MATHEMATICS

PART - 2

CLASS XI

SYLLABUS

(1) **MATRICES AND DETERMINANTS :** ***Matrix Algebra*** **–** Definitions, types, operations, algebraic properties. ***Determinants*** – Definitions, properties, evaluation, factor method, product of determinants, co-factor determinants. **(18 periods)**

(2) **VECTOR ALGEBRA :** Definitions, types, addition, subtraction, scalar multiplication, properties, position vector, resolution of a vector in two and three dimensions, direction cosines and direction ratios. **(15 periods)**

(3) **ALGEBRA :** ***Partial Fractions*** – Definitions, linear factors, none of which is repeated, some of which are repeated, quadratic factors (none of which is repeated). ***Permutations*** – Principles of counting, concept, permutation of objects not all distinct, permutation when objects can repeat, circular permutations**.** ***Combinations, Mathematical induction, Binomial theorem for positive integral index***–finding middle and particular terms. **(25 periods)**

(4) **SEQUENCE AND SERIES :** Definitions, special types of sequences and series, harmonic progression, arithmetic mean, geometric mean, harmonic mean. **Binomial theorem** for rational number other than positive integer, **Binomial series**, approximation, summation of Binomial series, **Exponential series, Logarithmic series** (simple problems) **(15 periods)**

(5) **ANALYTICAL GEOMETRY :** Locus, ***straight lines*** **–** normal form, parametric form, general form, perpendicular distance from a point, ***family of straight lines***, angle between two straight lines, **pair of straight lines**. ***Circle*** – general equation, parametric form, tangent equation, length of the tangent, condition for tangent. Equation of chord of contact of tangents from a point, **family of circles –** concetric circles, orthogonal circles. **(23 periods)**

(6) **TRIGONOMETRY :** Trigonometrical **ratios and identities**, signs of T-ratios, **compound angles** A ± B, multiple angles 2A, 3A, sub multiple (half) angle A/2, transformation of a product into a sum or difference, conditional identities, **trigonometrical equations**, **properties of triangles**, **solution of triangles** (SSS, SAA and SAS types only), **inverse trigonometrical functions**. **(25 periods)**

(7) **FUNCTIONS AND GRAPHS :** Constants, variables, intervals, neighbourhood of a point, Cartesian product, relation. **Function** – graph of a function, vertical line test. ***Types of functions*** – Onto, one-to-one, identity, inverse, composition of functions, sum, difference product, quotient of two functions, constant function, linear function, polynomial function, rational function, exponential function, reciprocal function, absolute value function, greatest integer function, least integer function, signum function, odd and even functions, trigonometrical functions, quadratic functions. ***Quadratic inequation*** – Domain and range. **(15 periods)**

(8) **DIFFERENTIAL CALCULUS :** ***Limit of a function*** – Concept, fundamental results, important limits, ***Continuity of a function*** – at a point, in an interval, discontinuous function. ***Concept of Differentiation*** – derivatives, slope, relation between continuity and differentiation. ***Differentiation techniques*** – first principle, standard formulae, product rule, quotient rule, chain rule, inverse functions, method of substitution, parametric functions, implicit function, third order derivatives. **(30 periods)**

(9) **INTEGRAL CALCULUS :** Concept, integral as anti-derivative, integration of linear functions, properties of integrals. ***Methods of integration*** – decomposition method, substitution method, integration by parts. ***Definite integrals*** – integration as summation, simple problems. **(32 periods)**

(10) **PROBABILITY :** Classical definitions, axioms, basic theorems, conditional probability, total probability of an event, Baye's theorem (statement only), simple problems. **(12 periods)**

CONTENTS

7. FUNCTIONS AND GRAPHS

7.1 Introduction:

The most prolific mathematician whoever lived, Leonhard Euler (1707–1783) was the first scientist to give the function concept the prominence in his work that it has in Mathematics today. The concept of functions is one of the most important tool in Calculus.

To define the concept of functions, we need certain pre-requisites.

Constant and variable:

A quantity, which retains the same value throughout a mathematical process, is called a constant. A variable is a quantity which can have different values in a particular mathematical process.

It is customary to represent constants by the letters a, b, c, … and variables by x, y, z.

Intervals:

The real numbers can be represented geometrically as points on a number line called the real line (fig. 7.1)

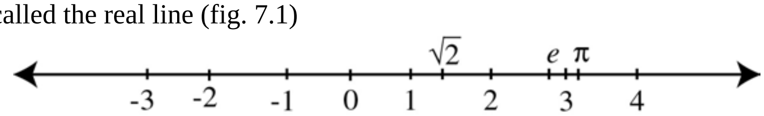


Fig 7. 1

The symbol R denotes either the real number system or the real line. A subset of the real line is called an interval if it contains atleast two numbers and contains all the real numbers lying between any two of its elements.

For example,

(a) the set of all real numbers x such that $x > 6$

(b) the set of all real numbers x such that $-2 \leq x \leq 5$

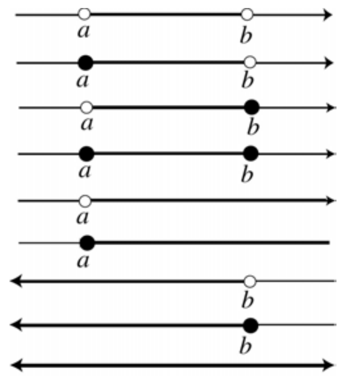
(c) the set of all real numbers x such that $x < 5$ are some intervals.

