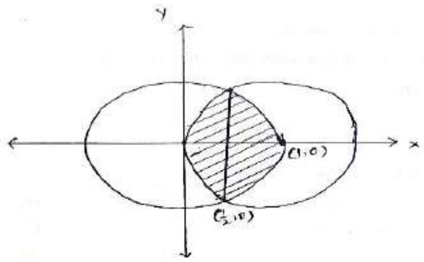
8. Find the area bounded by curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$ using integrating method. **(2015-J)**

Sol: Given circles are $x^2+y^2=1---(1)$

and $(x-1)^2+y^2=1--(2)$

Solve (1) and (2) we get

$x^2=(x-1)^2 \quad \Rightarrow x=\dfrac{1}{2}$



Required area

$$(A)=2\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+\int_{\frac{1}{2}}^1\sqrt{1-x^2}\,dx\right]$$

$$=2\left[\begin{array}{l}\left\{\dfrac{x-1}{2}\sqrt{1-(x-1)^2}+\dfrac{1}{2}\sin^{-1}(x-1)\right\}_0^{\frac{1}{2}}\\+\left\{\dfrac{x}{2}\sqrt{1-x^2}+\dfrac{1}{2}\sin^{-1}x\right\}_{\frac{1}{2}}^1\end{array}\right]$$

$$=2\left[\begin{array}{l}\left\{-\dfrac{1}{4}\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\left(\dfrac{\pi}{6}\right)+0+\dfrac{1}{2}\left(\dfrac{\pi}{2}\right)\right\}\\+\left\{\dfrac{1}{2}\left(\dfrac{\pi}{2}\right)-\dfrac{1}{4}\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\left(\dfrac{\pi}{6}\right)\right\}\end{array}\right]$$

$$=-\frac{\sqrt{3}}{2}-\frac{\pi}{6}+\frac{\pi}{2}+\frac{\pi}{2}-\frac{\sqrt{3}}{4}-\frac{\pi}{6}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\ sq.units$$

9. Find the area of the region of circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$

Sol: Given circle is $4x^2+4y^2=9--(1)$ and parabola $x^2=4y----(2)$

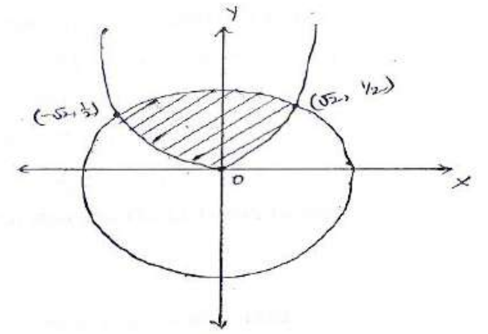
Solve (1) and (2) we get $4(4y)+4y^2=9$

$4y^2+16y-9=0 \Rightarrow 4y^2+18y-2y-9=0$

$\Rightarrow y=\dfrac{1}{2},-\dfrac{9}{2}$

(But y is non negative)

$\therefore x^2=4y \Rightarrow x^2=4\left(\dfrac{1}{2}\right)=2 \quad \Rightarrow x=\pm\sqrt{2}$



Required area $(A)=2\int_0^{\sqrt{2}}(y_2-y_1)\,dx$

$$=2\left\{\int_0^{\sqrt{2}}\sqrt{\frac{9-4x^2}{4}}\,dx-\int_0^{\sqrt{2}}\frac{x^2}{4}\,dx\right\}$$

$$=2\left\{\int_0^{\sqrt{2}}\sqrt{\frac{9}{4}-x^2}\,dx-\frac{1}{4}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}\right\}$$

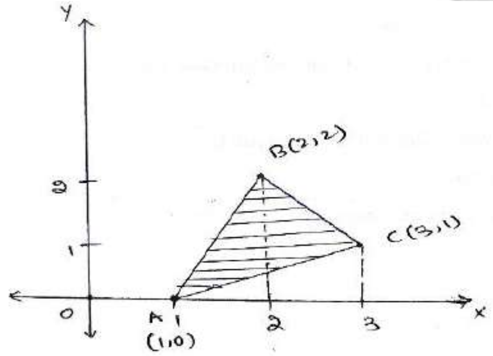
$$=2\left[\left\{\frac{x}{2}\sqrt{\frac{9}{4}-x^2}+\frac{9/4}{2}\sin^{-1}\frac{x}{3/2}\right\}_0^{\sqrt{2}}-\frac{1}{12}\left(\left(\sqrt{2}\right)^3-0\right)\right]$$

$$=2\left\{\frac{\sqrt{2}}{2}\sqrt{\frac{9}{4}-2}+\frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}-0-\frac{\sqrt{2}}{6}\right\}$$

$$=\frac{\sqrt{2}}{2}+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)-\frac{\sqrt{2}}{3}=\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)sq.units$$

10. Using integration, find the area of region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3, 1). **(2016-M, 2018-M)**

Sol: Let $A(1,0), B(2,2)\, and\, C(3,1)$ be the vertices of a triangle



Equation of AB is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$\Rightarrow y-0=\frac{2-0}{2-1}(x-1)\Rightarrow y=2x-2$$

Equation of AC is

$$y=\frac{1-0}{3-1}(x-1)\Rightarrow y=\frac{1}{2}(x-1)$$

Equation of BC is

$$y-2=\frac{1-2}{3-2}(x-2)\Rightarrow y=x-4$$

Required area is

$$A=\int_1^2(2x-2)\,dx+\int_2^3(4-x)\,dx-\int_1^3\frac{1}{2}(x-1)\,dx$$

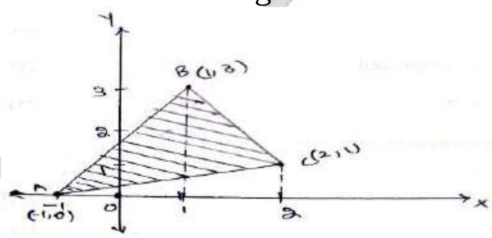
$$=\left[x^2-2x\right]_1^2+\left[-\frac{x^2}{2}+4x\right]_2^3-\frac{1}{2}\left[\frac{x^2}{2}-x\right]_1^3$$

$$=\left[4-4-1+2\right]+\left[-\frac{9}{2}+12+2-8\right]-\frac{1}{2}\left[\frac{9}{2}-3-\frac{1}{2}+1\right]$$

$$=\left[1-\frac{9}{2}+6-\frac{9}{4}-2+\frac{1}{4}\right]=\frac{3}{2}sq.unit$$

11. Using integration, find the area of region bounded by the triangle whose vertices are (-1, 0), (1, 3) and (3, 2).

Sol: Let $A(-1,0), B(1,3)\, and\, C(3,2)$ be the vertices of the triangle



Eqn of AB is

$$y-0=\frac{3-0}{1+1}(x+1)\Rightarrow y=\frac{3}{2}(x+1)---(1)$$

Eqn of AC is

$$y-0=\frac{2-0}{3+1}(x+1)\Rightarrow y=\frac{1}{2}(x+1)---(2)$$

Eqn of BC is

$$y-3=\frac{2-3}{3-1}(x-1)\Rightarrow y=-\frac{1}{2}x+\frac{7}{2}---(3)$$

Required area is

$$A=\int_{-1}^{1}\frac{3}{2}(x+1)\,dx+\int_{1}^{3}\left(-\frac{1}{2}x+\frac{7}{2}\right)dx-\int_{-1}^{3}\frac{1}{2}(x+1)\,dx$$

$$=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}+\left[-\frac{x^2}{4}+\frac{7}{2}x\right]_{1}^{3}-\left[\frac{1}{2}\left(\frac{x^2}{2}+x\right)\right]_{-1}^{3}$$

$$=\frac{3}{2}\left(\frac{1}{2}+1-\frac{1}{2}+1\right)+\left(-\frac{9}{4}+\frac{21}{2}+\frac{1}{4}-\frac{7}{2}\right)-\frac{1}{2}\left(\frac{9}{2}+3-\frac{1}{2}+1\right)$$

$$=3+5-4=4\,sq.unit$$

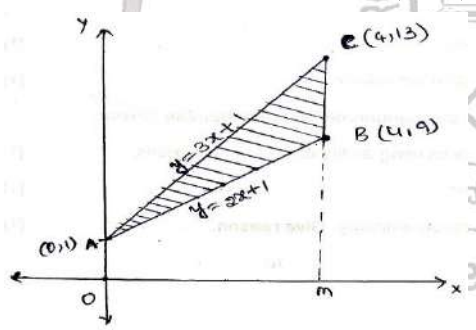
12. Using integration find the area of the triangular region whose sides have the equations $y=2x+1,\ y=3x+1\ \&\ x=4$ **(2015-M, 2017-M)**

Sol: The given lines are $y=2x+1----(1)$, $y=3x+1---(2)$ and $x=4--(3)$

Solving (1) and (2) we get $A(1,0)$

Solving (1) and (3) we get $B(4,9)$

Solving (2) and (3) we get $C(4,13)$



From the figure, required area is

$$A=\int_{0}^{4}\left[(3x+1)-(2x+1)\right]dx$$

$$=\int_{0}^{4}x\,dx=\left(\frac{x^2}{2}\right)_{0}^{4}=8\,sq.units$$

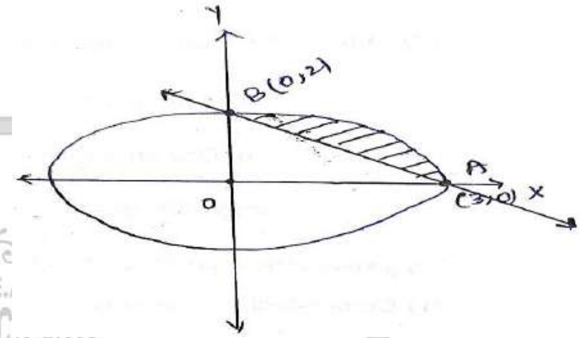
13. Using the method of integration, find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$ **(2020-M).**

Sol: The given ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1---(1)$

and the line is $\frac{x}{3}+\frac{y}{2}=1---(2)$

$$\therefore y=\frac{2}{3}\left(\sqrt{9-x^2}\right)\ \ and\ \ y=\frac{2}{3}(3-x)$$

The required area is as shown in figure



$$A=\int_{0}^{3}\frac{2}{3}\left(\sqrt{9-x^2}\right)dx-\int_{0}^{3}\frac{2}{3}(3-x)\,dx$$

$$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_{0}^{3}-\frac{2}{3}\left[3x-\frac{x^2}{2}\right]_{0}^{3}$$

$$=\frac{2}{3}\left[0+\frac{9}{2}\left(\sin^{-1}1\right)-0-0\right]-\frac{2}{3}\left[9-\frac{9}{2}-0+0\right]$$

$$=\frac{2}{3}\left[\frac{9}{2}\left(\frac{\pi}{2}\right)\right]-\frac{2}{3}\left(\frac{9}{2}\right)=\frac{3\pi}{2}-3\ sq.units$$

Unit- 9
DIFFERENTIAL EQUATIONS

$\left[Total\ marks:10,\ Q.No-20(2M)\ 34(3M)\ \&\ 46(5M)\right]$

Question No: 20 (2M)

Find the order and degree (if defined) of the differential equation

1. $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2-y\dfrac{dy}{dx}=0$ **(2014-M)**

Sol: Order: 2, Degree: 1

2. $\left(\dfrac{d^3y}{dx^3}\right)^2+\left(\dfrac{d^2y}{dx^2}\right)^3+\dfrac{dy}{dx}+y=0$ **(2014-J)**

Sol: Order: 3, Degree: 2

3. $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$ **(2016-J)**

Sol: Order: 3, Degree: 1

4. $\left(\dfrac{dy}{dx}\right)^2+\dfrac{dy}{dx}-\sin^2 y=0$ **(2017-M)**

Sol: Order: 1, Degree: 1

5. $\left(\dfrac{d^3y}{dx^3}\right)^2+\left(\dfrac{d^2y}{dx^2}\right)^3+\left(\dfrac{dy}{dx}\right)^4+y^5=0$ **(17-M, 2018-M)**

Sol: Order: 3, Degree: 2

6. $\left(\dfrac{dy}{dx}\right)^3-\left(\dfrac{dy}{dx}\right)^2-y=\sin x$

Sol: Order: 1, Degree: 3

7. $\left(\dfrac{d^3y}{dx^3}\right)+2\left(\dfrac{d^2y}{dx^2}\right)^2-\dfrac{dy}{dx}+y=0$

Sol: Order: 3, Degree: 1

8. $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$ **(2018-J)**

Sol: Order: 3, Degree: 1

9. $\dfrac{dy}{dx}-\cos x=0$

Sol: Order: 1, Degree: 1

10. $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2-y\dfrac{dy}{dx}=0$

Sol: Order: 2, Degree: 1

11. $y'''+y^2+e^y=0$

Sol: Order: 3, Degree: 1

12. $\left(\dfrac{ds}{dt}\right)^4+3s\left(\dfrac{d^2s}{dt^2}\right)=0$

Sol: Order: 2, Degree: 1

13. $\dfrac{d^2y}{dx^2}=\cos 3x+\sin 3x$

Sol: Order: 2, Degree: 1

14. $(y''')^2+(y'')^3+(y')^4+y^5=0$

Sol: Order: 3, Degree: 2

15. $2x^2\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+y=0$

Sol: Order: 2, Degree: 1

16. $\left(\dfrac{d^2y}{dx^2}\right)^3+\left(\dfrac{dy}{dx}\right)^2+\sin\left(\dfrac{dy}{dx}\right)+1=0$ **(15-M, 20-M)**

Sol: Order: 2, Degree: **Not defined**

17. $\left(\dfrac{d^2y}{dx^2}\right)^2+\cos\left(\dfrac{dy}{dx}\right)=0$ **(2015-J)**

Sol: Order: 2, Degree: **Not defined**

18. $\dfrac{d^4y}{dx^4}+\sin\left(\dfrac{d^3y}{dx^3}\right)=0$ **(16-M, 2017-J)**

Sol: Order: 4, Degree: **Not defined**

19. $\left(\dfrac{d^2y}{dx^2}\right)^2+\cos\left(\dfrac{dy}{dx}\right)=0$ **(2019-M)**

Sol: Order: 2, Degree: **Not defined**

20. $\dfrac{d^4y}{dx^4}-\sin\left(\dfrac{d^3y}{dx^3}\right)=0$

Sol: Order: 4, Degree: **Not defined**

21. $y''' + y^2 + e^{y'} = 0$ **(2019-J)**

Sol: Order:3, Degree: **Not defined**

Question No: 34 (3M)

1. Form the differential equation representing the family of curves $y = mx$, where m is arbitrary constant. **(2014-M)**

Sol: Given equation $y = mx---(1)$

Differentiating w. r. to. x we get

$\frac{dy}{dx} = m---(2)$

From (1) and (2) eliminating arbitrary constant we get $y = \frac{dy}{dx}x$

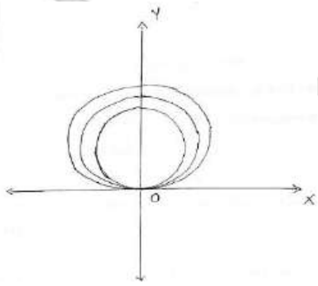
$i.e., x\frac{dy}{dx} - y = 0$ which is the required differential equation.

2. Form the differential equation of the family of circles touching the $x-axis$ at origin. **(2014-J, 2015-M)**

Sol: We know that the equation of a circle with centre $(0,a)$ and radius 'a' is given by $x^2 + (y-a)^2 = a^2$ or $x^2 + y^2 = 2ay$

So the general equation of the family of all circles touching the x-axis at the origin is given by $x^2 + y^2 = 2ay---(1)$

where 'a' is a parameter



Differentiate (1) w.r.to x

$2x + 2yy' = 2ay'$

$2a = \frac{2x + 2yy'}{y'}$

Substituting this in (1), we get

$$x^2 + y^2 = \left(\frac{2x + 2yy'}{y'}\right)y$$

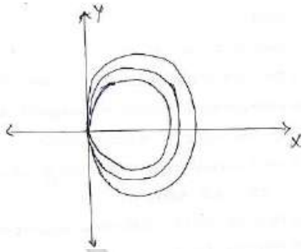
$$(x^2 + y^2)y' = 2xy + 2y^2y' \Rightarrow (x^2 - y^2)y' = 2xy$$

3. Form the differential equation of the family of circles touching the y-axis at the origin.

Sol: We know that the equation of a circle with centre $(a,0)$ and radius 'a' is given by $(x-a)^2 + y^2 = a^2$ or $x^2 + y^2 = 2ax$

So the general equation of the family of all circles touching the y-axis at the origin is given by $x^2 + y^2 = 2ax---(1)$

where 'a' is a parameter



Differentiate (1) w.r.to x

$2x + 2y\frac{dy}{dx} = 2a$

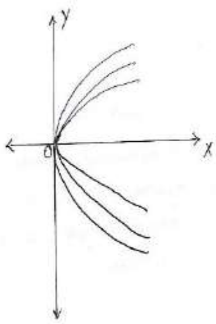
Substituting this in (1), we get

$$x^2 + y^2 = \left(2x + 2y\frac{dy}{dx}\right)x$$

$$x^2 - y^2 + 2xy\frac{dy}{dx} = 0$$

4. Form the differential equation of the family of parabolas having vertex at the origin and axis along positive direction of x-axis.

Sol: we know that family of parabolas having vertex at the origin and axis along positive direction of x-axis is $y^2 = 4ax---(1)$



Differentiating w.r.to.x we get

$2y\frac{dy}{dx} = 4a--(2)$

Eliminating 'a' from (1) and (2), we get

$y^2 = \left(2y\frac{dy}{dx}\right)x \Rightarrow y = 2x\frac{dy}{dx}$ which is required differential equation.

5. Form the differential equation of the family of circles having centre on y-axis and radius 3 units. **(2017-M)**

Sol: we know that, equation of the family of circles having centre on y-axis and radius 3 unit is $x^2+(y-k)^2=3^2 ---(1)$

Differentiating (1) w.r.to. x we get

$2x+2(y-k)\frac{dy}{dx}=0 \qquad \therefore y-k=\frac{-x}{\frac{dy}{dx}}$

Substituting this in equation (1), we get

$x^2+\left(-\frac{x}{y'}\right)=9$

$x^2\left[(y')^2+1\right]=9(y')^2$

this is required differential equation.

6. Form the differential equation representing the family of curves $y = a\sin(x+b)$ where a & b are arbitrary constant. **(2015-J, 2018-M, 2018-J)**

Sol: Given equation $y=a\sin(x+b)---(1)$

Differentiating (1) w. r. to x we get

$\frac{dy}{dx}=a\cos(x+b)$

Again Differentiating w. r. to x we get

$\frac{d^2y}{dx^2}=-a\sin(x+b)=-y$ (from (1))

$\therefore \frac{d^2y}{dx^2}+y=0$ is the required differential equation.

7. Form the differential equation representing family of curves $\frac{x}{a}+\frac{y}{b}=1$ where a & b are arbitrary constants. **(2016-J)**

Sol: Given $\frac{x}{a}+\frac{y}{b}=1$

Differentiating w. r. to x we get,

$\frac{1}{a}+\frac{1}{b}\frac{dy}{dx}=0$

Again, differentiating w.r.to.x we get,

$0+\frac{1}{b}\frac{d^2y}{dx^2}=0$

$\Rightarrow \frac{d^2y}{dx^2}=0$ this is the required differential equation.

8. Form the differential equation of the family of curves $y=ae^{3x}+be^{-2x}$ by eliminating arbitrary constants a & b. **(2019-M, 2019-J)**

Sol: Given $y=ae^{3x}+be^{-2x}---(1)$

Differentiating (1) w. r. to. x, we get

$y'=3ae^{3x}-2be^{-2x}$

$y'=3ae^{3x}-2be^{-2x}+3be^{-2x}-3be^{-2x}$

$=3(ae^{3x}+be^{-2x})-5be^{-2x}$

$y'=3y-5be^{-2x} \qquad (using\ (1))$

Again, differentiate w. r. to. x we get

$y''=3y'+10be^{-2x}$

$y''=3y'+2(3y-y') \qquad (using\ (2))$

$\therefore y''-y'-6y=0$

This is the required differential equation.

9. Form the differential equation of the curve $y^2=a(b^2-x^2)$ where a & b are arbitrary constants.

Sol: Given $y^2=a(b^2-x^2)---(1)$

Differentiating (1) w. r. to. x, we get

$2yy'=-2ax \qquad \Rightarrow \frac{yy'}{x}=-a$

Again, differentiate w. r. to. x we get

$\frac{x(yy''+y'y')-yy'}{x^2}=0$

$\Rightarrow x\left(yy''+(y')^2\right)=yy'$

This is the required differential equation.

10. Find the equation of curves passing through the point(1,1), given that the slope the tangent to the curve at any point is x. **(2016-M)**

Sol: WKT, the slope of the tangent to the curve at any point (x, y) is $\frac{dy}{dx}$.

$$\therefore \frac{dy}{dx} = x$$

By variable separable,

$$dy = xdx$$

$$\Rightarrow \int dy = \int x\,dx + c$$

$$y = \frac{x^2}{2} + c$$

It passes through (1,1)

$$\therefore 1 = \frac{1}{2} + c \Rightarrow c = \frac{1}{2}$$

The required equation of curve is $2y = x^2 + 1$

11. Find the equation of a curve passing through the points (-2, 3) given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$. **(2017-J, 2020-M)**

Sol: WKT, the slope of the tangent to the curve at any point (x, y) is $\frac{dy}{dx}$.

$$\therefore \frac{dy}{dx} = \frac{2x}{y^2}$$

By variable separable,

$$y^2 dy = 2xdx$$

$$\Rightarrow \int y^2 dy = \int 2x\,dx + c$$

$$\frac{y^3}{3} = x^2 + c$$

It passes through $(-2,3)$

$$\therefore 9 = 4 + c \Rightarrow c = 5$$

required equation of curve is $\frac{y^3}{3} = x^2 + 5$

Question No: 46 (5M)

1. Find the general solution of differential equation $e^x tany dx + (1-e^x) sec^2 y dy = 0$ **(2014-M)**

Sol: Given $e^x tany dx + (1-e^x) sec^2 y dy = 0$

By variable separable

$$\frac{e^x}{1-e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

$$\Rightarrow \int \frac{e^x}{1-e^x}dx + \int \frac{\sec^2 y}{\tan y}dy = \log C$$

$$\Rightarrow -\log(1-e^x) + \log\tan y = \log C$$

$$\therefore \tan y = C(1-e^x)$$

Which is required solution

2. Solve: $xlogx\frac{dy}{dx} + y = \frac{2}{x}logx$ **(2014 - J)**

Sol: Given differential equation is

$$\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x^2}$$

$$here\ P = \frac{1}{x\log x}, Q = \frac{2}{x^2}$$

$$I.F = e^{\int Pdx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

$Required\ solution\ is$

$$y.\log x = \int (\log x)\frac{2}{x^2}dx + C$$

$$= 2\left[(\log x)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx\right] + C$$

$$y.\log x = -\frac{2}{x}\log x - \frac{2}{x} + C$$

3. Solve $\frac{dy}{dx} + ysecx = tanx, \quad 0 \le x < \frac{\pi}{2}$ **(2015-M, 2019-M)**

Sol: Given differential equation is

$$\frac{dy}{dx} + (\sec x)y = \tan x$$

$$here\ P = \sec x, Q = \tan x$$

$$I.F = e^{\int Pdx} = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

Required solution is

$$y(I.F)=\int Q(I.F)dx+C$$

$$y.(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx+C$$

$$=\int(\sec x\tan x+\tan^2 x)+C$$

$$=\int(\sec x\tan x+\sec^2 x-1)+C$$

$$y.(\sec x+\tan x)=\sec x+\tan x-x+C$$

4. Find the general solution of the differential equation $\left(x+3y^2\right)\dfrac{dy}{dx}=y,\ (y>0)$ **(2015-J)**

Sol: Given differential equation can be written as

$$\frac{x+3y^2}{y}=\frac{dx}{dy}$$

$$\therefore \frac{dx}{dy}-\frac{1}{y}x=3y$$

$$here\ P=-\frac{1}{y},Q=3y$$

$$I.F=e^{\int Pdy}=e^{\int -\frac{1}{y}dy}=e^{-\log y}=\frac{1}{y}$$

Required solution is

$$x.(I.F)=\int Q(I.F)dy+C$$

$$x.\frac{1}{y}=\int 3y\left(\frac{1}{y}\right)dy+C$$

$$\frac{x}{y}=3y+C$$

$$x=3y^2+Cy$$

$$y.\tan x=\sec x+\tan x-x+C$$

5. Solve the differential equation $ydx+\left(x-ye^y\right)dy=0$ **(2016-M)**

Sol: Given differential equation can be written as

$$\frac{dx}{dy}=\frac{ye^y}{y}-\frac{x}{y}\qquad\Rightarrow\frac{dx}{dy}+\frac{1}{y}x=e^y$$

$$Here\ P=\frac{1}{y},Q=e^y$$

$$I.F=e^{\int Pdy}=e^{\int\frac{1}{y}dy}=e^{\log y}=y$$

$\therefore$ *Required solution is*

$$x.(I.F)=\int Q(I.F)dy+C$$

$$x.y=\int e^y y\,dy+C$$

$$=y\int e^y dy-\int (y'\int e^y dy)dy+C$$

$$xy=ye^y-e^y+c$$

6. Find the general solution of the differential equation $x\dfrac{dy}{dx}-2y=x^2,\ (x\neq 0)$ **(2016-J)**

Sol: Given differential equation can be written as

$$\frac{dy}{dx}-\frac{2}{x}y=x,\ (x\neq 0)$$

$$Here\ P=-\frac{2}{x},Q=x$$

$$I.F=e^{\int pdx}=e^{\int -\frac{2}{x}dx}=e^{-2\log x}=\frac{1}{x^2}$$

Required solution is

$$y\left(\frac{1}{x^2}\right)=\int x\left(\frac{1}{x^2}\right)dx+c$$

$$\frac{y}{x^2}=\int\frac{1}{x}dx+C$$

$$\frac{y}{x^2}=\log x+C$$

$$\therefore y=x^2\log x+Cx^2$$

7. Find the general solution of the differential equation $x\dfrac{dy}{dx}-y=2x^2$

Sol: Given differential equation can be written as

$$\frac{dy}{dx}-\frac{1}{x}y=2x,\ Here\ P=-\frac{1}{x},Q=2x$$

$$I.F=e^{\int Pdx}=e^{\int -\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$$

Required solution is

$$y\left(\frac{1}{x}\right)=\int 2x\left(\frac{1}{x}\right)dx+c$$

$$\frac{y}{x}=\int 2dx+C\Rightarrow\frac{y}{x}=2x+C$$

$$\therefore y=2x^2+Cx$$

8. Solve the differential equation

$\cos^2 x\frac{dy}{dx}+y=\tan x\ ,\ \left(0\le x<\frac{\pi}{2}\right)$ **(2017-M)**

Sol: Given differential equation can be written as

$$\frac{dy}{dx}+\left(\sec^2 x\right)y=\tan x\sec^2 x,$$

$$Here\ P=\sec^2 x, Q=\tan x\sec^2 x$$

$$I.F=e^{\int Pdx}=e^{\int \sec^2 xdx}=e^{\tan x}$$

$$Required\ solution\ is$$

$$y\left(e^{\tan x}\right)=\int \tan x\sec^2 x\left(e^{\tan x}\right)dx+c$$

$$put\ \tan x=t\quad D.w.t.x\ \sec^2 xdx=dt$$

$$ye^t=\int te^t dt+C=te^t-e^t+C$$

$$\therefore ye^{\tan x}=\tan xe^{\tan x}-e^{\tan x}+C$$

9. Find the general solution of the differential equation $ydx-\left(x+2y^2\right)dy=0$ **(2017-J)**

Sol: Given differential equation can be written as

$$\frac{dx}{dy}-\frac{x+2y^2}{y}=0$$

$$\Rightarrow \frac{dx}{dy}-\frac{1}{y}x=2y$$

$$Here\ P=-\frac{1}{y}, Q=2y$$

$$I.F=e^{\int Pdy}=e^{\int -\frac{1}{y}dy}=e^{-\log y}=\frac{1}{y}$$

$$Required\ solution\ is$$

$$x\left(\frac{1}{y}\right)=\int \frac{1}{y}2ydy+c$$

$$\frac{x}{y}=2y+C$$

$$\therefore x=2y^2+Cy$$

10. Find the general solution of the differential equation $x\frac{dy}{dx}+2y=x^2\log x$

(2018-M, 2020-M)

Sol: Given differential equation can be written as

$$\frac{dy}{dx}+\frac{2}{x}y=x\log x,$$

$$Here\ P=\frac{2}{x}, Q=x\log x$$

$$I.F=e^{\int Pdx}=e^{\int \frac{2}{x}dx}=e^{\log x^2}=x^2$$

$$Required\ solution\ is$$

$$yx^2=\int (x\log x)x^2dx+c$$

$$=\int x^3\log xdx+C$$

$$=(\log x)\frac{x^4}{4}-\int \frac{x^4}{4}.\frac{1}{x}dx+C$$

$$=\frac{x^4}{4}(\log x)-\frac{1}{4}\frac{x^4}{4}+C$$

$$\therefore yx^2=\frac{x^4}{4}(\log x)-\frac{x^4}{16}+C.$$

11. Solve the differential equation

$\frac{dy}{dx}+y\sec x=\tan x,\quad 0\le x<\frac{\pi}{2}$

Sol: Given differential equation can be written as

$$\frac{dy}{dx}+(\sec x)y=\tan x$$

$$here\ P=\sec x, Q=\tan x$$

$$I.F=e^{\int Pdx}=e^{\int \sec xdx}=e^{\log(\sec x+\tan x)}=\sec x+\tan x$$

$$Required\ solution\ is$$

$$y(I.F)=\int Q(I.F)dx+C$$

$$y.(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx+C$$

$$=\int \left(\sec x\tan x+\tan^2 x\right)+C$$

$$=\int \left(\sec x\tan x+\sec^2 x-1\right)+C$$

$$y.(\sec x+\tan x)=\sec x+\tan x-x+C$$

12. Find the general solution of the differential equation $x\frac{dy}{dx}+2y=x^2,\ (x\ne 0)$

Sol: Given differential equation can be written as

$$\frac{dy}{dx}+\frac{2}{x}y=x,\ (x\neq 0)$$

$$Here\ \ P=\frac{2}{x},Q=x$$

$$I.F=e^{\int pdx}=e^{\int \frac{2}{x}dx}=e^{2\log x}=x^2$$

Required solution is

$$y\left(x^2\right)=\int x\left(x^2\right)dx+c$$

$$yx^2=\int x^3dx+C \qquad \Rightarrow \therefore yx^2=\frac{x^4}{4}+C$$

13. Find the general solution of the differential equation $(x+y)\frac{dy}{dx}=1$ **(2019-J)**

Sol: Given differential equation can be written as

$$x+y=\frac{dx}{dy}$$

$$\frac{dx}{dy}-x=y$$

$$Here\ \ P=-1,Q=y$$

$$I.F=e^{\int Pdy}=e^{\int -1dy}=e^{-y}$$

Required solution is

$$x\left(e^{-y}\right)=\int e^{-y}ydy+C$$

$$=y\left(\frac{e^{-y}}{-1}\right)-1\left(\frac{-e^{-y}}{-1}\right)+C$$

$$=-ye^{-y}-e^{-y}+C$$

$$\Rightarrow x=-y-1+Ce^{y}$$

$$\therefore x+y+1=Ce^{y}$$

14. Find the particular solution of the differential equation $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x,\ x\neq 0$ given that $y=0\ when\ x=\frac{\pi}{2}$

Sol: Given differential equation is

$$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x$$

$$Here\ P=\cot x,Q=4x\operatorname{cosec} x$$

$$\therefore I.F=e^{\int Pdx}=e^{\int \cot xdx}=e^{\log\sin x}=\sin x$$

$$\therefore General\ solution\ is\ \ y\sin x=\int 4x\operatorname{cosec} x\sin xdx+C$$

$$=4\int xdx+C$$

$$y\sin x=2x^2+C$$

$$when\ \ x=\frac{\pi}{2},y=0$$

$$0=2\left(\frac{\pi^2}{4}\right)+C\Rightarrow C=-\frac{\pi^2}{2}$$

$\therefore$ Required particular solution is

$$y\sin x=2x^2-\frac{\pi^2}{2}.$$

15. Find the particular solution of the differential equation $\frac{dy}{dx}-3y\cot x=\sin 2x,\ y=2\ when\ \ x=\frac{\pi}{2}$

Sol: Given differential equation is

$$\frac{dy}{dx}-3y\cot x=\sin 2x$$

$$Here\ P=-3\cot x,Q=\sin 2x$$

$$\therefore I.F=e^{\int Pdx}=e^{\int -3\cot xdx}=e^{-3\log\sin x}$$

$$=e^{\log\operatorname{cosec}^3 x}=\operatorname{cosec}^3 x$$

$\therefore$ General solution is

$$y\operatorname{cosec}^3 x=\int \sin 2x\left(\operatorname{cosec}^3 x\right)dx+C$$

$$y\operatorname{cosec}^3 x=\int 2\left(\sin x\right)^{-2}\cos xdx+C$$

$$y\operatorname{cosec}^3 x=2\frac{\left(\sin x\right)^{-1}}{-1}+C=-\frac{2}{\sin x}+C$$

$$when\ \ x=\frac{\pi}{2},y=2$$

$$2(1)=-\frac{2}{1}+C\Rightarrow C=4$$

$\therefore$ Required particular solution is

$$y\operatorname{cosec}^3 x=-\frac{2}{\sin x}+4.$$

16. Find the particular solution of the differential equation

$$\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2} \text{ when } y=0 \ \& \ x=1$$

Sol: Given differential equation can be written as

$$\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{\left(1+x^2\right)^2}$$

$$Here\ P=\frac{2xy}{1+x^2},Q=\frac{1}{\left(1+x^2\right)^2}$$

$$I.F=e^{\int Pdx}=e^{\int \frac{2xy}{1+x^2}dx}=e^{\log\left(1+x^2\right)}=1+x^2$$

$\therefore$ *General solution is*

$$y\left(1+x^2\right)=\int \left(1+x^2\right)\frac{1}{\left(1+x^2\right)^2}dx+C$$

$$=\int \frac{1}{1+x^2}dx+C=\tan^{-1}x+C$$

$when\ x=1, y=0$

$$\therefore 0=\frac{\pi}{4}+C\Rightarrow C=-\frac{\pi}{4}$$

$\therefore$ Re*quired particular solution is*

$$\left(1+x^2\right)y=\tan^{-1}x-\frac{\pi}{4}.$$

17. Find the equation a curve passing through the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of 'x' coordinate and the product of 'x' coordinate and y coordinate of that point.

Sol: Given

$$\frac{dy}{dx}=x+xy \qquad \Rightarrow \frac{dy}{dx}-xy=x$$

$Here\ P=-x, Q=x$

$$I.F=e^{\int Pdx}=e^{\int -xdx}=e^{\frac{-x^2}{2}}$$

$\therefore$ *General solution is*

$$ye^{\frac{-x^2}{2}}=\int e^{\frac{-x^2}{2}}.xdx+C$$

$$=\int -e^t dt+C \quad where\ t=\frac{-x^2}{2}$$

$$=-e^t+C$$

$$ye^{\frac{-x^2}{2}}=-e^{\frac{-x^2}{2}}+C$$

It passes through $(0,1)$

$$1=-1+C\Rightarrow C=2$$

$\therefore$ Re*quired equation of curve is*

$$ye^{\frac{-x^2}{2}}=-e^{\frac{-x^2}{2}}+2$$

$$\Rightarrow y=-1+2e^{\frac{x^2}{2}}$$

UNIT-10
VECTOR ALGEBRA

$$\begin{bmatrix}\text{Total marks}: 11, Q.No-7(1M), \\ 21\ \&\ 22(2M\ \text{each}), 35\ \&\ 36(3M\ \text{each})\end{bmatrix}$$

Question No: 7 (1M)

1. Define Unit Vector. **(2014 - J)**

Ans: A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector.

2. Define Negative of a vector. **(2015–M, 2018-J, 2020-M)**

Ans: A vector whose magnitude is the same as that of given vector, but direction is opposite to that of it, is called negative of the given vector.

3. Define collinear vectors. **(2017 - M)**

Ans: Two or more vectors said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

4. Define coplanar vectors.

Ans: The vectors $\vec{a}, \vec{b}$ *and* $\vec{c}$ are said to be coplanar, if vectors lie in same plane.

5. Find the direction cosines of the vector $\hat{i}+2\hat{j}+3\hat{k}$. **(2014 - M)**

Sol: Here $a=1, b=2$ and $c=3$

$$\therefore \sqrt{a^2+b^2+c^2} = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

Direction cosines $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

6. If $\hat{a} = \frac{1}{\sqrt{14}}\left(2\hat{i}+3\hat{j}+\hat{k}\right)$, then write the direction cosines of $\hat{a}$. **(2019 - J)**

Sol: $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$

7. Compute the magnitude of $\vec{a} = 2\hat{i}-7\hat{j}-3\hat{k}$

Sol: Magnitude of $\vec{a}$ is

$$\left|\vec{a}\right| = \sqrt{(2)^2+(-7)^2+(-3)^2} = \sqrt{4+49+9} = \sqrt{62}$$

8. Find the position vector of a point R which divides the joining two points P and Q whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ & $-\hat{i}+\hat{j}+\hat{k}$ respectively, in the ratio $2:1$ internally.

Sol: Here $m:n = 2:1$ and

$$\vec{a} = \hat{i}+2\hat{j}-\hat{k}\ \&\ \vec{b} = -\hat{i}+\hat{j}+\hat{k}$$

The position vector of a point R is

$$\vec{OR} = \frac{m\vec{b}+n\vec{a}}{m+n}$$

$$= \frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1} = \frac{-\hat{i}+4\hat{j}+\hat{k}}{3}$$

9. Show that vectors $\vec{a} = 2\hat{i}-3\hat{j}+4\hat{k}$ & $\vec{b} = -4\hat{i}+6\hat{j}-8\hat{k}$ are collinear. **(2015 - J)**

Sol: Given $\vec{b} = -4\hat{i}+6\hat{j}-8\hat{k}$

$$\Rightarrow \vec{b} = -2\left(2\hat{i}-3\hat{j}+4\hat{k}\right) \quad \Rightarrow \vec{b} = -2\left(\vec{a}\right)$$

$\therefore \vec{a}$ *and* $\vec{b}$ *are collinear*

10. Find the unit vector in the direction of the vectors $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$. **(17–J, 2018-M)**

Sol: The unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{\left|\vec{a}\right|}$$

$$\therefore \left|\vec{a}\right| = \sqrt{(1)^2+(1)^2+(2)^2} = \sqrt{6}$$

$$\hat{a} = \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$$

10. Find the unit vector in the direction of the vectors $\vec{a} = 2\hat{i}-3\hat{j}-\hat{k}$. **(2016 - J)**

Sol: The unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{\left|\vec{a}\right|}$$

$$\therefore \left|\vec{a}\right| = \sqrt{(2)^2 + (-3)^2 + (-1)^2} = \sqrt{14}$$

$$\hat{a} = \frac{2\hat{i} - 3\hat{j} - \hat{k}}{\sqrt{14}} = \frac{2}{\sqrt{14}}\hat{i} - \frac{3}{\sqrt{14}}\hat{j} - \frac{1}{\sqrt{14}}\hat{k}$$

12. If the vectors $2\hat{i} + 3\hat{j} - 6\hat{k}$ & $4\hat{i} - m\hat{j} - 12\hat{k}$ are parallel find m.

Sol: If two vectors parallel then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \frac{2}{4} = \frac{3}{-m} = \frac{-6}{-12} \Rightarrow \frac{1}{2} = \frac{3}{-m} = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{3}{-m} \Rightarrow m = -6$$

13. Find the vector joining the points $P(2,3,0)$ & $Q(-1,-2,-4)$ directed from P to Q

Sol: Vector joining two points is

$$(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\Rightarrow (-1-2)\hat{i} + (-2-3)\hat{j} + (-4-0)\hat{k}$$

$$\Rightarrow -3\hat{i} - 5\hat{j} - 4\hat{k}$$

14. If Vector $\overrightarrow{AB} = 2\hat{i} - \hat{j} + \hat{k}$ & $\overrightarrow{OB} = 3\hat{i} - 4\hat{j} + 4\hat{k}$, find the position vector $\overrightarrow{OA}$.**(2016 - M)**

Sol: WKT; $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \Rightarrow \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{AB}$

$$\Rightarrow \overrightarrow{OA} = 3\hat{i} - 4\hat{j} + 4\hat{k} - \left(2\hat{i} - \hat{j} + \hat{k}\right) \Rightarrow \overrightarrow{OA} = \hat{i} - 3\hat{j} + 3\hat{k}$$

15. Find the value of x for which $x\left(\hat{i} + j + k\right)$ is a unit vector. **(2019 - M)**

Sol: $\textit{By given} \left|x\left(\hat{i} + \hat{j} + \hat{k}\right)\right| = 1$

$$\Rightarrow x\sqrt{(1)^2 + (1)^2 + (1)^2} = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$$

16. Find $\left|\vec{x}\right|$, if for a unit vector $\vec{a}$, $\left(\vec{x} - \vec{a}\right).\left(\vec{x} + \vec{a}\right) = 12$

Sol: Given $\left(\vec{x} - \vec{a}\right).\left(\vec{x} + \vec{a}\right) = 12$

$$\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}.\vec{x} - \vec{a}.\vec{a} = 12 \Rightarrow \left|\vec{x}\right|^2 - \left|\vec{a}\right|^2 = 12$$

$$\Rightarrow \left|\vec{x}\right|^2 - 1 = 12 \Rightarrow \boxed{\left|\vec{x}\right| = \sqrt{13}}$$

$(\textit{magnitude non negative})$

17. If $\left|\vec{a}.\vec{b}\right| = \left|\vec{a} \times \vec{b}\right|$ then find the angle between $\vec{a}$ & $\vec{b}$.

Sol: Given $\left|\vec{a}.\vec{b}\right| = \left|\vec{a} \times \vec{b}\right|$

$$\Rightarrow \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta = \left|\vec{a}\right|\left|\vec{b}\right|\sin\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$$

18. Write down a unit vector XY plane, making an angle 30° with positive direction of X-axis.

Sol: If $\vec{r}$ is unit vector in XY-plane,

then $\vec{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Here angle 30° ,then

$$\vec{r} = \cos 30\,\hat{i} + \sin 30\,\hat{j} \Rightarrow \vec{r} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

19. Find the angle between two vectors $\vec{a}$ & $\vec{b}$ with magnitude $\sqrt{3}$ & 2, respectively having $\vec{a}.\vec{b} = \sqrt{6}$.

Sol: Given $\left|\vec{a}\right| = \sqrt{3}, \left|\vec{b}\right| = 2$ & $\vec{a}.\vec{b} = \sqrt{6}$

WKT; $\vec{a}.\vec{b} = \left|\vec{a}\right|.\left|\vec{b}\right|\cos\theta \Rightarrow \sqrt{6} = \sqrt{3}.2\cos\theta$

$$\Rightarrow \sqrt{3}.\sqrt{2} = \sqrt{3}.2\cos\theta \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

20. If $\vec{a}$ is a non-zero vector of magnitude 'a' and $\lambda\vec{a}$ is a unit vector then find the value of λ .

Sol: Given $\left|\vec{a}\right| = a$ & $\left|\lambda\vec{a}\right| = 1 \Rightarrow \left|\lambda\right|\left|\vec{a}\right| = 1$

$$\Rightarrow \left|\lambda\right| a = 1 \Rightarrow \boxed{\lambda = \frac{1}{a}}$$

21. Find the vector components of the vector with initial point (2,1) and terminal point (-5,7). **(MQP-2021)**

Sol: The vector with the initial point P (2,1) and terminal point Q (-5,7) is

$$\overrightarrow{PQ}=(-5-2)\hat{i}+(7-1)\hat{j}=-7\hat{i}+6\hat{j}$$

The required vector components are

$-7\hat{i}$ and $6\hat{j}$

Question No: 21 & 22 (2M Each)

1. If $\vec{a}$ is a unit vector such that $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=8$ find $|\vec{x}|$.

(2017– M, 2019-J)

Sol: Given $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=8$

$$\Rightarrow \vec{x}.\vec{x}+\vec{x}.\vec{a}-\vec{a}.\vec{x}-\vec{a}.\vec{a}=8 \Rightarrow |\vec{x}|^2-|\vec{a}|^2=8$$

$$\Rightarrow |\vec{x}|^2-1=8 \Rightarrow |\vec{x}|^2=9 \Rightarrow \boxed{|\vec{x}|=3}$$

$(magnitude\ non\ negative)$

2. If $\vec{a}$ is a unit vector such that $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$ find $|\vec{x}|$. **(2017 - M)**

Sol: Given $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=12$

$$\Rightarrow \vec{x}.\vec{x}.+\vec{x}.\vec{a}-\vec{a}.\vec{x}-\vec{a}.\vec{a}=12 \quad \Rightarrow |\vec{x}|^2-|\vec{a}|^2=12$$

$$\Rightarrow |\vec{x}|^2-1=12 \Rightarrow \boxed{|\vec{x}|=\sqrt{13}}$$

$(magnitude\ non\ negative)$

3. Find $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ & $|\vec{a}|=8|\vec{b}|$

(2015-M, 2019-M)

Sol: Given $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ & $|\vec{a}|=8|\vec{b}|$

$$\Rightarrow \vec{a}.\vec{a}-\vec{a}.\vec{b}+\vec{b}.\vec{a}-\vec{b}.\vec{b}=8 \Rightarrow |\vec{a}|^2-|\vec{b}|^2=8$$

$$\Rightarrow \left(8|\vec{b}|\right)^2-|\vec{b}|^2=8$$

$$\Rightarrow 64|\vec{b}|^2-|\vec{b}|^2=8 \Rightarrow 63|\vec{b}|^2=8$$

$$\Rightarrow |\vec{b}|^2=\frac{8}{63} \qquad \Rightarrow |\vec{b}|=\sqrt{\frac{8}{63}}$$

4. If two vectors $\vec{a}$ & $\vec{b}$ such that $|\vec{a}|=2, |\vec{b}|=3, \vec{a}\cdot\vec{b}=4$ Find $|\vec{a}-\vec{b}|$.

(2015-J 2016-J)

Sol: Given $|\vec{a}|=2, |\vec{b}|=3, \vec{a}\cdot\vec{b}=4$

Now

$$|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b}).(\vec{a}-\vec{b}) = \vec{a}.\vec{a}-\vec{a}.\vec{b}-\vec{b}.\vec{a}+\vec{b}.\vec{b}$$

$$=|\vec{a}|^2-2(\vec{a}.\vec{b})+|\vec{b}|^2=(2)^2-2(4)+(3)^3=5$$

$$\Rightarrow \boxed{|\vec{a}-\vec{b}|=\sqrt{5}}$$

5. If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$ prove that $\vec{a}$ & $\vec{b}$ are perpendicular. **(2016-M)**

Sol: $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$ square on both side

$$|\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$$

$$|\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}.\vec{b}$$

$$2\vec{a}.\vec{b}=-2\vec{a}.\vec{b}$$

$$4\vec{a}.\vec{b}=0 \qquad \Rightarrow \vec{a}.\vec{b}=0$$

Therefore, scalar product is zero so $\vec{a}$ & $\vec{b}$ are perpendicular to each other.

6. Find the angle between the vectors $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ & $\vec{b}=\hat{i}+\hat{j}+\hat{k}$. **(2016-M)**

Sol: Given $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ & $\vec{b}=\hat{i}+\hat{j}+\hat{k}$

Angle b/w two vectors

$$\vec{a} \text{ \& } \vec{b} \text{ is } \cos\theta=\left|\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right|$$

$$\Rightarrow \cos\theta=\frac{(\hat{i}+\hat{j}-\hat{k}).(\hat{i}+\hat{j}+\hat{k})}{\sqrt{1^2+1^2+(-1)^2}.\sqrt{1^2+(1)^2+1^2}}$$

$$\Rightarrow \cos\theta=\frac{1+1-1}{\sqrt{3}.\sqrt{3}}$$

$$\Rightarrow \cos\theta=\frac{1}{3} \Rightarrow \boxed{\theta=\cos^{-1}\left(\frac{1}{3}\right)}$$

7. Find the angle between the vectors $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ & $\vec{b}=\hat{i}-\hat{j}+\hat{k}$.

Sol: Given $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ & $\vec{b}=\hat{i}+\hat{j}+\hat{k}$

Angle b/w two vectors

$$\vec{a} \text{ \& } \vec{b} \text{ is } \cos\theta=\left|\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right|$$

$$\Rightarrow \cos\theta=\frac{(\hat{i}+\hat{j}-\hat{k}).(\hat{i}-\hat{j}+\hat{k})}{\sqrt{1^2+1^2+(-1)^2}.\sqrt{1^2+(-1)^2+1^2}}$$

$\Rightarrow \cos\theta = \frac{1-1-1}{\sqrt{3}.\sqrt{3}} \quad \Rightarrow \cos\theta = -\frac{1}{3} \Rightarrow \boxed{\theta = \cos^{-1}\left(-\frac{1}{3}\right)}$

8. Find the sine of angle between the vectors $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ & $\vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$. **(2016-M)**

Sol: Given $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ & $\vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

Angle b/w two vectors

$\vec{a}$ & $\vec{b}$ is $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

$$\therefore \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 3 & 2 & 6 \end{vmatrix} = 8\hat{i} - 4\hat{k}$$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{(8)^2 + (-4)^2} = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$

& $|\vec{a}| = \sqrt{1^2+2^2+2^2} = 3, |\vec{b}| = \sqrt{3^2+2^2+6^2} = 7$

$\therefore \sin\theta = \frac{4\sqrt{5}}{3(7)} \Rightarrow \boxed{\sin\theta = \frac{4\sqrt{5}}{21}}$

9. Let $|\vec{a}| = 3, |\vec{b}| = \frac{\sqrt{2}}{3}$ and $|\vec{a}\times\vec{b}| = 1$ Find tha angle between $\vec{a}$ and $\vec{b}$. **(2015-J)**

Sol: Given $|\vec{a}| = 3, |\vec{b}| = \frac{\sqrt{2}}{3}$ and $|\vec{a}\times\vec{b}| = 1$

Angle b/w two vectors

$\vec{a}$ & $\vec{b}$ is $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

$\Rightarrow \sin\theta = \frac{1}{\cancel{3}\left(\frac{\sqrt{2}}{\cancel{3}}\right)} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

10. Obtain the projection of the vector

$\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector

$\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ **(2014-J, 2019-M)**

Sol: Given vectors $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

The projection of the vector

$\vec{a}$ on the vector $\vec{b}$ is $\frac{(\vec{a}.\vec{b})}{|\vec{b}|}$

$\Rightarrow \frac{(2\hat{i}+3\hat{j}+2\hat{k}).(\hat{i}+2\hat{j}+\hat{k})}{\sqrt{1^2+2^2+1^2}} = \frac{2+6+2}{\sqrt{6}} = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}.$

11. Obtain the projection of the vector $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ on the vector $\vec{b} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ **(2018-J)**

Sol: Given vectors $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ &

$\vec{b} = 2\hat{i} + 3\hat{j} + 2\hat{k}$

The projection of the vector $\vec{a}$ on the vector $\vec{b}$ is $\frac{(\vec{a}.\vec{b})}{|\vec{b}|} \quad \Rightarrow \frac{(\hat{i}+2\hat{j}+\hat{k}).(2\hat{i}+3\hat{j}+2\hat{k})}{\sqrt{2^2+3^2+2^2}}.$

$= \frac{2+6+2}{\sqrt{17}} = \frac{10}{\sqrt{17}}.$

12. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$. **(2015-J, 2018-M, 2020-M)**

Sol: Let given vectors $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ & $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

The projection of the vector $\vec{a}$ on the vector $\vec{b}$ is $\frac{(\vec{a}.\vec{b})}{|\vec{b}|}$

$\Rightarrow \frac{(\hat{i}+3\hat{j}+7\hat{k}).(7\hat{i}-\hat{j}+8\hat{k})}{\sqrt{7^2+(-1)^2+8^2}}.$

$= \frac{7-3+56}{\sqrt{114}} = \frac{60}{\sqrt{114}}.$

13. Find the value of $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}.(\hat{i}\times\hat{j})$

Sol: Given $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}.(\hat{i}\times\hat{j})$

$\Rightarrow \hat{i}.\hat{i} + \hat{j}.(-\hat{j}) + \hat{k}.\hat{k} \quad (\because \hat{j}\times\hat{k} = \hat{i}, \hat{i}\times\hat{k} = -\hat{j}, \hat{i}\times\hat{j} = \hat{k})$

$\Rightarrow \hat{i}.\hat{i} - \hat{j}.\hat{j} + \hat{k}.\hat{k} = 1-1+1 = 1 \quad (\because \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1)$

14. Find the value of

$$\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{k}\times\hat{i}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$$

Sol: Given $\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{k}\times\hat{i}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$

$$\Rightarrow \hat{i}.\hat{i}+\hat{j}.\hat{j}+\hat{k}.\hat{k} \quad \left(\because \hat{j}\times\hat{k}=\hat{i}, \hat{k}\times\hat{i}=\hat{j}, \hat{i}\times\hat{j}=\hat{k}\right)$$

$$\Rightarrow \hat{i}.\hat{i}+\hat{j}.\hat{j}+\hat{k}.\hat{k}=1+1+1=3 \quad \left(\because \hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1\right)$$

15. Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a}=3\hat{\imath}+\hat{\jmath}+4\hat{k}$ And $\vec{b}=\hat{\imath}-\hat{\jmath}+\hat{k}$.
(2014-M, 2017-J, 2018-M, 2019-J)

Sol: Given vectors $\vec{a}=3\hat{\imath}+\hat{\jmath}+4\hat{k}$ and $\vec{b}=\hat{\imath}-\hat{\jmath}+\hat{k}$

The area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 4\\ 1 & -1 & 1\end{vmatrix}$$

$$=\hat{i}\left(1-(-4)\right)-\hat{j}\left(3-4\right)+\hat{k}\left(-3-1\right)=5\hat{i}+\hat{j}-4\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(5)^2+(1)^2+(-4)^2}=\sqrt{25+1+16}=\sqrt{42}.$$

16. Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a}=\hat{\imath}+\hat{\jmath}-\hat{k}$ And $\vec{b}=\hat{\imath}-\hat{\jmath}+\hat{k}$
(2014-J)

Sol: Given vectors $\vec{a}=\hat{\imath}+\hat{\jmath}-\hat{k}$ and $\vec{b}=\hat{\imath}-\hat{\jmath}+\hat{k}$

The area of the a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -1\\ 1 & -1 & 1\end{vmatrix}$$

$$=\hat{i}\left(1-1\right)-\hat{j}\left(1-(-1)\right)+\hat{k}\left(-1-1\right)=0\hat{i}-2\hat{j}-2\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(0)^2+(-2)^2+(-2)^2}=\sqrt{0+4+4}=\sqrt{8}=2\sqrt{2}.$$

17. Find the area of the parallelogram whose adjacent sides are given by the vectors $\vec{a}=\hat{\imath}-\hat{\jmath}+3\hat{k}$ and $\vec{b}=2\hat{\imath}-7\hat{\jmath}+\hat{k}$
(2015-M, 2018-J, 2020-M)

Sol: Given vectors $\vec{a}=\hat{\imath}-\hat{\jmath}+3\hat{k}$ and $\vec{b}=2\hat{\imath}-7\hat{\jmath}+\hat{k}$

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}$$

$$=\hat{i}\left(-1-(-21)\right)-\hat{j}\left(1-6\right)+\hat{k}\left(-7-(-2)\right)=20\hat{i}+5\hat{j}-5\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(20)^2+(5)^2+(-5)^2}$$

$$=\sqrt{400+25+25}=\sqrt{450}=\sqrt{225\times 2}=15\sqrt{2}.$$

18. Find the area of the a parallelogram whose adjacent sides are given by the vectors $\vec{a}=\hat{\imath}-\hat{\jmath}-3\hat{k}$ and $\vec{b}=2\hat{\imath}-7\hat{\jmath}+\hat{k}$
(2016-J, 2017-M)

Sol: Given vectors $\vec{a}=\hat{\imath}-\hat{\jmath}-3\hat{k}$ and $\vec{b}=2\hat{\imath}-7\hat{\jmath}+\hat{k}$

The area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & -3\\ 2 & -7 & 1\end{vmatrix}$$

$$=\hat{i}\left(-1-21\right)-\hat{j}\left(1-(-6)\right)+\hat{k}\left(-7-(-2)\right)$$

$$=-22\hat{i}-7\hat{j}-5\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(-22)^2+(-7)^2+(-5)^2}$$

$$=\sqrt{484+49+25}=\sqrt{558}.$$

19. Find the position vector of a point R which divides the line joining two points P & Q whose position vectors are $\hat{\imath}+2\hat{\jmath}-\hat{k}$ & $-\hat{\imath}+\hat{\jmath}+\hat{k}$ respectively in the ratio 2:1 (i) internally (ii) Externally

Sol: Let $\vec{a} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{b} = -\hat{\imath} + \hat{\jmath} + \hat{k}$, $m:n = 2:1$

(i) internally: The position vector of a point R which divides the line joining two points P & Q whose position vectors are $\vec{a}$ & $\vec{b}$ respectively in the ratio m:n internally

$$\vec{r} = \frac{m\vec{b}+n\vec{a}}{m+n} = \frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1}$$

$$= \frac{-\hat{i}+4\hat{j}+\hat{k}}{3} = -\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}.$$

(ii) Externally: The position vector of a point R which divides the line joining two points P & Q whose position vectors are $\vec{a}$&$\vec{b}$respectively in the ratio m:n externally

$$\vec{r} = \frac{m\vec{b}-n\vec{a}}{m-n} = \frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)-1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2-1}$$

$$= \frac{-3\hat{i}+0\hat{j}+3\hat{k}}{1} = -3\hat{i}+3\hat{k}.$$

20. Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the positive direction of coordinate axes. **(MQP-2021)**

Sol: The vector equally inclined if and only if direction cosines are equal

Let $\vec{a} = \hat{i}+\hat{j}+\hat{k} \Rightarrow |\vec{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$$Dc\text{'s}\quad l = \cos\alpha = \frac{a_1}{|\vec{a}|} = \frac{1}{\sqrt{3}},$$

$$m = \cos\beta = \frac{a_2}{|\vec{a}|} = \frac{1}{\sqrt{3}},$$

$$n = \cos\gamma = \frac{a_3}{|\vec{a}|} = \frac{1}{\sqrt{3}}$$

$\therefore$ given vector equally inclined to the positive direction of axes.

21. Find a vector in the direction of vector $\vec{a} = \hat{i}-2\hat{j}$ that has magnitude 7 units. **(MQP-2021)**

Sol: Given $\vec{a} = \hat{i}-2\hat{j} \Rightarrow |\vec{a}| = \sqrt{1^2+(-2)^2} = \sqrt{5}$

Unit vector in the direction $\hat{a}$ is $\frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}-2\hat{j}}{\sqrt{5}}$

The vector having magnitude 7 and in the direction of $\vec{a}$ is

$$7\hat{a} = 7\left(\frac{\hat{i}-2\hat{j}}{\sqrt{5}}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$$

Question No: 35 & 36 (3M Each)

1. Find the unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ where $\vec{a} = \hat{\imath}+\hat{\jmath}+\hat{k}$ and $\vec{b} = \hat{\imath}+2\hat{\jmath}+3\hat{k}$. **(2014-J, 2015-J, 2019-M, MQP-2021)**

Sol: Given $\vec{a} = \hat{\imath}+\hat{\jmath}+\hat{k}$ and $\vec{b} = \hat{\imath}+2\hat{\jmath}+3\hat{k}$

$$\vec{a}+\vec{b} = \left(\hat{i}+\hat{j}+\hat{k}\right)+\left(\hat{i}+2\hat{j}+3\hat{k}\right) = 2\hat{i}+3\hat{j}+4\hat{k}$$

$$\vec{a}-\vec{b} = \left(\hat{i}+\hat{j}+\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right) = 0\hat{i}-\hat{j}-2\hat{k}$$

$$Now,\ \left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= \hat{i}\left(-6-(-4)\right) - \hat{j}\left(-4-0\right) + \hat{k}\left(-2-0\right)$$

$$= -2\hat{i}+4\hat{j}-2\hat{k}$$

$$\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right| = \sqrt{(-2)^2+(4)^2+(-2)^2} = \sqrt{24} = 2\sqrt{6}$$

The unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is

$$\pm\frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|} = \pm\frac{-2\hat{i}+4\hat{j}-2\hat{k}}{2\sqrt{6}}$$

$$= \pm\left(\frac{-1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}\right).$$

2. Find the unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ *and* $(\vec{a}-\vec{b})$ where $\vec{a}=3\hat{\imath}+2\hat{\jmath}+2\hat{k}$ and $\vec{b}=\hat{\imath}+2\hat{\jmath}-2\hat{k}$ **(2014-J, 2015-J, 2020-M)**

Sol: Given $\vec{a}=3\hat{\imath}+2\hat{\jmath}+2\hat{k}$ and $\vec{b}=\hat{\imath}+2\hat{\jmath}-2\hat{k}$

$$\vec{a}+\vec{b}=\left(3\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-2\hat{k}\right)=4\hat{i}+4\hat{j}+0\hat{k}$$

$$\vec{a}-\vec{b}=\left(3\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-2\hat{k}\right)=2\hat{i}+0\hat{j}+4\hat{k}$$

Now $\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}$

$$=\hat{i}(16-0)-\hat{j}(16-0)+\hat{k}(0-8)=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\sqrt{(16)^2+(-16)^2+(-8)^2}$$

$$=\sqrt{256+256+64}=\sqrt{576}=24$$

The unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ *and* $(\vec{a}-\vec{b})$ is

$$\pm\frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|}=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$$

$$=\pm\left(\frac{8\left(2\hat{i}-2\hat{j}-\hat{k}\right)}{24}\right)=\pm\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}\right).$$

3. Three vectors $\vec{a},\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=0$ evaluate $\mu=\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$, if $\left|\vec{a}\right|=1,\ \left|\vec{b}\right|=4$ *and* $\left|\vec{c}\right|=2$ **(2017-M)**

Sol: Given $\vec{a}+\vec{b}+\vec{c}=0$ and $\left|\vec{a}\right|=1,\ \left|\vec{b}\right|=4$ *and* $\left|\vec{c}\right|=2$,

$$\mu=\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$$

$$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)$$

$$=\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}+\vec{b}.\vec{a}+\vec{b}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}+\vec{c}.\vec{b}+\vec{c}.\vec{c}$$

$$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)\ (\because given)$$

$$(0)^2=(1)^2+(4)^2+(2)^2+2(\mu)$$

$$0=21+2(\mu)\Rightarrow\boxed{\mu=-\frac{21}{2}}$$

4. If two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}|=2, |\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=4$ Find $|\vec{a}-\vec{b}|$ **(2014-J)**

Sol: Given $|\vec{a}|=2, |\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=4$

Now

$$\left|\vec{a}-\vec{b}\right|^2=\left(\vec{a}-\vec{b}\right).\left(\vec{a}-\vec{b}\right)$$

$$=\vec{a}.\vec{a}-\vec{a}.\vec{b}-\vec{b}.\vec{a}+\vec{b}.\vec{b}=\left|\vec{a}\right|^2-2\left(\vec{a}.\vec{b}\right)+\left|\vec{b}\right|^2$$

$$\left|\vec{a}-\vec{b}\right|^2=(2)^2-2(4)+(3)^2=4-8+9=5$$

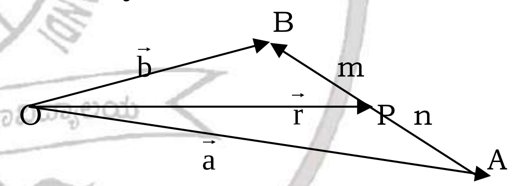
$$\therefore\left|\vec{a}-\vec{b}\right|=\sqrt{5}.$$

5. Show that position vector of the point *P* divides the line joining the points A & B having position vectors $\vec{a}$ & $\vec{b}$ internally in the ratio m:n is $\frac{m\vec{b}+n\vec{a}}{m+n}$ **(2015-M, 16-J, 2016-J, 2018-M, 2019-J, MQP-2021)**

Sol: Let the points A & B having position vector $\vec{a}$ & $\vec{b}$

i.e., $\overrightarrow{OA}=\vec{a}$ & $\overrightarrow{OB}=\vec{b}$

Then the line segment joining the point A & B divided by a third point P Internally in the ratio m:n



By diagram $\frac{\overrightarrow{AP}}{\overrightarrow{PB}}=\frac{m}{n}$

$$\Rightarrow m\overrightarrow{PB}=n\overrightarrow{AP}$$

$$\Rightarrow m\left(\overrightarrow{OB}-\overrightarrow{OP}\right)=n\left(\overrightarrow{OP}-\overrightarrow{OA}\right)$$

$$\Rightarrow m\left(\vec{b}-\vec{r}\right)=n\left(\vec{r}-\vec{a}\right)$$

$$\Rightarrow m\vec{b}-m\vec{r}=n\vec{r}-n\vec{a}$$

$$\Rightarrow m\vec{b}+n\vec{a}=m\vec{r}+n\vec{r}\Rightarrow m\vec{b}+n\vec{a}=\vec{r}\left(m+n\right)$$

$$\Rightarrow\boxed{\vec{r}=\frac{m\vec{b}+n\vec{a}}{m+n}}.$$

6. If $\vec{a}=2\hat{\imath}+2\hat{\jmath}+3\hat{k}, \vec{b}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$ and $\vec{c}=3\hat{\imath}+\hat{\jmath}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendiculear to $\vec{c}$. Then find the value of λ. **(2014-M)**

Sol: Given $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ and $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$. i.e., $\left(\vec{a}+\lambda\vec{b}\right).\vec{c}=0$

Now $\vec{a}+\lambda\vec{b}=2\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(-\hat{i}+2\hat{j}+\hat{k}\right)$

$$=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$$

By given $\left(\vec{a}+\lambda\vec{b}\right).\vec{c}=0$

$$\Rightarrow\left((2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right).\left(3\hat{i}+\hat{j}\right)=0$$

$$\Rightarrow(2-\lambda).3+(2+2\lambda).1+(3+\lambda)0.=0$$

$$\Rightarrow 6-3\lambda+2+2\lambda=0$$

$$\Rightarrow 8-\lambda=0\Rightarrow\boxed{\lambda=8}$$

7. Show that the four points with position vectors $4\hat{i} + 8\hat{j} + 12\hat{k}$,$2\hat{i} + 4\hat{j} + 6\hat{k}$,$3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ are coplanar. **(2015-M, 2019-M)**

Sol: Let $\overrightarrow{OA}=4\hat{i}+8\hat{j}+12\hat{k},\overrightarrow{OB}=2\hat{i}+4\hat{j}+6\hat{k},$

$$\overrightarrow{OC}=3\hat{i}+5\hat{j}+4\hat{k},\overrightarrow{OD}=5\hat{i}+8\hat{j}+5\hat{k}$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=2\hat{i}+4\hat{j}+6\hat{k}-\left(4\hat{i}+8\hat{j}+12\hat{k}\right)$$

$$=-2\hat{i}-4\hat{j}-6\hat{k}.$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=3\hat{i}+5\hat{j}+4\hat{k}-\left(4\hat{i}+8\hat{j}+12\hat{k}\right)$$

$$=-\hat{i}-3\hat{j}-8\hat{k}.$$

$$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=5\hat{i}+8\hat{j}+5\hat{k}-\left(4\hat{i}+8\hat{j}+12\hat{k}\right)$$

$$=\hat{i}+0\hat{j}-7\hat{k}.$$

$$\therefore\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \ \overrightarrow{AD}\right]=\begin{vmatrix}-2&-4&-6\\-1&-3&-8\\1&0&-7\end{vmatrix}$$

$$=-2(21-0)+4(7-(-8))-6(0-(-3))$$

$$=-42+60-18=0$$

Given position vectors are coplanar.

8. Show that the four points A, B, C & D with position vector $4\hat{i}+5j+k,\ -j-k,\ 3\hat{i}+9j+4k$ & $4\left(-\hat{i}+j+k\right)$ are coplanar.

Sol: Let $\overrightarrow{OA}=4\hat{i}+5\hat{j}+\hat{k},\overrightarrow{OB}=-\hat{j}-\hat{k},$ $\overrightarrow{OC}=3\hat{i}+9\hat{j}+4\hat{k},\overrightarrow{OD}=\left(-\hat{i}+\hat{j}+\hat{k}\right)$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-\hat{j}-\hat{k}-\left(4\hat{i}+5\hat{j}+\hat{k}\right)$$

$$=-4\hat{i}-6\hat{j}-2\hat{k}.$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=3\hat{i}+9\hat{j}+4\hat{k}-\left(4\hat{i}+5\hat{j}+\hat{k}\right)$$

$$=-\hat{i}+4\hat{j}+3\hat{k}.$$

$$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=4\left(-\hat{i}+\hat{j}+\hat{k}\right)-\left(4\hat{i}+5\hat{j}+\hat{k}\right)$$

$$=-8\hat{i}-\hat{j}+3\hat{k}.$$

$$\therefore\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \ \overrightarrow{AD}\right]=\begin{vmatrix}-4&-6&-2\\-1&4&3\\-8&-1&3\end{vmatrix}$$

$$=-4(12-(-3))+6(-3-(-24))-2(1-(-32))$$

$$=-60+126-66=0$$

Given position vectors are coplanar.

9. Find λ ,if the vectors$\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} + 7\hat{j} + 3\hat{k}$ are coplanar. **(2015-J)**

Sol: Given vectors $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} + 7\hat{j} + 3\hat{k}$ are coplanar i.e., $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]=0$

$$\therefore\begin{vmatrix}1&3&1\\2&-1&-1\\\lambda&7&3\end{vmatrix}=0$$

$$\Rightarrow 1(-3-(-7))-3(6-(-\lambda))+1(14-(-\lambda))=0$$

$$\Rightarrow 4-18-3\lambda+14+\lambda=0$$

$$\Rightarrow -2\lambda-0=0\Rightarrow\boxed{\lambda=0}.$$

10. Find 'x' such that the four points A (3, 2, 1), B (4, X, 5), C (4, 2, -2) and D (6, 5, -1) are coplanar.

(2017-M, J, 2018-M, 2020-M)

Sol: Given four points A (3, 2, 1), B (4, X, 5), C (4, 2, -2) and D (6, 5, -1) are coplanar

Let the position vectors

$$\overrightarrow{OA}=3\hat{i}+2\hat{j}+\hat{k},\overrightarrow{OB}=4\hat{i}+x\hat{j}+5\hat{k},$$

$$\overrightarrow{OC}=4\hat{i}+2\hat{j}-2\hat{k},\overrightarrow{OD}=6\hat{i}+5\hat{j}-\hat{k}$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=4\hat{i}+x\hat{j}+5\hat{k}-\left(3\hat{i}+2\hat{j}+\hat{k}\right)$$

$$=\hat{i}+(x-2)\hat{j}+4\hat{k}.$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=4\hat{i}+2\hat{j}-2\hat{k}-\left(3\hat{i}+2\hat{j}+\hat{k}\right)$$

$$=\hat{i}+0\hat{j}-3\hat{k}.$$

$$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=6\hat{i}+5\hat{j}-\hat{k}-\left(3\hat{i}+2\hat{j}+\hat{k}\right)$$

$$=3\hat{i}+3\hat{j}-2\hat{k}.$$

$$\therefore \text{By given}\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0$$

$$\begin{vmatrix}1 & x-2 & 4\\ 1 & 0 & -3\\ 3 & 3 & -2\end{vmatrix}=0$$

$$1(0-(-9))-(x-2)(-2-(-9))+4(3-0)=0$$

$$9-7x+14+12=0$$

$$7x=35 \Rightarrow \boxed{x=5}$$

11. For any three vectors $\vec{a},\vec{b}$ and $\vec{c}$, prove that vectors $\vec{a}-\vec{b},\vec{b}-\vec{c}\ \&\ \vec{c}-\vec{a}$ are coplanar. **(2016-M)**

Sol: Given vectors $\vec{a}-\vec{b},\vec{b}-\vec{c}\ \&\ \vec{c}-\vec{a}$

$$\text{Now}\left[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}\right]=\left(\vec{a}-\vec{b}\right).\left(\left(\vec{b}-\vec{c}\right)\times\left(\vec{c}-\vec{a}\right)\right)$$

$$=\left(\vec{a}-\vec{b}\right).\left(\left(\vec{b}\times\vec{c}\right)-\left(\vec{b}\times\vec{a}\right)-\left(\vec{c}\times\vec{c}\right)+\left(\vec{c}\times\vec{a}\right)\right)$$

$$=\vec{a}.\left(\vec{b}\times\vec{c}\right)-\vec{a}.\left(\vec{b}\times\vec{a}\right)-\vec{a}(0)$$

$$+\vec{a}\left(\vec{c}\times\vec{a}\right)-\vec{b}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{b}(0)-\vec{b}\left(\vec{c}\times\vec{a}\right)$$

$$=\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]-\left[\vec{a}\ \ \vec{b}\ \ \vec{a}\right]+\left[\vec{a}\ \ \vec{c}\ \ \vec{a}\right]$$

$$-\left[\vec{b}\ \ \vec{b}\ \ \vec{c}\right]+\left[\vec{b}\ \ \vec{b}\ \ \vec{a}\right]-\left[\vec{b}\ \ \vec{c}\ \ \vec{a}\right]$$

$$=\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]-0+0-0+0-\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]=0$$

$$\therefore \textit{given vectors are coplaner.}$$

12. Find the area of a triangle having the points A (1, 1, 1), B (1, 2, 3), C (2, 3, 1) as its vertices using vector method.

Sol: Let $\overrightarrow{OA}=\hat{i}+\hat{j}+\hat{k},\overrightarrow{OB}=\hat{i}+2\hat{j}+3\hat{k},\overrightarrow{OC}=2\hat{i}+3\hat{j}+\hat{k}$

Now

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\hat{j}+2\hat{k}\ \&\ \overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\hat{i}+2\hat{j}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 2\\ 1 & 2 & 0\end{vmatrix}=-4\hat{i}+2\hat{j}-\hat{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(-4)^2+(2)^2+(-1)^2}=\sqrt{16+4+1}=\sqrt{21}$$

$$\textit{The area of triangle}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{\sqrt{21}}{2}.$$

13. Find the area of a triangle having the points A (1, 1, 2), B (2, 3, 5), C (1, 5, 5) as its vertices using vector method.

Sol: Let

$$\overrightarrow{OA}=\hat{i}+\hat{j}+2\hat{k},\overrightarrow{OB}=2\hat{i}+3\hat{j}+5\hat{k},\overrightarrow{OC}=\hat{i}+5\hat{j}+5\hat{k}$$

Now

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k}\ \&\ \overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=4\hat{j}+3\hat{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 0 & 4 & 3\end{vmatrix}=-6\hat{i}-3\hat{j}+4\hat{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(-6)^2+(-3)^2+(4)^2}=\sqrt{36+9+16}=\sqrt{61}$$

$$\textit{The area of triangle}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{\sqrt{61}}{2}.$$

14. Find the area of the rectangle, whose vertices are

$$A\left(-\hat{i}+\frac{1}{2}j+4k\right), B\left(\hat{i}+\frac{1}{2}j+4k\right),$$
$$C\left(\hat{i}-\frac{1}{2}j+4k\right)\ D\left(-\hat{i}-\frac{1}{2}j+4k\right)$$

Sol: Let

$$\overrightarrow{OA}=-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}, \overrightarrow{OB}=\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},$$
$$\overrightarrow{OC}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}, \overrightarrow{OD}=-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=2\hat{i}\ \ \&\ \ \overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=2\hat{i}-\hat{j}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&0&0\\2&-1&0\end{vmatrix}=-2\hat{k}\Rightarrow\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

$$=\sqrt{(-2)^2}=\sqrt{4}=2$$

$$The\ area\ of\ reactangle=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=2.$$

15. Prove that $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

(2016-J, 2018-J, and 2019-J)

Sol: $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=\left(\vec{a}+\vec{b}\right).\left(\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right)$

$$=\left(\vec{a}+\vec{b}\right).\left(\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{c}\right)+\left(\vec{c}\times\vec{a}\right)\right)$$
$$=\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}(0)+\vec{a}\left(\vec{c}\times\vec{a}\right)$$
$$+\vec{b}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{b}(0)+\vec{b}\left(\vec{c}\times\vec{a}\right)$$
$$=\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]+\left[\vec{a}\ \ \vec{b}\ \ \vec{a}\right]+\left[\vec{a}\ \ \vec{c}\ \ \vec{a}\right]$$
$$+\left[\vec{b}\ \ \vec{b}\ \ \vec{c}\right]+\left[\vec{b}\ \ \vec{b}\ \ \vec{a}\right]+\left[\vec{b}\ \ \vec{c}\ \ \vec{a}\right]$$
$$=\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]+0+0+0+0+\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$$

$=2\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$ *Hence proved*

16. Prove that $\left[\vec{a}\ \vec{b}\ \ \vec{c}+\vec{d}\right]=\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]+\left[\vec{a}\ \vec{b}\ \ \vec{d}\right]$

(2014-M)

Sol: $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}+\vec{d}\right]=\vec{a}.\left(\vec{b}\times\left(\vec{c}+\vec{d}\right)\right)$

$$=\vec{a}.\left(\vec{b}\times\vec{c}+\vec{b}\times\vec{d}\right)$$
$$=\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{d}\right)=\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]+\left[\vec{a}\ \ \vec{b}\ \ \vec{d}\right]$$

Hence proved

17. Show that the points

$$A\left(-2\hat{i}+3\hat{j}+5\hat{k}\right), B\left(\hat{i}+2\hat{j}+3\hat{k}\right)\ and\ C\left(7\hat{i}-\hat{k}\right)$$

are collinear. **(MQP-2021)**

Sol: We have

$$\overrightarrow{AB}=(1+2)\hat{i}+(2-3)\hat{j}+(3-5)\hat{k}=3\hat{i}-\hat{j}-2\hat{k}$$
$$\overrightarrow{BC}=(7-1)\hat{i}+(0-2)\hat{j}+(-1-3)\hat{k}=6\hat{i}-2\hat{j}-4\hat{k}$$
$$\overrightarrow{AC}=(7+2)\hat{i}+(0-3)\hat{j}+(-1-5)\hat{k}=9\hat{i}-3\hat{j}-6\hat{k}$$
$$\left|\overrightarrow{AB}\right|=\sqrt{14}, \left|\overrightarrow{BC}\right|=2\sqrt{14}\ and\ \left|\overrightarrow{AC}\right|=3\sqrt{14}$$
$$\therefore\left|\overrightarrow{AC}\right|=\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|$$

Hence the points A, B and C are collinear

18. Let $\vec{a}, \vec{b}$ *and* $\vec{c}$ are three vectors such that $\left|\vec{a}\right|=3, \left|\vec{b}\right|=4, \left|\vec{c}\right|=5$ and each one of them being perpendicular to the sum of the other, find $\left|\vec{a}+\vec{b}+\vec{c}\right|$. **(MQP-2021)**

Sol: Given

$$\vec{a}.\left(\vec{b}+\vec{c}\right)=0, \vec{b}.\left(\vec{c}+\vec{a}\right)=0, \vec{c}.\left(\vec{a}+\vec{b}\right)=0$$

Now $\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)$

$$=\vec{a}.\vec{a}+\vec{a}.\left(\vec{b}+\vec{c}\right)+\vec{b}.\vec{b}+\vec{b}.\left(\vec{a}+\vec{c}\right)+\vec{c}.\left(\vec{a}+\vec{b}\right)+\vec{c}.\vec{c}$$
$$=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2=9+16+25$$
$$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=50\qquad\therefore\left|\vec{a}+\vec{b}+\vec{c}\right|=\sqrt{50}=5\sqrt{2}.$$

UNIT-11

THREE-DIMENSIONAL GEOMETRY

[Total marks: 11, Q.No − 8(1M), 23(2M), 37(3M) & 47(5M)]

Question No: 8 (1M)

1. Define skew lines.

Sol: Skew lines are lines in space which are neither parallel nor intersecting.

2. If a line makes angle $90^\circ, 60^\circ$ & 30° with positive direction of x, y and z axis respectively, find its direction cosines. **(2014-J, 2018-M)**

Sol: The direction cosines of given line are

$\cos 90^0, \cos 60^0, \cos 30^0 \quad i.e\ 0, \frac{1}{2}, \frac{\sqrt{3}}{2}.$

3. If a line makes angle $90^\circ, 135^\circ$ & 45° with positive direction of x, y and z axis respectively, find its direction cosines. **(2020-M)**

Sol: The direction cosines of given line are

$\cos 90^0, \cos 135^0, \cos 45^0 \quad i.e\ 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}.$

4. Write the direction cosines of X-axis. **(2015-M)**

Sol: The direction cosines of X-axis are $\cos 0^0, \cos 90^0, \cos 90^0 \quad i.e\ 1, 0, 0.$

5. Write the direction cosines of Z-axis. **(2016-J)**

Sol: The direction cosines of Z-axis are $\cos 90^0, \cos 90^0, \cos 0^0 \quad i.e\ 0, 0, 1.$

6. Write the direction cosines of Y-axis. **(2017-J)**

Sol: The direction cosines of Y-axis are $\cos 90^0, \cos 0^0, \cos 90^0 \quad i.e\ 0, 1, 0.$

7. Find the direction cosines of a line which makes equal angles with the positive co-ordinate axis. **(2017-M)**

Sol: Given $\alpha = \beta = \gamma$

$But \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore 3\cos^2\alpha = 1 \Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

$\therefore$ The direction cosines of a line are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

8. Find direction cosines of a line parallel to x-axis.

Sol: The direction cosines of a line parallel to X-axis are 1,0,0

9. If a line has direction ratios 2, -1, -2 then determine its direction cosines. **(2019-M)**

Sol: The direction cosines of given line are

$\pm\frac{2}{\sqrt{4+1+4}}, \pm\frac{-1}{\sqrt{4+1+4}}, \pm\frac{-2}{\sqrt{4+1+4}} \ i.e \pm\frac{2}{3}, \mp\frac{1}{3}, \mp\frac{2}{3}.$

10. Find the direction cosines of the normal to the plane $x + y + z = 1$

Sol: The direction ratios of the normal are 1, 1 & 1

Then direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

11. Find the equation of the plane having intercept 4 on Z-axis and parallel to XOY plane. **(2014-M)**

Sol: Any plane parallel to XOY plane is of the form $z = a$

It passes through $(0, 0, 4)$

$\therefore z = 4$ is required equation of the plane.

12. Find the intercepts cut off by the plane $2x + y - z = 5$ **(2015-J, 2019-J)**

Sol: Intercepts cut by the plane are $\frac{5}{2}, 5$ & -5

13. Find the distance of the point (-6, 0, 0) from the plane $2x - 3y + 6z = 2$ **(2016-M)**

Sol: The distance of point (-6, 0, 0) from the plane $2x-3y+6z=2$ is

$$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|=\left|\frac{2(-6)-3(0)+6(0)-2}{\sqrt{2^2+3^2+6^2}}\right|=\frac{14}{\sqrt{49}}=2\,units$$

14. Find the distance of the plane $2x-3y+4z-6=0$ from the origin. **(2018-M)**

Sol: $$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|=\left|\frac{2(0)-3(0)+4(0)-6}{\sqrt{2^2+3^2+4^2}}\right|=\frac{6}{\sqrt{29}}units$$

15. What is the equation of the plane that cuts co-ordinate axes at (a,0,0),(0, b, 0) & (0, 0, c)

Sol: The equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

16. Find the equation of the plane having intercept 3 on Y-axis and parallel to ZOX plane.

Sol: Any plane parallel to ZOX plane is of the form y=a

It passes through $(0,3,0)$

$\therefore$ Required equation of the plane is $y=3$

17. The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y-4}{7}=\frac{z-6}{2}$. Write its vector form. **(2018-J)**

Sol: Here $\vec{a}=5\hat{i}+4\hat{j}+6\hat{k}$ and $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

Now its vector form is $\vec{r}=\vec{a}+\lambda\vec{b}$

$$\Rightarrow \vec{r}=5\hat{i}-4\hat{j}+6\hat{k}+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

18. The co-ordinates of the foot of the perpendicular drawn from the point (2,5,7) on the x-axis are?

Sol: 2,0,0

19. The distance of a point $P(a,\ b,\ c)$ from x-axis is?

Sol: Let A(a,0,0) point on X-axis

Then distance of a point $P(a,\ b,\ c)$ from x-axis is

$$\sqrt{(a-a)^2+(b-0)^2+(c-0)^2}=\sqrt{b^2+c^2}$$

20. The equation of x-axis in space are?

Sol: $y=0, z=0$

Question No: 23 (2M)

1. Find the angle between the pair of lines $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(\hat{i}+2\hat{j}+2\hat{k}\right)$ & $\vec{r}=5\hat{i}-2\hat{j}+\mu\left(3\hat{i}+2\hat{j}+6\hat{k}\right)$ **(2014-M, 2015-M)**

Sol:

$\vec{b_1}=\hat{i}+2\hat{j}+2\hat{k}$ $\quad\left|\vec{b_1}\right|=\sqrt{1+4+4}=\sqrt{9}=3$

$\vec{b_2}=3\hat{i}+2\hat{j}+6\hat{k}$ $\quad\left|\vec{b_2}\right|=\sqrt{9+4+36}=\sqrt{49}=7$

$\vec{b_1}.\vec{b_2}=3+4+12=19$

$$\cos\theta=\left|\frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}\right|=\frac{19}{3.7}=\frac{19}{21}\Rightarrow\theta=\cos^{-1}\left(\frac{19}{21}\right).$$

2. Find the angle between the pair of lines $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda\left(3\hat{i}+2\hat{j}+6\hat{k}\right)$ & $\vec{r}=7\hat{i}-6\hat{k}+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$ **(2017-J)**

Sol:

$\vec{b_1}=3\hat{i}+2\hat{j}+6\hat{k}$ $\quad\left|\vec{b_1}\right|=\sqrt{9+4+36}=\sqrt{49}=7$

$\vec{b_2}=\hat{i}+2\hat{j}+2\hat{k}$ $\quad\left|\vec{b_2}\right|=\sqrt{1+4+4}=\sqrt{9}=3$

$\vec{b_1}.\vec{b_2}=3+4+12=19$

$$\cos\theta=\left|\frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}\right|=\frac{19}{3.7}=\frac{19}{21}\Rightarrow\theta=\cos^{-1}\left(\frac{19}{21}\right).$$

3. Find the angle between the planes whose vector eq^ns^ are $\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ & $\vec{r}.\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$ **(2018-M)**

Sol:

$\vec{n_1}=2\hat{i}+2\hat{j}-3\hat{k}$ $\quad\left|\vec{n_1}\right|=\sqrt{4+4+9}=\sqrt{17}$

$\vec{n_2}=3\hat{i}-3\hat{j}+5\hat{k}$ $\quad\left|\vec{n_2}\right|=\sqrt{9+9+25}=\sqrt{43}$

$\vec{n_1}.\vec{n_2}=6-6-15=-15$

$$\cos\theta=\left|\frac{\vec{n_1}.\vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|}\right|=\left|\frac{-15}{\sqrt{17}\sqrt{43}}\right|\Rightarrow\theta=\cos^{-1}\left(\frac{15}{\sqrt{731}}\right).$$

4. Find the angle between the two planes $2x+y-2z=5$ & $3x-6y-2z=7$

Sol:

$\vec{N_1}=2\hat{i}+\hat{j}-2\hat{k}$ $\quad |\vec{N_1}|=\sqrt{4+1+4}=\sqrt{9}=3$

$\vec{N_2}=3\hat{i}-6\hat{j}-2\hat{k}$ $\quad |\vec{N_2}|=\sqrt{9+36+4}=\sqrt{49}=7$

$\vec{N_1}.\vec{N_2}=6-6+4=4$

$$\cos\theta=\left|\frac{\vec{N_1}.\vec{N_2}}{|\vec{N_1}||\vec{N_2}|}\right|=\left|\frac{4}{3.7}\right|\Rightarrow\theta=\cos^{-1}\left(\frac{4}{21}\right).$$

5. Find the angle between the line $\frac{x+1}{2}=\frac{y}{2}=\frac{z-3}{6}$ and the plane $10x+2y-11z=3$. **(2018-J)**

Sol:

$\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ $\quad |\vec{b}|=\sqrt{4+9+36}=\sqrt{49}=7$

$\vec{n}=10\hat{i}+2\hat{j}-11\hat{k}$ $\quad |\vec{n}|=\sqrt{100+4+121}=\sqrt{225}=15$

$\vec{b}.\vec{n}=20+6-66=-40$

$$\sin\theta=\left|\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right|=\left|\frac{-40}{7.15}\right|\Rightarrow\theta=\sin^{-1}\left(\frac{8}{21}\right).$$

6. Find the angle between the pair of lines $\frac{x+3}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ & $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$ **(2019-J)**

Sol:

$\vec{b_1}=3\hat{i}+5\hat{j}+4\hat{k}$ $\quad |\vec{b_1}|=\sqrt{9+25+16}=\sqrt{50}$

$\vec{b_2}=\hat{i}+\hat{j}+2\hat{k}$ $\quad |\vec{b_2}|=\sqrt{1+1+4}=\sqrt{6}$

$\vec{b_1}.\vec{b_2}=3+5+8=16$

$$\cos\theta=\left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|=\frac{16}{\sqrt{50}.\sqrt{6}}\Rightarrow\theta=\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right).$$

7. If the co-ordinates of the points A, B, C & D be $(1,2,3),(4,5,7),(-4,3,-6)\&(2,9,2)$ respectively, then find the angle between the lines AB & CD.

Sol: The direction ratios of AB are

4-1,5-2,7-3 i.e.,3,3,4

The direction ratios of CD are

2+4, 9-3, 2+6 i.e.,6,6,8 i.e.,3,3,4

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{9+9+16}{\sqrt{9+9+16}\sqrt{9+9+16}}=1$$

$\therefore\theta=0\,(AB\parallel CD)$

8. Find the vector equation the line passing through the points $(-1,0,2)$ and $(3,4,6)$ **(2015-J)**

Sol:

$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}=-\hat{i}+0\hat{j}+2\hat{k}$

$\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}=3\hat{i}+4\hat{j}+6\hat{k}$

$\vec{b}-\vec{a}=4\hat{i}+4\hat{j}+4\hat{k}$

Vector equation of a line $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

$\vec{r}=\left(-\hat{i}+0\hat{j}+2\hat{k}\right)+\lambda\left(4\hat{i}+4\hat{j}+4\hat{k}\right).$

9. Find the vector equation the line passing through the points $(3,-2,-5)$ & $(3,-2,6)$. **(2017-M)**

Sol:

$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}=3\hat{i}-2\hat{j}-5\hat{k}$

$\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}=3\hat{i}-2\hat{j}+6\hat{k}$

$\vec{b}-\vec{a}=0\hat{i}+0\hat{j}+11\hat{k}$

Vector equation of a line $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

$\vec{r}=\left(3\hat{i}-2\hat{j}-5\hat{k}\right)+\lambda\left(0\hat{i}+0\hat{j}+11\hat{k}\right).$

10. Find the Cartesian equation of the line parallel to $y-axis$ and passing through the point (1,1,1) **(2016-M)**

Sol:

$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}=\hat{i}+\hat{j}+\hat{k}$

$\vec{b}=a\hat{i}+b\hat{j}+c\hat{k}=0\hat{i}+1\hat{j}+0\hat{k}$ $\left(\because\|^{el}\ to\,Y-axis\right)$

Cartesian equation of a line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

$$\frac{x-1}{0}=\frac{y-1}{1}=\frac{z-1}{0}.$$

11. Find the equation of the plane through the intersection of planes $3x-y+2z-4=0$ & $x+y-z-2=0$ and the point (2,2,1). **(2014-J)**

Sol: The equation of the plane passing through the intersection of two planes

$$(3x-y+2z-4)+\lambda(x+y+z-2)=0----(1)$$

Equation (1) is passing through the point (2,2,1)

$$(3(2)-(2)+2(1)-4)+\lambda((2)+(2)+(1)-2)=0$$

$$\Rightarrow 2+3\lambda=0 \Rightarrow \lambda=-\frac{2}{3}$$

Substituting λ *in equation* (1)

$$(3x-y+2z-4)-\frac{2}{3}(x+y+z-2)=0$$

$$9x-3y+6z-12-2x-2y-2z+4=0$$

$$7x-5y+4z-8=0$$

12. Show that the lines $\frac{x-5}{7}=\frac{y-2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other. **(2016-J)**

Sol: $\vec{b_1}=7\hat{i}-5\hat{j}+\hat{k}$, $\vec{b_2}=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{b_1}.\vec{b_2}=7-10+3=0$

$\therefore$ *Given two lines are perpendicular to each other.*

13. Find the distance of the point $(3,-2,1)$ from the plane $2x-y+2z+3=0$. **(2019-M)**

Sol: The distance of the point $(3,-2,1)$ from the plane $2x-y+2z+3=0$ is

$$d=\left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|=\left|\frac{2(3)-(-2)+2(1)+3}{\sqrt{4+1+4}}\right|=\frac{13}{3}$$

14. Show that the lines

$$\frac{x+3}{-3}=\frac{y-1}{1}=\frac{z-5}{5} \;\&\; \frac{x+1}{-1}=\frac{y-2}{2}=\frac{z-5}{5}$$

are coplanar.

Sol:

Here $x_1=-3, y_1=1, z_1=5; a_1=-3, b_1=1, c_1=5$

$x_2=-1, y_2=2, z_2=5; a_2=-1, b_2=2, c_2=5$

$$Consider \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}=0$$

$\therefore$ *Given lines are coplanar.*

15. Find the equation of the plane with intercepts 2,3 and 4 on the X, Y and Z-axis respectively. **(2020-M)**

Sol: Required equation of a plane is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$$

Question No: 37 (3M)

1. Find the distance of a point $(2,5,-3)$ from the plane $\vec{r}.\left(6\hat{i}-3\hat{j}+2\hat{k}\right)=4$ **(2014-M)**

Sol:

WKT $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ *and point* $(2,5,-3)$

Given $\vec{r}.\left(6\hat{i}-3\hat{j}+2\hat{k}\right)=4$

$$\Rightarrow\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(6\hat{i}-3\hat{j}+2\hat{k}\right)=4$$

$$\Rightarrow 6x-3y+2z=4 \Rightarrow 6x-3y+2z-4=0$$

$$d=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|=\left|\frac{6(2)-3(5)+2(-3)-4}{\sqrt{6^2+3^2+2^2}}\right|$$

$$=\frac{13}{\sqrt{49}}=\frac{13}{7} \text{ units.}$$

2. Find the distance of a point $(2,5,-7)$ from the plane $\vec{r}.\left(\hat{i}-2\hat{j}-2\hat{k}\right)=9$

Sol:

WKT $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ *and point* $(2,5,-7)$

Given $\vec{r}.\left(\hat{i}-2\hat{j}-2\hat{k}\right)=9$

$$\Rightarrow\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(\hat{i}-2\hat{j}-2\hat{k}\right)=9$$

$$\Rightarrow x-2y-2z=9 \Rightarrow x-2y-2z-9=0$$

$$d=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|=\left|\frac{1(2)-2(5)-2(-7)-9}{\sqrt{1^2+(-2)^2+(-2)^2}}\right|$$

$$=\frac{4}{\sqrt{9}}=\frac{4}{3} \text{ units.}$$

3. Find the shortest distance between the lines L_1 and L_2 whose vector equations are $\vec{r} = \left(\hat{i}+\hat{j}\right)+\lambda\left(2\hat{i}-\hat{j}+\hat{k}\right)$ and

$\vec{r} = 2\hat{i}+\hat{j}-\hat{k}+\mu\left(3\hat{i}-5\hat{j}+2\hat{k}\right)$ **(2014-J, 2019-J)**

Sol: Given lines compare with

$$\vec{r} = \vec{a_1}+\lambda\vec{b_1} \text{ and } \vec{r} = \vec{a_2}+\lambda\vec{b_2}$$

$$\text{We have } \vec{a_1} = \hat{i}+\hat{j}+0\hat{k}\ ,\ \vec{b_1} = 2\hat{i}-\hat{j}+\hat{k}$$

$$\vec{a_2} = 2\hat{i}+\hat{j}-\hat{k}\ ,\ \vec{b_2} = 3\hat{i}-5\hat{j}+2\hat{k}$$

$$\vec{a_2}-\vec{a_1} = \hat{i}+0\hat{j}-\hat{k}$$

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= \hat{i}(-2+5)-\hat{j}(4-3)+\hat{k}(-10+3) = 3\hat{i}-\hat{j}-7\hat{k}$$

$$\left|\vec{b_1}\times\vec{b_2}\right| = \sqrt{9+1+49} = \sqrt{59}$$

$$Now\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right) = 3+0+7 = 10$$

$$\text{Shortest distance } d = \left|\frac{\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right| = \frac{10}{\sqrt{59}} \text{ units}$$

4. Find the shortest distance between the lines L_1 and L_2 whose vector equations are $\vec{r} = \left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$ and

$\vec{r} = 3\hat{i}+3\hat{j}-5\hat{k}+\mu\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$ **(15-J, 2016-M)**

Sol: Given lines compare with

$$\vec{r} = \vec{a_1}+\lambda\vec{b} \text{ and } \vec{r} = \vec{a_2}+\lambda\vec{b}$$

$$\text{We have } \vec{a_1} = \hat{i}+2\hat{j}-4\hat{k}\ ,\ \vec{a_2} = 3\hat{i}+3\hat{j}-5\hat{k}\ ,\ \vec{b} = 2\hat{i}+3\hat{j}+6\hat{k}$$

$$\vec{a_2}-\vec{a_1} = 2\hat{i}+\hat{j}-\hat{k} \text{ and } \left|\vec{b}\right| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$\vec{b}\times\left(\vec{a_2}-\vec{a_1}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-3-6)-\hat{j}(-2-12)+\hat{k}(2-6) = -9\hat{i}+14\hat{j}-4\hat{k}$$

$$\left|\vec{b}\times\left(\vec{a_2}-\vec{a_1}\right)\right| = \sqrt{81+196+16} = \sqrt{293}$$

$$\text{Shortest distance } d = \left|\frac{\vec{b}\times\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b}\right|}\right| = \frac{\sqrt{293}}{7} \text{ units}$$

5. Find the shortest distance between the lines $\vec{r} = \left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and

$\vec{r} = \left(2\hat{i}-\hat{j}-\hat{k}\right)+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$. **(2017-M)**

Sol: Given lines compare with

$$\vec{r} = \vec{a_1}+\lambda\vec{b_1} \text{ and } \vec{r} = \vec{a_2}+\lambda\vec{b_2}$$

$$\text{We have } \vec{a_1} = \hat{i}+2\hat{j}+\hat{k}\ ,\ \vec{b_1} = \hat{i}-\hat{j}+\hat{k}$$

$$\vec{a_2} = 2\hat{i}-\hat{j}-\hat{k}\ ,\ \vec{b_2} = 2\hat{i}+\hat{j}+2\hat{k}$$

$$\vec{a_2}-\vec{a_1} = \hat{i}-3\hat{j}-2\hat{k}$$

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= \hat{i}(-2-1)-\hat{j}(2-2)+\hat{k}(1+2) = -3\hat{i}-0\hat{j}+3\hat{k}$$

$$\left|\vec{b_1}\times\vec{b_2}\right| = \sqrt{9+0+9} = \sqrt{18} = 3\sqrt{2}$$

$$Now\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right) = -3+0-6 = -9$$

$$S.D\ \ d = \left|\frac{\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right| = \left|\frac{-9}{3\sqrt{2}}\right| = \frac{3}{\sqrt{2}} \text{ units}$$

6. Find the equation of the line passing through the points (−1, 0 ,2) and (3,4,6) in both vector and Cartesian forms. **(2016-J)**

Sol:

$$\vec{a} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k} = -\hat{i}+0\hat{j}+2\hat{k}$$

$$\vec{b} = x_2\hat{i}+y_2\hat{j}+z_2\hat{k} = 3\hat{i}+4\hat{j}+6\hat{k}$$

$$\vec{b}-\vec{a} = 4\hat{i}+4\hat{j}+4\hat{k}$$

$$\text{Vector equation of a line } \vec{r} = \vec{a}+\lambda\left(\vec{b}-\vec{a}\right)$$

$$\vec{r} = \left(-\hat{i}+0\hat{j}+2\hat{k}\right)+\lambda\left(4\hat{i}+4\hat{j}+4\hat{k}\right)$$

Cartesian equation of a line

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x+1}{3+1}=\frac{y-0}{4-0}=\frac{z-2}{6-2}$$

$$\frac{x+1}{4}=\frac{y}{4}=\frac{z-2}{4}$$

7. Find the Cartesian and vector equation of the line that passes through the points (3, -2, -5) and (3, -2, 6)

Sol:

$$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}=3\hat{i}-2\hat{j}-5\hat{k}$$

$$\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}=3\hat{i}-2\hat{j}+6\hat{k}$$

$$\vec{b}-\vec{a}=0\hat{i}+0\hat{j}+11\hat{k}$$

Vector equation of a line $\vec{r}=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right)$

$$\vec{r}=\left(3\hat{i}-2\hat{j}-5\hat{k}\right)+\lambda\left(0\hat{i}+0\hat{j}+11\hat{k}\right)$$

Cartesian equation of a line

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x-3}{3-3}=\frac{y+2}{-2+2}=\frac{z+5}{6+5}$$

$$\frac{x+1}{0}=\frac{y+2}{0}=\frac{z+5}{11}$$

8. Find the vector and Cartesian equations of the plane which passes through the point (5, 2, -4) and perpendicular to the line with direction ratios 2, 3, -1 (**2017-J)**

Sol:

$$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}=5\hat{i}+2\hat{j}-4\hat{k}$$

$$\vec{N}=A\hat{i}+B\hat{j}+C\hat{k}=2\hat{i}+3\hat{j}-\hat{k}$$

Vector equation of a line $\left(\vec{r}-\vec{a}\right).\vec{N}=0$

$$\left(\vec{r}-\left(5\hat{i}+2\hat{j}-4\hat{k}\right)\right).\left(2\hat{i}+3\hat{j}-\hat{k}\right)=0$$

Cartesian equation of a line

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

$$2(x-5)+3(y-2)-1(z+4)=0$$

$$2x-10+3y-6-z-4=0$$

$$2x+3y-z=20.$$

9. Find the equation of the plane passing through the intersection of the planes $3x-y+2z-4=0$ and $x+y+z-2=0$ and point $(2,2,1)$

(2015-M, 2018-M, 2018-J, 2020-M)

Sol: Equation of a plane passing through the intersection of two planes

$$(3x-y+2z-4)+\lambda(x+y+z-2)=0-----(1)$$

Equation passing through the point (2,2,1)

$$(3(2)-(2)+2(1)-4)+\lambda(2+2+1-2)=0$$

$$2+3\lambda=0\Rightarrow\lambda=-\frac{2}{3}$$

Substituting the value of λ in equation (1)

$$(3x-y+2z-4)-\frac{2}{3}(x+y+z-2)=0$$

$$9x-3y+6z-12-2x-2y-2z+4=0$$

$$7x-5y+4z-8=0$$

10. Find the equation of plane passing through the intersection of planes $x+y+x=1$ & $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$

Sol: Equation of a plane passing through the intersection of two planes

$$(x+y+z-1)+\lambda(2x+3y+4z-5)=0-----(1)$$

$$(1+2\lambda)x+(1+3\lambda)y+(1+4\lambda)z+(-1-5\lambda)=0$$

Equation (1) is perpendicular to $x-y+z=0$

$$\therefore(1+2\lambda)-1(1+3\lambda)+(1)(1+4\lambda)=0$$

$$1+2\lambda-1-3\lambda+1+4\lambda=0$$

$$1+3\lambda=0\Rightarrow\lambda=-\frac{1}{3}$$

Substituting the value of λ in equation (1)

$$(x+y+z-1)-\frac{1}{3}(2x+3y+4z-5)=0$$

$$3x+3y+3z-3-2x-3y-4z+5=0$$

$$x-z+2=0.$$

11. Find the vector equation of the plane passing through the points $R(2,5,-3), S(-2,-3,5) \& T(5,3,-3)$

Sol: $\vec{a} = 2\hat{i}+5\hat{j}-3\hat{k}$

$\vec{b} = -2\hat{i}-3\hat{j}+5\hat{k}, \vec{c} = 5\hat{i}+3\hat{j}-3\hat{k}$

Now $\vec{b}-\vec{a} = -4\hat{i}-8\hat{j}+8\hat{k}$ & $\vec{c}-\vec{a} = 3\hat{i}-2\hat{j}$

Vector equation of plane passing through the three non collinear points

$$(\vec{r}-\vec{a}).\left[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right]=0$$

$$\left(\vec{r}-\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\right).\left[\left(-4\hat{i}-8\hat{j}+8\hat{k}\right)\times\left(3\hat{i}-2\hat{j}\right)\right]=0.$$

12. Find the shortest distance between the lines

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \;\&\; \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

Sol: *We have* $\vec{a_1} = -\hat{i}-\hat{j}-\hat{k}$, $\vec{b_1} = 7\hat{i}-6\hat{j}+\hat{k}$

$\vec{a_2} = 3\hat{i}+5\hat{j}+7\hat{k}$, $\vec{b_2} = \hat{i}-2\hat{j}+\hat{k}$

$\vec{a_2}-\vec{a_1} = 4\hat{i}+6\hat{j}+8\hat{k}$

$$\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -4\hat{i}-6\hat{j}-8\hat{k}$$

$$\left|\vec{b_1}\times\vec{b_2}\right| = 2\sqrt{4+9+16} = 2\sqrt{29}$$

$$Now\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right) = -16-36-64 = -116$$

$$Shortest\ distance\quad d = \left|\frac{\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right|$$

$$= \left|\frac{-116}{2\sqrt{29}}\right| = \frac{58}{\sqrt{29}} = \frac{2\sqrt{29}\sqrt{29}}{\sqrt{29}} = 2\sqrt{29}\,units$$

Question No: 47 (5M)

1. Derive the equation of a line in a space through a given point and parallel to a vector both in the vector and Cartesian form. **(2014-J, 2015-M, 2019-M, 2020-M)**

Sol: Let A be a given point whose position vector $\vec{a}$ and $\vec{b}$ be given vector.

Let 'l' be the line passing through the point A and is parallel to a given vector $\vec{b}$

Let 'P' be any point on the line 'l' whose position vector is $\vec{r}$

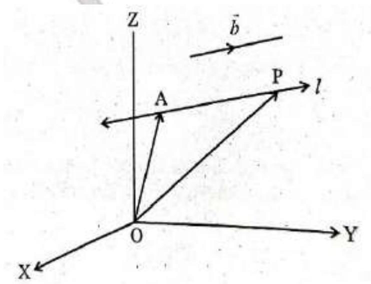
Then $\overrightarrow{AP}$ is parallel to the vector $\vec{b}$.

$\overrightarrow{AP} = \lambda\vec{b}$, where λ is a real number.

$\overrightarrow{OP}-\overrightarrow{OA} = \lambda\vec{b}$

$\vec{r}-\vec{a} = \lambda\vec{b}$

$\therefore \vec{r} = \vec{a}+\lambda\vec{b} --(1)$



Which is the required vector equation of the line.

Cartesian form: Let (x_1, y_1, z_1) be the co-ordinates of the point A. Let a, b, c be the direction ratios of the given line $\vec{b}$. Let $P(x,y,z)$ be a point on the required line.

$\therefore\ \vec{r} = x\hat{i}+y\,j+zk$ $\quad \vec{a} = x_1\hat{i}+y_1\,j+z_1k$, $\quad \vec{b} = a\hat{i}+b\,j+ck$

Substitute these values in $\vec{r} = \vec{a}+\lambda\vec{b}$ and equating the co-efficient of $\hat{i}, j, k$ we get

$x = x_1+\lambda a \quad \Rightarrow x-x_1 = \lambda a$

$y = y_1+\lambda b \quad \Rightarrow y-y_1 = \lambda b$

$z = z_1+\lambda c \quad \Rightarrow z-z_1 = \lambda c$

Equating the parameter λ , we get

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$, is the required equation.

2. Derive the equation of a line in a space through two given point both in the vector and Cartesian form.

(2017-J, 2018-M, 2018-J)

Sol: Let $\vec{a}$ and $\vec{b}$ be the position vector of two points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ respectively that are lying in a line.

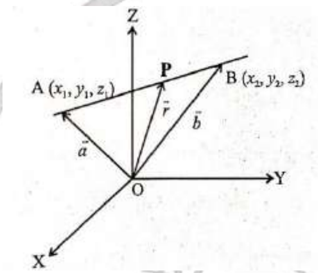
Let $\vec{r}$ be the position vector of an arbitrary point $P(x, y, z)$

Now, P lies on AB iff

$\overrightarrow{AP} = \vec{r} - \vec{a}$ & $\overrightarrow{AB} = \vec{b} - \vec{a}$ are collinear vectors.

$\Rightarrow \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\therefore \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$



Which is the required vector equation of the line.

Cartesian form:

We have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Equation of a line in vector form is

$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

Equating the co-efficient of $\hat{i}, \hat{j}$ and $\hat{k}$ we get,

$$x = x_1 + \lambda(x_2 - x_1)$$

$$y = y_1 + \lambda(y_2 - y_1)$$

$$z = z_1 + \lambda(z_2 - z_1)$$

On eliminating λ, we get

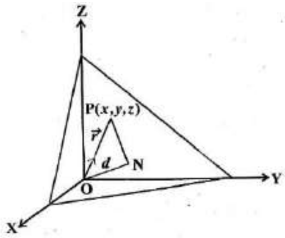
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Which is the cartesian form of the equation of a line.

3. Derive the equation of a plane in normal form both in the vector and Cartesian form. **(2014-M, 2016-J, 2019-J)**

Sol: Consider a plane whose perpendicular distance from the origin is d$(d \neq 0)$.

If $\overrightarrow{ON}$ is the normal from the origin to the plane and $\hat{n}$ is the unit normal vector along $\overrightarrow{ON}$, then $\overrightarrow{ON} = d\hat{n}$.



Let P be any point on the plane.

Therefore $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$.

Therefore $\overrightarrow{NP}.\overrightarrow{ON} = 0 ----(1)$

Let $\vec{r}$ be the position vector of the point P.

Then $\overrightarrow{NP} = \vec{r} - d\hat{n}$, now equation (1) becomes

$(\vec{r} - d\hat{n}).d\hat{n} = 0$

$(\vec{r} - d\hat{n}).\hat{n} = 0 \qquad (\because d \neq 0)$

$\vec{r}.\hat{n} - d(\hat{n}.\hat{n}) = 0$

$\vec{r}.\hat{n} - d(1) = 0 \Rightarrow \vec{r}.\hat{n} = d ---(2) \qquad (\because \hat{n}.\hat{n} = 1)$

This is the vector form of the equation of the plane.

Cartesian form:

Let $P(x, y, z)$ be any point on the plane.

$\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Let l, m, n be the direction cosines of $\hat{n}$

then $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$

Equation (2) becomes

$(x\hat{i} + y\hat{j} + z\hat{k}).(l\hat{i} + m\hat{j} + n\hat{k}) = d$

$lx + my + nz = d ---(2)$

This is the cartesian equation of the plane in the normal form.

4. Derive the equation of a plane perpendicular to a given vector and passing through a given point in both vector form and Cartesian form.

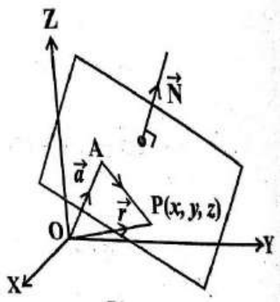
(2015-J, 2016-M, 2017-M)

Sol: Let a plane pass through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$.

Let $\vec{r}$ be the position vector of any point $p(x, y, z)$ in the plane.

Then the point P lies in the plane iff $\overrightarrow{AP}$ is perpendicular to $\vec{N}$.

i.e., $\overrightarrow{AP}.\vec{N} = 0$



But

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\Rightarrow \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$

$\therefore (\vec{r} - \vec{a}).\vec{N} = 0 --- (1)$

This is vector equation of the plane.

Cartesian plane:

Let the given point A be (x_1, y_1, z_1), P be (x, y, z) and direction ratios of $\vec{N}$ are $A, B \& C$, then

$\vec{a} = x_1\hat{i} + y_1 j + z_1 k$

$\vec{r} = x\hat{i} + y j + zk$

$\& \vec{N} = A\hat{i} + B j + Ck$

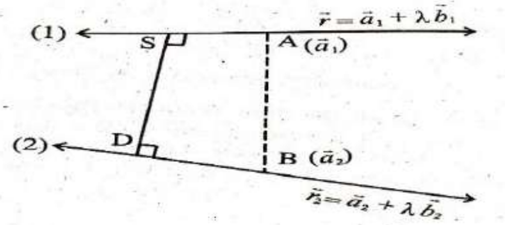
Now, $(\vec{r} - \vec{a}).\vec{N} = 0$

$\left[(x - x_1)\hat{i} + (y - y_1) j + (z - z_1)k\right].(A\hat{i} + B j + Ck) = 0$

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

5. Derive the formula to find the shortest distance between the two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ in vector form.

Sol:



Line $\vec{r} = \vec{a_1} + \lambda\vec{b_1} ---- (1)$

Passes through $A(\vec{a_1})$ and parallel to $\vec{b_1}$

Line $\vec{r} = \vec{a_2} + \mu\vec{b_2} ---- (2)$

Passes through $B(\vec{a_2})$ and parallel to $\vec{b_2}$

Let SD be the shortest distance between the line (1) and (2)

$\Rightarrow \overrightarrow{SD}$ is $\perp^r$ line (1) and line (2)

$\Rightarrow \overrightarrow{SD}$ *is* $\perp^r$ *to both* $\vec{b}_1$ *and* $\vec{b_2}$

$\therefore \overrightarrow{SD}$ is parallel to $\vec{b_1} \times \vec{b_2}$

$\Rightarrow \overrightarrow{SD} = |\overrightarrow{SD}|$ unit vector

$= |\overrightarrow{SD}| \dfrac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|}$

$|\overrightarrow{SD}|$ projection of

$\overrightarrow{AB}$ *on* $\overrightarrow{SD} = (\vec{a_2} - \vec{a_1})$.*unit vector of* $\overrightarrow{SD}$

$= (\vec{a_2} - \vec{a_1}).\dfrac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|}$

$\therefore SD = \dfrac{(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}$

Which is the required vector equation.

Cartesian form:

The shortest distance between the lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}----(1)$$

$$and \quad \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}----(2)$$

Is

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(b_1c_2-b_2c_2\right)^2+\left(c_1a_2-a_1c_2\right)^2+\left(a_1b_2-b_1a_2\right)^2}}\right|$$

UNIT-12

LINEAR PROGRAMING PROBLEM

$\left[Total\ marks:7,\ Q.No-9(1M)\&49.a(6M)\right]$

Question No: 9 (1M)

1. Define Linear Programming Problems.

Ans: A linear programming consists of a linear function to be maximized or minimized subject to certain constraints in the form of linear equations or inequalities

2. Define feasible region.

(2014-M, 2015-M, 2016-M, 2017-M, 2019-J)

Sol: The common region determined by all the constraints including the non-negative constraints $x \geq 0, y \geq 0$ of LPP is called Feasible region.

3. In a linear programming problem. Define objective function. **(2014-J, 2019-M)**

Sol: Linear function $Z = ax + by$,where a & b are constants and x and y are decision variables, which has to be maximised or minimised is called a objective function.

4. Define optimal solution in LPP.

(2015-J, 2016-J, 2017-J & 2018-M, 2018-J, 2020-M)

Sol: A feasible solution which leads to maximise or minimise the objective function is called optimal solution in LPP.

5. Define optimization problem in LPP.

Sol: A problem which seeks to maximise or minimise profit or cost for a general class of problems called optimisation problem in LPP.

6. Define decision variables in LPP.

Sol: The variables x and y in objective function $Z = ax + by$ in LPP are called decision variable in LPP

7. Define the term constraints in the LLP.

Sol: The linear inequalities or equations or restrictions on the variables of LPP are called constraints in LPP.

8.Define the term corner points in the LLP.

Sol: A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.

Question No: 49.a (6M)

1. Minimise & maximise $Z = -3x + 4y$

Subjected to the constraints

$x + 2y \leq 8$, & $3x + 2y \leq 12$ $\quad x, y \geq 0$

by graphical method. **(2017-J)**

Sol: (i)

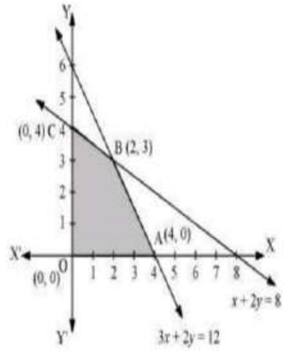
Conversion:

$x + 2y = 8$

X	0	8
Y	4	0

$3x + 2y = 12$

x	0	4
y	6	0



(ii) Shading side:

$x + 2y \leq 8$	$3x + 2y \leq 12$
put $(0,0), 0 + 2(0) \leq 8$	put $(0,0), 3(0) + 2(0) \leq 12$
true, so shading side including (0,0).	true, so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O(0,0), A(4,0),B(2,3),C(0,4)

(iv) Optimal solution:

Corner points	Z= $-3x + 4y$	
O(0,0)	0	
A(4,0)	-12	Minimum
B(2,3)	6	
C(0,4)	16	Maximum

The minimum value is -12 at the point A (4,0) and maximum value is 16 at the point C (0,4).

2. Maximise $Z = 3x + 2y$ Subjected to constraints $x + 2y \le 10$, $3x + y \le 15$ & $x \ge 0, y \ge 0$ by graphical method.

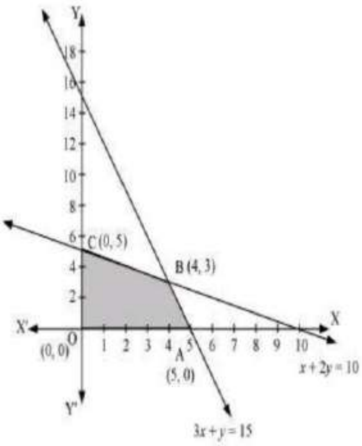
Sol: (i) Conversion:

$x + 2y = 10$

X	0	10
Y	5	0

$3x + y = 15$

x	0	5
y	15	0



(ii) Shading side:

$x + 2y \le 10$	$3x + y \le 15$
put (0,0), $0 + 2(0) \le 10$	put (0,0), $3(0) + (0) \le 15$
true, so shading side including (0,0).	true, so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O (0,0), A (5,0), B (4,3), C (0,5)

(iv) Optimal solution:

Corner points	Z= $3x + 2y$	
O(0,0)	0	
A(5,0)	15	
B(4,3)	18	Maximum
C(0,5)	10	

The maximum value is 18 at the point B (4,3).

3. Solve the following problem graphically: Maximise and minimise $Z = 10500x + 9000y$ subject to the constraints $x + y \le 50$, $2x + y \le 80$ $x \ge 0, y \ge 0$ **(2018-J)**

Sol: (i) Conversion:

$x + y = 50$

x	0	50
y	50	0

$2x + y = 80$

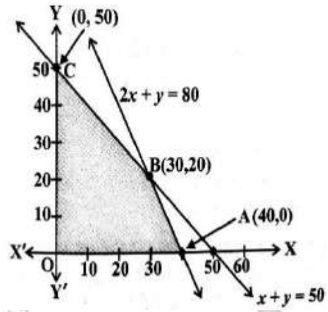
x	0	40
y	80	0

(ii) Shading side:

$x + y \le 50$	$2x + y \le 80$
put (0,0), $0 + 0 \le 50$	put (0,0), $2(0) + 0 \le 80$
true, so shading side including (0,0).	true, so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O (0,0), A (40,0), B (30,20), C (0,50)



(iv) Optimal solution:

Corner points	Z= $10500x + 9000y$	
O (0,0)	0	
A (40,0)	420000	
B (30,20)	495000	Maximum
C (0,50)	450000	

The maximum value is 495000 at the point B (30,20).

4. Minimise and maximise $z = 200x + 500y$ subjected to the constraints $x + 2y \ge 10$, $3x + 4y \le 24$, $x, y \ge 0$ by the graphical method.

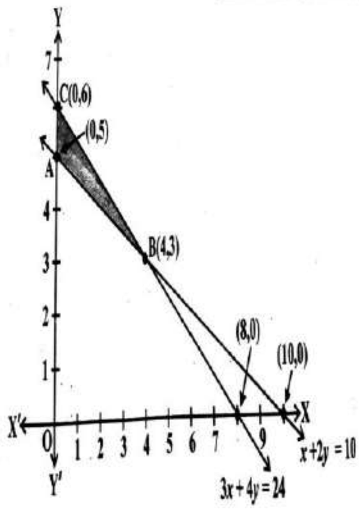
Sol: (i) Conversion:

$x + 2y = 10$

x	0	10
y	5	0

$3x + 4y = 24$

x	0	8
y	6	0



(ii) Shading side:

$x+2y\ge 10$	$3x+4y\le 24$
put (0,0), $0+2(0)\ge 10$	put (0,0) $3(0)+4(0)\le 24$
false, so shading side excluding (0,0).	true, so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are A (0,5), B (4,3), C (0,6)

(iv) Optimal solution:

Corner points	Z= $200x+500y$	
A(0,5)	2500	
B(4,3)	2300	Minimum
C(0,6)	3000	Maximum

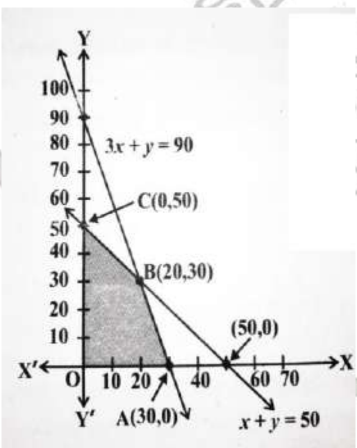
The minimum value is 2300 at the point B (4,3) and maximum value is 3000 at the point C (0,6).

5. Maximise $z=4x+y$ subject to constraints $x+y\le 50,\ 3x+y\le 90,\ x\ge 0,\ y\ge 0$ by graphical method. **(2020-M)**

Sol: (i) Conversion:

$x+y=50$

x	0	50
y	50	0



$3x+y=90$

x	0	30
y	90	0

(ii) Shading side:

$x+y\le 50$	$3x+y\le 90$
put (0,0), $0+0\le 50$	put (0,0), $3(0)+0\le 90$
true, so shading side including (0,0).	true, so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O (0,0), A (30,0), B (20,30), C (0,50)

(iv) Optimal solution:

Corner points	Z= $4x+y$	
O(0,0)	0	
A(30,0)	120	Maximum
B(20,30)	110	
C(0,50)	50	

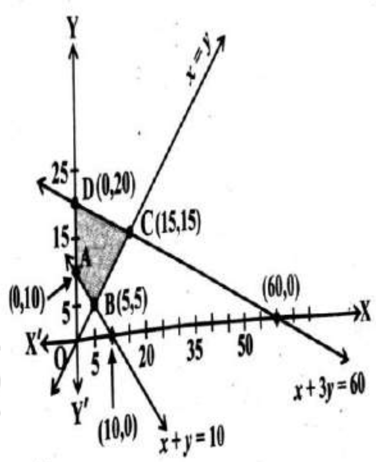
The maximum value is 120 at the point A (30,0).

6. Minimise & maximise $Z=3x+9y$ Subjected to the constraints $x+3y\le 60$, $x+y\ge 10$ & $x\le y$ $x,y\ge 0$ by graphical method. **(2016-J, 2018-M)**

Sol: (i) Conversion:

$x+3y=60$

x	0	60
y	20	0



$x+y=10$

x	0	10
y	10	0

$x-y=0$

(ii) Shading side:

$x+3y\le 60$

put (0,0), $0+3(0)\le 60$

true, so shading side including (0,0).

$x+y\ge 10$

put (0,0), $0+0\ge 10$

false, so shading side excluding (0,0).

$x-y\le 0$

put (1,0), $1-0\le 0$

false, so shading side excluding (1,0).

(iii) Corner points:

The corner points of feasible region are A (0,10), B (5,5), C (15,15), D (0,20)

(iv) Optimal solution:

Corner points	Z= $3x+9y$	
A(0,10)	90	
B(5,5)	60	Minimum
C(15,15)	180	Maximum
D(0,20)	180	Maximum

The Minimum value is 60 at the point B (5,5) and Maximum value is 180 at the two points C (15,15) and D (0, 20).

7. Minimise and Maximise $Z=5x+10y$ Subjected to constraints $x+2y\le 120$, $x+y\ge 60$ & $x-2y\ge 0$ $x\ge 0, y\ge 0$ by graphical method. **(2016-M, 2019-M)**

Sol:

(i) Conversion:

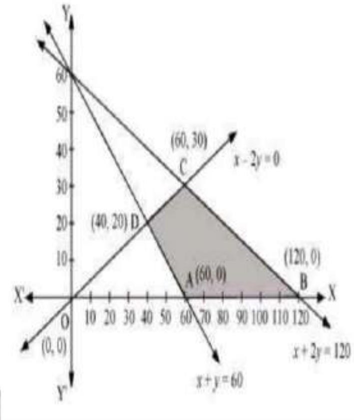
$x+2y=120$

X	0	120
Y	60	0

$x+y=60$

x	0	60
y	60	0

$x-2y=0$



(ii) Shading side:

$x+2y\le 120$

put (0,0), $0+2(0)\le 120$

true, so shading side including (0,0).

$x+y\ge 60$

put (0,0), $0+0\ge 60$

false, so shading side excluding (0,0).

$x-2y\ge 0$

put (1,0), $1-2(0)\ge 0$ true,

so shading side including (1,0).

(iii) Corner points:

The corner points of feasible region are A (60,0), B (120,0), C (60,30), D (40,20)

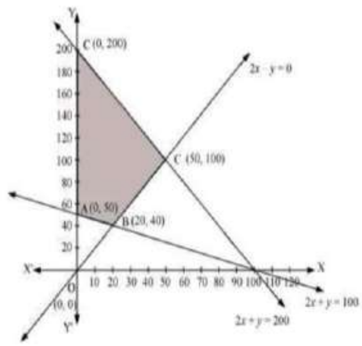
(iv) Optimal solution:

Corner points	Z= $5x+10y$	
A(60,0)	300	Minimum
B(120,0)	600	Maximum
C(60,30)	600	Maximum
D(40,20)	400	

The Minimum value is 300 at the point A(60,0) and Maximum value is 180 at the line segment joining two points B(120,0) and C(60,30).

8. Maximise and Minimise $Z=x+2y$ Subjected to constraints $x+2y\ge 100$, $2x-y\le 0$ & $2x+y\le 200$ $x, y\ge 0$ by graphical method. **(2014-M, 2014-J)**

Sol: (i) Conversion:



$x+2y=100$

$2x-y=0$

x	0	100
y	50	0

$2x+y=200$

x	0	100
y	200	0

(ii) Shading side:

$x+2y\ge 100$

put (0,0), $0+2(0)\ge 100$

false, so shading side excluding (0,0).

$2x-y\le 0$

put (1,0), $2(1)-0\le 0$

false, so shading side excluding (1,0).

$2x+y\le 200$

put (0,0), $2(0)+0\le 200$

true, so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are A (0,50), B (20,40), C (50,100), D (0,200)

(iv) Optimal solution:

Corner points	Z= $x+2y$	
A(0,50)	100	Minimum
B(20,40)	100	
C(50,100)	250	
D(0,200)	400	Maximum

The Minimum value is 100 at the line segment joining two points A(0,50) and B(20,40) and Maximum value is 400 at the point D(0,200).

9. Minimise $Z=600x+400y$ Subjected to the constraints $x+2y\le 12$, $2x+y\le 12$ & $4x+5y\ge 20$ $x,y\ge 0$ by graphical method **(2017-M)**

Sol: (i) Conversion:

$x+2y=12$

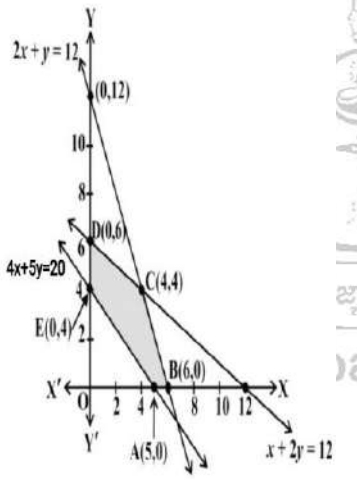
x	0	12
y	6	0

$2x+y=12$

x	0	6
y	12	0

$4x+5y=20$

x	0	5
y	4	0



(ii) Shading side:

$x+2y\le 12$

put (0,0), $0+2(0)\le 12$

true, so shading side including (0,0).

$2x+y\le 12$

put (0,0), $2(0)+0\le 12$

true,so shading side including (0,0).

$4x+5y\ge 20$

put (0,0), $4(0)+5(0)\ge 20$

false, so shading side excluding (0,0).

(iii) Corner points:

The corner points of feasible region are A (5,0),B(6,0),C(4,4),D(0,6),E(0,4).

(iv) Optimal solution:

Corner points	Z= $600x+400y$	
A(5,0)	3000	
B(6,0)	3600	
C(4,4)	4000	
D(0,6)	2400	
E(0,4)	1600	Minimum

The Minimum value is 1600 at the point E (0,4)

10. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes. **(2015-J)**

Sol:

Types	No. of items	Flour	Fat
I	x	200x	25x
II	y	100y	50y
	Z=x+y	≤ 5000	≤ 1000

Maximise Z=x+y

s.t.c: $200x+100y\le 5000\Rightarrow 2x+y\le 50$

$25x+50y\le 1000\Rightarrow x+2y\le 40$

And $x\ge 0, y\ge 0$

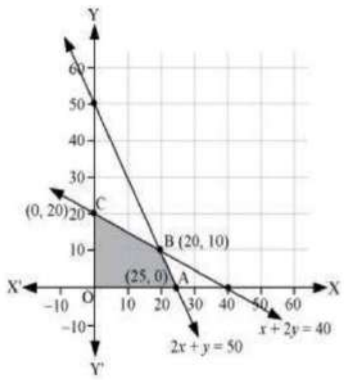
(i) Conversion:

$2x+y=50$

x	0	25
y	50	0

$x+2y=40$

x	0	40
y	20	0



(ii) Shading side:

$2x+y\le 50$

put (0,0), $2(0)+0\le 50$

true, so shading side including (0,0).

$x+2y\le 40$

put (0,0), $0+2(0)\le 40$

true,so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O(0,0), A (25,0),B(20,10),C(0,20)

(iv) Optimal solution:

Corner points	Z= $x+y$	
O(0,0)	0	
A(25,0)	25	
B(20,10)	30	Maximum
C(0,20)	20	

The Maximum value is 30 at the point B (20,10).
We can make I-type cakes 20 and II-type cakes 10.

11. A manufacturing company produces two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realize a maximum profit? What is the maximum profit per week? **(2015-M)**

Sol:

Types	No.of items	Fabricating	Finishing	Profit
A	x	9x	1x	8000x
B	y	12y	3y	12000y
		≤ 180	≤ 30	$Z=8000x+12000y$

Maximise Z=8000x+12000y

s.t.c: $9x+12y \le 180 \Rightarrow 3x+4y \le 60$

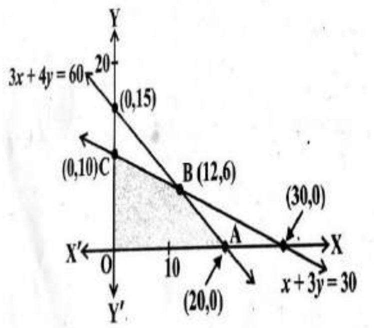
$x+3y \le 30$

And $x \ge 0, y \ge 0$

(i) Conversion:

$3x+4y=60$

x	0	20
y	15	0



$x+3y=30$

x	0	30
y	10	0

(ii) Shading side:

$3x+4y \le 60$

put (0,0), $3(0)+4(0) \le 60$

true, so shading side including (0,0).

$x+3y \le 30$

put (0,0), $0+3(0) \le 30$

true,so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O(0,0), A (20,0),B(12,6),C(0,10)

(iv) Optimal solution:

Corner points	Z= $8000x+12000y$	
O(0,0)	0	
A(20,0)	160000	
B(12,6)	168000	Maximum
C(0,10)	120000	

The Maximum value is 168000 at the point B(12,6).

Company should produce 12 pices of Model A and 6 pieces of Model Bto get maximum profit.

12. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximize his profit, if he operates his machines for at the most 12 hours a day?

Sol:

Types	No.of items	Machine A	Machine B	Profit
Nuts	x	1.x	3x	17.50x
Bolts	y	3y	1.y	7y
		≤ 12	≤ 12	$Z=17.5x+7y$

Maximise $Z=17.5x+7y$

s.t.c: $x+3y \le 12$

$3x+y \le 12$ And $x \ge 0, y \ge 0$

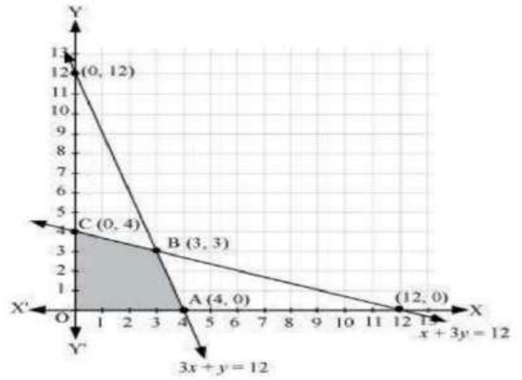
(i) Conversion:

$x+3y=12$

x	0	4
y	12	0

$3x+y=12$

x	0	12
y	4	0



(ii) Shading side:

$x+3y\le 12$

put (0,0), $0+3(0)\le 12$

true, so shading side including (0,0).

$3x+y\le 12$

put (0,0), $3(0)+0\le 12$

true,so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O(0,0), A (4,0),B(3,3),C(0,4)

(iv) Optimal solution:

Corner points	**Z=** $17.5x+7y$	
O(0,0)	0	
A(4,0)	70	
B(3,3)	73.50	Maximum
C(0,4)	28	

The Maximum value is 73.50 at the point B (3,3).

i.e., 3 packages of nuts and 3 packages of bolts should produce day to get maximum profit.

13. A dietician wishes to mix two types of foods in such away that vitamin contents of the mixture contain at least 8 units of vitamin A and 10units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C.Food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food 'I' and Rs 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimize the cost of such a mixture.

Sol:

Types	Vitamin A	Vitamin B	Cost
Food I (x)	2x	x	50x
Food II (y)	y	2y	70y
	≥ 8	≥ 10	$Z=50x+70y$

Minimise $Z=50x+70y$

s.t.c: $2x+y\ge 8$

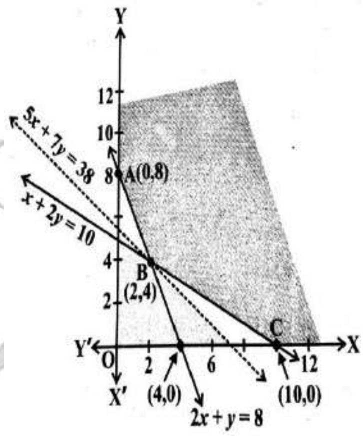
$x+2y\ge 10$

And $x\ge 0, y\ge 0$

(i) Conversion:

$2x+y=8$

x	0	4
y	8	0



$x+2y=10$

x	0	10
y	5	0

(ii) Shading side:

$2x+y\ge 8$

put (0,0), $2(0)+0\ge 8$

false, so shading side excluding (0,0).

$x+2y\ge 10$

put (0,0), $0+2(0)\ge 10$

false,so shading side excluding (0,0).

(iii) Corner points:

The corner points of feasible region are A (0,8), B (2,4), C (10,0)

(iv) Optimal solution:

Corner points	Z= $50x+70y$	
A(0,8)	560	
B(2,4)	380	Minimum
C(10,0)	500	

The Minimum value is 380 at the point B(2,4).

i.e The dietician would be to mix 2 kg of Food I and 4 kg of Food II to minimum cost of the mixture.

14. A corporative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 liters and 10 liters per hectare. Further no more than 800 liters of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much should be allocated to each crop so as to maximize the total profit of the society?

Sol:

Types	No.of hectares	Liquid herbicide	Profit
Crop X	x	20x	10500x
Crop Y	y	10y	9000y
	≤ 50	≤ 800	$Z = 10500x+9000y$

Maximise $Z = 10500x+9000y$

s.t.c: $x+y \le 50$

$20x+10y \le 800 \Rightarrow 2x+y \le 80$

And $x \ge 0, y \ge 0$

(i) Conversion:

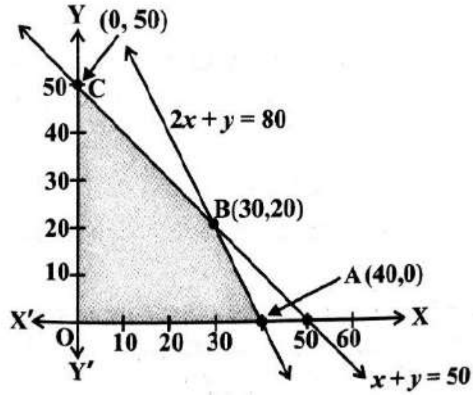
$x+y=50$

x	0	50
y	50	0

$2x+y=80$

x	0	40
y	80	0

(ii)



Shading side:

$x+y \le 50$	$2x+y \le 80$
put (0,0), $0+0 \le 50$	put (0,0), $2(0)+0 \le 80$
true, so shading side including (0,0).	true,so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O (0,0), A (40,0), B (30,20), C (0,50)

(iv) Optimal solution:

Corner points	Z= $10500x+9000y$	
O(0,0)	0	
A(40,0)	420000	
B(30,20)	495000	Maximum
C(0,50)	450000	

The maximum value is 495000 at the point B (30,20).

i.e The society by allocating 30 hectares crop X and 20 hectares for crop Y to get Maximum profit.

15. There are two types of fertilizers F_1 and F_2. F_1 Consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If F_1 costs Rs 6/kg and F_2 costs Rs 5/kg, determine how much of each type of fertilizer should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

Sol:

Types	Quantities	Nitrogen	Phosphoric acid	Cost
F_1	x	10% of x= $\frac{10}{100}x=\frac{x}{10}$	6% of x= $\frac{6}{100}x$	6x
F_2	y	5% of y= $\frac{5}{100}x=\frac{x}{20}$	10% of x= $\frac{10}{100}x$	5y
		≥ 14	≥ 14	$Z=6x+5y$

Maximise $Z=6x+5y$

s.t.c: $\frac{x}{10}+\frac{y}{20}\geq 14 \Rightarrow 2x+y\geq 280$

$\frac{6x}{100}+\frac{10y}{100}\geq 14 \Rightarrow 3x+5y\geq 700$

And $x\geq 0, y\geq 0$

(i) Conversion:

$2x+y=280$

x	0	140
y	280	0

$3x+5y=700$

x	0	700/3
y	140	0

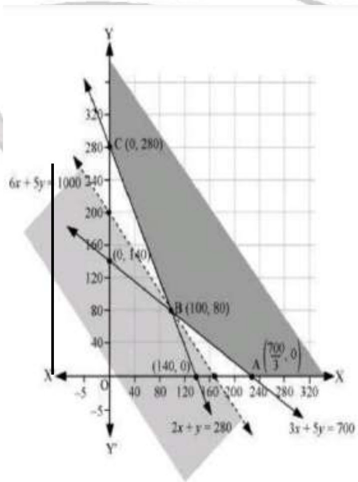
(ii) Shading side:

$2x+y\geq 280$

$3x+5y\geq 700$

put (0,0), $2(0)+0\geq 280$ put (0,0), $3(0)+5(0)\geq 700$

false, so shading side false, so shading side excluding (0,0). excluding (0,0).



(iii) Corner points:

The corner points of feasible region are

$A\left(\frac{700}{3},0\right), B(100,80)\, and\, C(0,280)$

(iv) Optimal solution:

Corner pts	Z= $6x+5y$	
$A\left(\frac{700}{3},0\right)$	1400	
$B(100,80)$	1000	Minimum
$C(0,280)$	1400	

The Minimum value is 1000 at the point B(100,80).

i.e Use 100 kgs of the fertilizer F_1 and 80 kgs of the fertilizer F_2 to minimise the cost.

16. A factory Manufactures two types of screws A and B. each type of screws requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at profit of Rs. 7 and screws B at profit of Rs.10. assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day order to maximize his profit ? determine the maximum profit. **(2019-J)**

Sol:

Types	No.of items	Automatic Machine	Hand operated Machine	Profit
Screw A	x	4x	6x	7x
Screw B	y	6y	3y	10y
		$\leq 4\times 60=240$	$\leq 4\times 60=240$	$Z=7x+10y$

Maximise $Z=7x+10y$

s.t.c: $4x+6y\leq 240 \Rightarrow 2x+3y\leq 120$

$6x+3y\leq 240 \Rightarrow 2x+y\leq 80$

And $x\geq 0, y\geq 0$

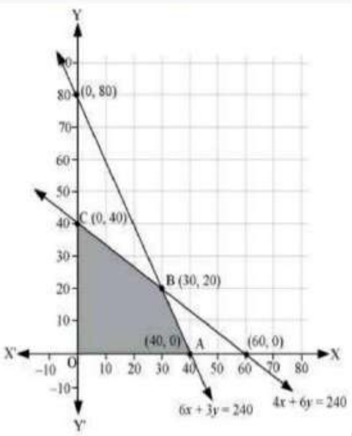
(i) Conversion:

$2x+3y=120$

x	0	60
y	40	0

$2x+y=80$

x	0	40
y	80	0



(ii) Shading side:

$2x+3y\le 120$

put (0,0), $2(0)+3(0)\le 120$

true, so shading side including (0,0).

$2x+y\le 80$

put (0,0), $2(0)+0\le 80$

true,so shading side including (0,0).

(iii) Corner points:

The corner points of feasible region are O (0,0), A (40,0), B (30,20), C (0,40)

(iv) Optimal solution:

Corner points	Z= $7x+10y$	
O(0,0)	0	
A(40,0)	280	
B(30,20)	410	Maximum
C(0,40)	400	

The Maximum value is 400 at the point B (30,20).

i.e., factory should produce 30 packages of screws A and 20 packages of screws B to get maximum profit.

UNIT-13
PROBABILITY

$\left[Total\ marks: 11,\ Q.No - 10(1M), 24(2M), 38(3M)\ \&\ 48(5M)\right]$

Question No: 10 (1M)

1. If $P(A) = 0.6,\ P(B) = 0.3\ \&\ P(A \cap B) = 0.2,\ find\ P(A/B).$

(2014-J, 2019-M)

Sol: $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.2}{0.3} = \dfrac{2}{3}.$

2. If $P(A) = \frac{7}{13},\ P(B) = \frac{9}{13}\ and\ P(A \cap B) = \frac{4}{13}.\ Find\ P(A/B)$

(2016-J, 2018-M)

Sol: $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{4}{13}}{\frac{9}{13}} = \dfrac{4}{9}.$

3. Find
$P(A/B),\ if\ P(B) = 0.5\ and\ P(A \cap B) = 0.32.$

(2018-J)

Sol: $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.32}{0.5} = \dfrac{\frac{32}{100}}{\frac{5}{10}}$

$= \dfrac{32}{100} \times \dfrac{10}{5} = \dfrac{6.4}{10} = 0.64.$

4. A fair die is rolled. If
$E = \{1, 2, 4, 6\}, F = \{1, 3\},\ find\ P(E/F).$

Sol: Given $E = \{1, 2, 4, 6\}, F = \{1, 3\} \Rightarrow E \cap F = \{1\}$

$P(E) = \dfrac{4}{6} = \dfrac{2}{3}, P(F) = \dfrac{2}{6} = \dfrac{1}{3}\ and\ P(E \cap F) = \dfrac{1}{6}$

$Now\ P(E/F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\frac{1}{6}}{\frac{1}{3}} = \dfrac{1}{2}.$

5. If $P(A) = 0.3\ and\ P(B) = 0.4\ find\ P(A/B),$

if A and B are independent events.

Sol: Given A and B are independent events
i.e $P(A \cap B) = P(A).P(B)$

Now
$P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A).P(B)}{P(B)} = P(A) = 0.3$

6. If $P(A) = \dfrac{1}{2},\ P(B) = \dfrac{1}{3}\ \&\ P(A \cap B) = \dfrac{1}{6}$,
show that A & B are independent events.

Sol: $P(A).P(B) = \dfrac{1}{2}.\dfrac{1}{3} = \dfrac{1}{6} = P(A \cap B)$

$\therefore P(A \cap B) = P(A).P(B)$

then A and B are independent events.

7. If $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(A \cap B) = \frac{1}{4}\ find\ P(A/B)$

Sol: $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{4}}{\frac{1}{3}} = \dfrac{3}{4}.$

8. If $P(A) = \dfrac{1}{2},\ P(B) = 0\ then\ find\ P\left(\dfrac{B}{A}\right).$

Sol: $P(B/A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{0}{\frac{1}{2}} = 0.\ \ (\because B \cap A = \phi)$

9. If $P(A) = \dfrac{4}{5}\ \&\ P(B/A) = \dfrac{2}{5},\ Find\ P(A \cap B).$

(2014-M)

Sol:
$P(B/A) = \dfrac{2}{5} \Rightarrow \dfrac{P(B \cap A)}{P(A)} = \dfrac{2}{5} \Rightarrow \dfrac{P(A \cap B)}{\frac{4}{5}} = \dfrac{2}{5}$

$\Rightarrow P(A \cap B) = \dfrac{2}{5}.\dfrac{4}{5} = \dfrac{8}{25}$

10. If $P(A) = \dfrac{3}{5},\ P(B) = \dfrac{1}{5}\ find\ P(A \cap B)$ if A & B are independent events.

(2015-M, 2017- M, 2019-J 2020-M)

Sol: Given A and B are independent events

i.e., $P(A \cap B) = P(A).P(B)$

$\Rightarrow P(A \cap B) = \dfrac{3}{5}.\dfrac{1}{5} = \dfrac{3}{25}.$

11. If
$P(A) = 0.8, P(B) = 0.5\ and\ P(B/A) = 0.4\ then\ find\ P(A \cap B).$

(2015-J)

Sol: $P(B/A) = 0.4 \Rightarrow \dfrac{P(B \cap A)}{P(A)} = 0.4$

$\Rightarrow \dfrac{P(A\cap B)}{0.8} = 0.4 \Rightarrow P(A\cap B) = 0.4\times 0.8 = 0.32$

12. If $P(A)=0.8, P(B/A)=0.4$ *then find* $P(A\cap B)$.

(2016-M, 2017-J)

Sol: $P(B/A)=0.4 \Rightarrow \dfrac{P(B\cap A)}{P(A)} = 0.4$

$\Rightarrow \dfrac{P(A\cap B)}{0.8} = 0.4 \Rightarrow P(A\cap B) = 0.4\times 0.8 = 0.32$

13. If A & B are two events such that $P(A)\neq 0$ find $P(B/A)$ if A is the subset of B.

Sol: Given

$A\subset B \Rightarrow A\cap B = A \Rightarrow P(A\cap B) = P(A) \quad \rightarrow (1)$

Now $P(B/A) = \dfrac{P(B\cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1.(\because \text{ from } (1))$

14. If E is the event of sample space S of an experiment, then find $P(S/E)$.

Sol: Given E is the event of sample space S of an experiment

$\Rightarrow E\cap S = E \Rightarrow P(E\cap S) = P(E) \rightarrow (1)$

Now $P(S/E) = \dfrac{P(S\cap E)}{P(E)} = \dfrac{P(E)}{P(E)} = 1.(\because \text{ from } (1))$

15. If A & B are two events such that $P(A)\neq 0$ find $P(B/A)$ if $A\cap B = \phi$.

Sol: Given $A\cap B = \phi \Rightarrow P(A\cap B) = 0 \rightarrow (1)$

Now $P(B/A) = \dfrac{P(B\cap A)}{P(A)} = \dfrac{0}{P(A)} = 0.(\because \text{ from } (1))$

16. If E and F be events of a sample space S of an experiment, then prove $P(S/F)=1$.

Sol: Given E and F be events of a sample space S of an experiment

$\Rightarrow P(S\cap E) = P(E)$ & $P(S\cap F) = P(F) \rightarrow (1)$

Now

$P(S/F) = \dfrac{P(S\cap F)}{P(F)} = \dfrac{P(F)}{P(F)} = 1. \quad (\because \text{ from } (1))$

17. Given that the events A & B are such that

$P(A) = \dfrac{1}{2}$, $P(A\cup B) = \dfrac{3}{5}$ & $P(B) = p$.

Find p if they are mutually exclusive.

Sol: Given that the events A & B are mutually exclusive

i.e., $A\cap B = \phi \Rightarrow P(A\cap B) = 0$

WKT $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow \dfrac{3}{5} = \dfrac{1}{2} + p - 0 \Rightarrow p = \dfrac{3}{5} - \dfrac{1}{2} = \dfrac{1}{10} \Rightarrow \boxed{p = \dfrac{1}{10}}$

18. Define independent events.

Sol: The E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other.

19. Define Conditional probability.

Sol: If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, i.e. (E/F)is given by

$P(E/F) = \dfrac{P(E\cap F)}{P(F)}$ *provided* $P(F)\neq 0$.

20. State Law of total Probability.

Sol: Let $\{E_1, E_2, E_3 ....E_n\}$ be a partition of the sample space S and suppose that each of the events $E_1, E_2, E_3 ....E_n$ has non zero probability of occurrence. Let A be any event associated with S, then

$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + ...+ P(E_n)P(A/E_n)$

i.e. $P(A) = \sum_{j=1}^{n} P(E_j)P(A/E_j)$.

21. State Bayes' theorem.

Sol: If $E_1, E_2, E_3 ....E_n$ are n non empty events which constitute a partition of sample space S, i.e. $E_1, E_2, E_3.........E_n$ are pairwise disjoint and $E_1 \cup E_2 \cup E_3 \cup ....\cup E_n = S$ and A is any event of nonzero probability, then

$$P(E_i / A) = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)} \quad for\, i = 1,2,3,......,n$$

22. Define Random Variable and Probability Distribution.

Sol: The probability distribution of a random variable X is the system of numbers

$$\begin{array}{cccccc} X & : & x_1 & x_2 & ...... & x_n \\ P(X) & : & p_1 & p_2 & ...... & p_n \end{array}$$

where $p_i > 0, \sum_{i=1}^{n} p_i = 1, i = 1,2,.....,n$

The real numbers $x_1, x_2, ...x_n$ are the possible values of the random variable X and p_i is the probability of the random variable X taking the value x_i i.e. $P(X = x_i) = p_i$

23. Find k, if random variable X has the probability distribution

X	0	1	2
P(X)	0.4	0.4	k

Sol: WKT $\sum_{i=1}^{n} p_i = 1, i = 1,2,.....,n \Rightarrow$

0.4+0.4+k=1 $\Rightarrow k = 1 - 0.8 \Rightarrow \boxed{k = 0.2}$

24. Given is not probability distribution, why?

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

Sol: WKT $p_i > 0$ but here $p(3) = -0.1 < 0$

$\therefore$ Given is not probability distribution.

25. Given is not a probability distribution, why?

Sol: Here sum of all probabilities $0.3 + 0.2 + 0.4 + 0.1 + 0.05 = 1.05 \neq 1.$

$\therefore$ Given is not probability distribution.

26. Given is not probability distribution of random variable, why ?

X	-1	0	1
P(X)	0.6	0.1	0.2

Sol: Here sum of all probabilities $0.6 + 0.1 + 0.2 = 0.9 \neq 1.$

$\therefore$ Given is not probability distribution.

27. Let X be a discrete random variable. The probability distribution of X is given below

X	30	10	-10
P(X)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{2}$

Then $E(X) = ?$

Sol:

$$Mean\ E(X) = \sum_{i=1}^{n} x_i p(x_i) = 30 \times \frac{1}{5} + 10 \times \frac{3}{10} + (-10)\frac{1}{2} = 6 + 3 - 5 = 4.$$

<u>Question No: 24 (2M)</u>

1. Find the probability distribution of number of heads in two tosses of a coin. **(14-M)**

Sol: Here sample space S={HH, HT, TH, TT}

Let X denotes the no. of heads, clearly we can take the X values 0,1 and 2

$P(X=0) = P(TT) = \frac{1}{4}$; $P(X=1) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$; $P(X=2) = P(HH) = \frac{1}{4}$

Probability distribution of X is

X	0	1	2
P(X)	1/4	1/2	¼

2. A die is thrown. If E is the event 'the number appearing is a multiple of 3 & F be the event 'the number appearing is even' then prove that E & F are independent events. **(2014-J)**

Sol: Here sample space S={1,2,3,4,5,6}

Given E: the number appearing is a multiple of 3 i.e E= {3,6} , $P(E) = \frac{2}{6} = \frac{1}{3}$

F: the number appearing is even i.e F= {2,4,6} , $P(F) = \frac{3}{6} = \frac{1}{2}$

$E\cap F=\{6\}, P(E\cap F)=\frac{1}{6}$

Now $P(E).P(F)=\frac{1}{3}.\frac{1}{2}=\frac{1}{6}=P(E\cap F)$

∴ E & F are independent events.

3. Let X denote the number of hours, you study during a randomly selected school day. The probability that X can take the values of x, has the following form. where k is some constant.

$$P(X=x)=\begin{cases}0.1 & , \text{ if } x=0\\ kx & , \text{ if } x=1 \text{ or } x=2\\ k(5-x), & \text{ if } x=3 \text{ or } 4\\ 0 & , \text{ otherwise}\end{cases}$$

Find the value of k. **(2015-M)**

Sol: The probability distribution of X is

X	0	1	2	3	4	Otherwise
P(X)	0.1	k	2k	2k	k	0

WKT;

$\sum_{i=1}^{n} p_i=1\Rightarrow 0.1+k+2k+2k+k+0=1\Rightarrow \boxed{k=0.15}$

4. Find the probability distribution of the number of tails in simultaneous tosses of three coins. **(2015-J)**

Sol: Here sample space

S={HHH,HHT,HTH,THH,HTT,THT,TTH,TTT}

Let X represent the no. of tails, then we can take X values 0,1,2 and 3.

$P(X=0)=P(HHH)=\frac{1}{8}$

$P(X=1)=P(HHT)+P(HTH)+P(THH)=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$

$P(X=2)=P(HTT)+P(THT)+P(TTH)=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$

$P(X=3)=P(TTT)=\frac{1}{8}$

The probability distribution of X is

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

5. The random variable X has a probability distribution $P(X)$ of the following form where k is some number

$$P(X=x)=\begin{cases}k & \text{if } x=0\\ 2k & \text{if } x=1\\ 3k & \text{if } x=2\\ 0 & \text{otherwise}\end{cases}$$

Determine the value of k & $P(X\le 2)$ **(2016-M, 2018-J)**

Sol: The probability distribution of X is

X	0	1	2	Other wise
P(X)	k	2k	3k	0

WKT; $\sum_{i=1}^{n} p_i=1\Rightarrow k+2k+3k=1\Rightarrow \boxed{k=\frac{1}{6}}$

$P(X\le 2)=P(X=0)+P(X=1)+P(X=2)=k+2k+3k=6k=6\left(\frac{1}{6}\right)=1.$

6. Probability of solving specific problem independently by A & B are $\frac{1}{2}$ & $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that the problem is solved. **(2019-M)**

Sol: Given $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$

Here A & B are independent

$\Rightarrow P(A\cap B)=P(A).P(B)=\frac{1}{2}.\frac{1}{3}=\frac{1}{6}$

P(the problem solved)=

$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{2}+\frac{1}{3}-\frac{1}{6}=\frac{3+2-1}{6}=\frac{4}{6}=\frac{2}{3}.$

7. Find the probability distribution of number of heads in two tosses of a coin. **(2016-J)**

Sol: Here sample space S={HH,HT,TH,TT}

Let X denotes the no. of heads, clearly, we can take the X values 0,1 and 2

$P(X=0)=P(TT)=\frac{1}{4}$; $P(X=1)=P(HT)+P(TH)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$; $P(X=2)=P(HH)=\frac{1}{4}$

Probability distribution of X is

X	0	1	2
P(X)	1/4	1/2	1/4

8. A random variable X has the following probability distribution.

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Determine (i) k (ii) $P(0<X<3)$

Sol: **(i)** WKT;

$$\sum_{i=1}^{n} p_i = 1 \Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$\Rightarrow 10k^2+9k-1=0$$

$$(10k-1)(k+1)=0 \Rightarrow k=-1, \frac{1}{10}$$

$$but\ \ k=-1\ not\ possible \Rightarrow \boxed{k=\frac{1}{10}}$$

(ii) $P(0<X<3)=P(X=1)+P(X=2)=k+2k=3k=\frac{3}{10}.$

9. A random variable X has the following probability distribution

X	0	1	2	3	4
P(X)	0.1	k	2k	2k	k

Find the value of k. **(2020-M)**

Sol: WKT; $\sum_{i=1}^{n} p_i = 1 \Rightarrow 0.1+k+2k+2k+k=1$

$$\Rightarrow 6k=1-0.1 \Rightarrow k=\frac{0.9}{6} \Rightarrow \boxed{k=0.15}$$

10. A random variable X has the following probability distribution.

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Find the expectation of X. **(2019-J)**

Sol: Expectation of X = E(X)=

$$\sum_{i=1}^{n} x_i p(x_i) = 0\times\frac{188}{221}+1\times\frac{32}{221}+2\times\frac{1}{221}=\frac{34}{221}.$$

Question No: 38 (3M)

1. A die is tossed thrice. Find the probability of getting an odd number at least once. **(2014-M, 2016-J)**

Sol: Let A be the event of getting an odd number on a single throw of die.

$$\therefore P(A)=\frac{3}{6}=\frac{1}{2}\ and\ P(A')=1-P(A)=1-\frac{1}{2}=\frac{1}{2}$$

Let AAA denote the event of getting an odd number in each of the 3throws.

$\therefore$ Required probability

$$=P(atleast\ one\ odd\ number)$$

$$=1-P(no\ odd\ number)$$

$$=1-P(A'A'A')=1-P(A')P(A')P(A')$$

$$=1-\frac{1}{2}.\frac{1}{2}.\frac{1}{2}=1-\frac{1}{8}=\frac{7}{8}$$

2. Bag-I contains 3 red and 4 black balls while another Bag-II contains 5 red and 6 Black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag-II. **(2014-J, 2015-J, 2019-J)**

Sol: Let A: Bag I ; B:Bag II

$$\therefore P(A)=\frac{1}{2}, P(B)=\frac{1}{2}$$

Let E: drawn ball is red $\therefore P(E|A)=\frac{3}{7}, P(E|B)=\frac{5}{11}$

Required probability

$$P(B|E)=\frac{P(B).P(E|B)}{P(B).P(E|B)+P(A)P(E|A)}$$

$$=\frac{\frac{1}{2}.\frac{5}{11}}{\frac{1}{2}.\frac{5}{11}+\frac{1}{2}.\frac{3}{7}}=\frac{5X7}{5X7+3X11}=\frac{35}{68}.$$

3. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver? **(2015-M, 2019-M)**

Sol: Let

E_1: Insured person is a scooter driver

E_2: Insured person is a car driver

E_3: Insured person is a truck driver

$$\therefore P(E_1)=\frac{2000}{2000+4000+6000}=\frac{2}{12}=\frac{1}{6};$$

$$P(E_2)=\frac{4}{12}=\frac{1}{3}\ and\ P(E_3)=\frac{6}{12}=\frac{1}{2}$$

Let E: a person meets with an accident.

$$\therefore P(E|E_1)=0.01; P(E|E_2)=0.03$$

$$and\ P(E|E_3)=0.15$$

Required probability

$$P(E_1|E)=\frac{P(E_1).P(E_1|E)}{P(E_1).P(E|E_1)+P(E_2).P(E|E_2)+P(E_3).P(E|E_3)}$$

$$=\frac{\frac{1}{6}\times 0.01}{\frac{1}{6}\times 0.01+\frac{1}{3}\times 0.03+\frac{1}{2}\times 0.15}$$

$$=\frac{0.01}{0.01+0.06+0.45}=\frac{0.01}{0.52}=\frac{1}{52}.$$

4. Box-I contains 2 gold coins, while another Box-II contains 1 gold and 1 silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold ? **(2016-M)**

Sol: Let A and B be the events of selecting I and II boxes respectively.

$$\therefore P(A)=P(B)=\frac{1}{2}$$

Let E: selected coin is gold coin

$$\therefore P(E|A)=\frac{2_{C_1}}{2_{C_1}}=\frac{2}{2}=1; P(E|B)=\frac{1_{C_1}}{2_{C_1}}=\frac{1}{2}$$

$$P(A|E)=\frac{P(A).P(E|A)}{P(A).P(E|A)+P(B).P(E|B)}$$

$$=\frac{\frac{1}{2}.1}{\frac{1}{2}.1+\frac{1}{2}.\frac{1}{2}}=\frac{2}{3}$$

5. A man is known to speak truth 4 out of 5 times. He tossed a coin and reports that it is head. Find the probability that it is actually head. **(2018-J)**

Sol: Let E_1: coin shows a head

E_2: coin shows a tail

$$P(E_1)=P(E_2)=\frac{1}{2}\ (\because S=\{H,T\},E_1=\{H\},E_2=\{T\})$$

Let E: A reports that a head appears.

$$\therefore P(E|E_1)=\frac{4}{5}, P(E|E_2)=1-\frac{4}{5}=\frac{1}{5}$$

$\therefore$ Required probability

$$P(E_1|E)=\frac{P(E|E_1).P(E_1)}{P(E|E_1).P(E_1)+P(E|E_2).P(E_2)}$$

$$=\frac{\frac{4}{5}.\frac{1}{2}}{\frac{4}{5}.\frac{1}{2}+\frac{1}{5}.\frac{1}{2}}=\frac{\frac{4}{5}}{\frac{4}{5}+\frac{1}{5}}=\frac{4}{5}$$

Question No: 48 (5M)

1. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize?
(a) at least once (b) exactly once (c) at least twice? **(2014-M, 2018-J, 2020-M)**

Sol: Here $n=50, p=\frac{1}{100}, q=\frac{99}{100}$

$$P(X=x)={}^nC_x(q)^{n-x}(p)^x={}^{50}C_x\left(\frac{99}{100}\right)^{50-x}\left(\frac{1}{100}\right)^x$$

$$\text{Required}:(a)P(X\geq 1)=1-P(0)=1-\left(\frac{99}{100}\right)^{50}$$

$$(b)P(X=1)=\frac{1}{2}.\left(\frac{99}{100}\right)^{49}$$

$$(c)P(X\geq 2)=1-P(X=0,1)$$

$$=1-P(0)-P(1)$$

$$=1-\frac{149(99)^{49}}{(100)^{50}}$$

2. If a fair coin is tossed 10 times, find the probability of (i) exactly six heads (ii) at least six heads (iii) at most six heads **(2014-J, 2016-J)**

Sol: The repeated tosses of a coin are Bernoulli trails. Let X denote the number of heads in an experiment of 10 trails. Clearly, X has binomial distribution with

$$n=10, p=\frac{1}{2}\ and\ q=1-p=1-\frac{1}{2}=\frac{1}{2}$$

$$\therefore P(X=x)={}^{10}C_x\left(\frac{1}{2}\right)^{10-x}\left(\frac{1}{2}\right)^x$$

$$(i)P(X=6)={}^{10}C_6\left(\frac{1}{2}\right)^{10-6}\left(\frac{1}{2}\right)^6$$

$$={}^{10}C_6\left(\frac{1}{2}\right)^{10}=\frac{10!}{6!\times 4!2^{10}}=\frac{105}{512}$$

$(ii)\, P(atleast\ six\ heads) = P(X \geq 6)$

$= P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$

$= \left[\frac{10!}{6!\times 4!} + \frac{10!}{7!\times 3!} + \frac{10!}{8!\times 2!} + \frac{10!}{9!\times 1!} + \frac{10!}{10!\times 0!}\right]\frac{1}{2^{10}}$

$= \frac{193}{512}$

$(iii)\, P(atmost\ six\ heads) = P(X \leq 6)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= {}^{10}C_0\left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10} + {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$

$= \left[1 + 10 + \frac{10\times 9}{2\times 1} + \frac{10\times 9\times 8}{3\times 2\times 1} + \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} + \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} + \frac{10\times 9\times 8\times 7\times 6\times 5}{6\times 5\times 4\times 3\times 2\times 1}\right]\frac{1}{2^{10}}$

$= \frac{848}{1024} = \frac{53}{63}.$

3. A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of (i) 5 successes? (ii) At least 5 successes? (iii) At most 5 successes? **(2015-M, 2019-J)**

Sol: The trails are Bernoullian trails with n=6.
Here success is getting an odd number.

$p = P(success) = P(getting\ an\ odd\ number) = \frac{3}{6} = \frac{1}{2}$

$and\ q = P(failure) = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Let X be the number of successes (getting an odd number) in 6 trails.

$\therefore$X is a binomial random variable with

$n = 6,\ p = \frac{1}{2}, q = \frac{1}{2}$

$Now\ P(X = x) = {}^{n}C_x (q)^{n-x}(p)^x$

$= {}^{6}C_x\left(\frac{1}{2}\right)^{6-x}\left(\frac{1}{2}\right)^x = {}^{6}C_x\left(\frac{1}{2}\right)^6$

$(i)\, P(X=5) = {}^{6}C_5\left(\frac{1}{2}\right)^6 = 6\times\frac{1}{64} = \frac{3}{32}$

$(ii)\, P(X \geq 5) = P(X=5) + P(X=6)$

$= {}^{6}C_5\left(\frac{1}{2}\right)^6 + {}^{6}C_6\left(\frac{1}{2}\right)^6$

$= 6\times\frac{1}{64} + 1\times\frac{1}{64} = \frac{7}{64}$

$(iii)\, P(X \leq 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 1 - P(X=6) = 1 - \frac{1}{64} = \frac{63}{64}.$

4. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that (i) all the five cards are spades? (ii) Only 3 cards are spades? (iii) None is a spade? **(2015-J, 2019-M)**

Sol: Here

$n = 5,\ p = P(spade\ cards) = \frac{13}{52} = \frac{1}{4}; q = 1 - p = \frac{3}{4}$

$where\ P(X = x) = {}^{n}C_x (q)^{n-x}(p)^x$

$= {}^{5}C_x\left(\frac{3}{4}\right)^{5-x}\left(\frac{1}{4}\right)^x = {}^{5}C_x\frac{(3)^{5-x}}{(4)^5}$

$(i)\, P(X=5) = {}^{5}C_5\frac{(3)^0}{(4)^5} = \frac{1}{(4)^5} = \frac{1}{1024}$

$(ii)\, P(X=3) = {}^{5}C_3\frac{(3)^{5-3}}{(4)^5} = 10\times\frac{(3)^2}{(4)^5} = \frac{90}{1024}$

$(iii)\, P(X=0) = {}^{5}C_0\frac{(3)^{5-0}}{(4)^5} = \frac{(3)^5}{(4)^5} = \frac{243}{1024}$

5. The probability that a student is not a swimmer is $\frac{1}{5}$. Then the probability that out of five students, (i) at least four are swimmers(ii) at most three are swimmers. **(2016-M)**

Sol: Here $n=5, p=\frac{4}{5}; q=\frac{1}{5}$

$$\therefore P(X=x)={}^nC_x(q)^{n-x}(p)^x={}^5C_x\left(\frac{1}{5}\right)^{5-x}\left(\frac{4}{5}\right)^x$$

(i) *At least four are swimmers* :

$\therefore$ Required probability

$$=P(X\geq 4)=P(X=4)+P(X=5)$$

$$={}^5C_4\left(\frac{1}{5}\right)^{5-4}\left(\frac{4}{5}\right)^4+{}^5C_5\left(\frac{1}{5}\right)^{5-5}\left(\frac{4}{5}\right)^5$$

$$=\not{5}\left(\frac{1}{\not{5}}\right)\left(\frac{4}{5}\right)^4+\left(\frac{4}{5}\right)^5=\left(\frac{4}{5}\right)^4+\left(\frac{4}{5}\right)^5$$

(ii) *At most three are swimmers* :

$\therefore$ Required probability

$$=P(X\leq 3)=P(X=0)+P(X=1)$$
$$+P(X=2)+P(X=3)$$

$$=1-P(X=4)+P(X=5)$$

$$=1-\left(\frac{4}{5}\right)^4-\left(\frac{4}{5}\right)^5$$

6. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs (i) none will fuse after 150 days. (ii) at most two will fuse after 150 days. **(2017-M)**

Sol: Here

$n=5, p=0.05; q=1-p=1-0.05=0.95$

$$\therefore P(X=x)={}^nC_x(q)^{n-x}(p)^x={}^5C_x(0.95)^{5-x}(0.05)^x$$

Required :

(i) $P(none)=P(X=0)$

$$={}^5C_0(0.95)^{5-0}(0.05)^0=(0.95)^5$$

(ii) $P(not\ more\ than\ one)$

$$=P(X\leq 1)=P(X=0)+P(X=1)$$

$$=(0.95)^5+{}^5C_1(0.95)^4(0.05)$$

$$=(0.95)^4\left[0.95+5(0.05)\right]$$

$$=(0.95)^4(1.20)$$

(iii) $P(more\ than\ one)$

$$=P(X>1)=1-P(X\leq 1)$$

$$=1-(0.95)^4(1.20)$$

(iv) $P(atleast\ one)=P(X\geq 1)$

$$=1-P(X=0)=1-(0.95)^5.$$