But the set of all natural numbers is not an interval. Between any two rational numbers there are infinitely many real numbers which are not included in the given set. Hence the set of natural numbers is not an interval. Similarly the set of all non zero real numbers is also not an interval. Here the real number 0 is absent. It fails to contain every real number between any two real numbers say – 1 and 1.

Geometrically, intervals correspond to rays and line segments on the real line. The intervals corresponding to line segments are finite intervals and intervals corresponding to rays and the real line are infinite intervals. Here finite interval does not mean that the interval contains only a finite number of real numbers.

A finite interval is said to be closed if it contains both of its end points and open if it contains neither of its end points. To denote the closed set, the square bracket [   ] is used and the paranthesis (   ) is used to indicate open set. For example 3$\notin$ (3, 4), 3$\in$[3, 4]

Type of intervals

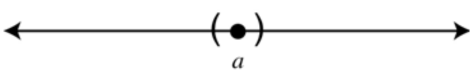
	Notation	Set	Graph
Finite	(a, b)	$\{x / a < x < b\}$	
	$[a, b)$	$\{x / a \le x < b\}$	
	$(a, b]$	$\{x / a < x \le b\}$	
	$[a, b]$	$\{x / a \le x \le b\}$	
Infinite	(a, ∞)	$\{x / x > a\}$	
	$[a, \infty)$	$\{x / x \ge a\}$	
	$(-\infty, b)$	$\{x / x < b\}$	
	$(-\infty, b]$	$\{x / x \le b\}$	
	$(-\infty, \infty)$	$\{x / -\infty < x < \infty\}$ or the set of real numbers	

Note :

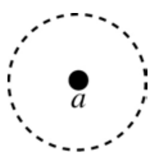
We can’t write a closed interval by using ∞ or $-\infty$. These two are not representatives of real numbers.

Neighbourhood

In a number line the neighbourhood of a point (real number) is defined as an open interval of very small length.



In the plane the neighbourhood of a point is defined as an open disc with very small radius.



In the space the neighbourhood of a point is defined as an open sphere with very small radius.

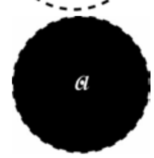


Fig 7. 2

Independent / dependent variables:

In the lower classes we have come across so many formuale. Among those, let us consider the following formulae:

(a) $V = \frac{4}{3}\pi r^3$ (volume of the sphere) (b) $A = \pi r^2$ (area of a circle)

(c) $S = 4\pi r^2$ (surface area of a sphere) (d) $V = \frac{1}{3}\pi r^2 h$ (volume of a cone)

Note that in (a), (b) and (c) for different values of r, we get different values of V, A and S. Thus the quantities V, A and S depend on the quantity r. Hence we say that V, A and S are dependent variables and r is an independent variable. In (d) the quantities r and h are independent variables while V is a dependent variable.

A variable is an independent variable when it has any arbitrary (independent) value.

A variable is said to be dependent when its value depends on other variables (independent).

"Parents pleasure depends on how their children score marks in Examination"

Cartesian product:

Let $A=\{a_1, a_2, a_3\}$, $B=\{b_1, b_2\}$. The Cartesian product of the two sets A and B is denoted by A × B and is defined as

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$

Thus the set of all ordered pairs (a, b) where $a \in A$, $b \in B$ is called the Cartesian product of the sets A and B.

It is noted that $A \times B \neq B \times A$ (in general), since the ordered pair (a, b) is different from the ordered pair (b, a). These two ordered pairs are same only if $a = b$.

Example 7.1: Find A × B and B × A if A = {1, 2}, B = {a, b}

Solution:

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

Relation:

In our everyday life we use the word 'relation' to connect two persons like 'is son of', 'is father of', 'is brother of', 'is sister of', etc. or to connect two objects by means of 'is shorter than', 'is bigger than', etc. When comparing (relate) the objects (human beings) the concept of relation becomes very important. In a similar fashion we connect two sets (set of objects) by means of relation.

Let A and B be any two sets. A relation from A → B (read as A to B) is a subset of the Cartesian product A × B.

Example 7.2: Let A = {1, 2}, B = {a, b}. Find some relations from A → B and B → A.

Solution:

Since relation from A to B is a subset of the Cartesian product

A × B = {(1 , a) , (1, b) , (2 , a) , (2 , b)} any subset of A × B is a relation from A → B.

∴ {(1 , a), (1 , b), (2 , a), (2 , b)}, {(1, a), (1, b)}, {(1, b, (2, b)}, {(1 , a)} are some relations from A to B.

Similarly any subset of B × A = {(a , 1), (a , 2), (b , 1), (b , 2)} is a relation from B to A.

{(a , 1), (a , 2), (b , 1), (b , 2)}, {(a, 1), (b, 1)}, {(a, 2), (b, 1)} are some relations from B to A.

7.2 Function:

A function is a special type of relation. In a function, no two ordered pairs can have the same first element and a different second element. That is, for a function, corresponding to each first element of the ordered pairs, there must be a different second element. i.e. In a function we cannot have ordered pairs of the form (a_1, b_1) and (a_2, b_2) with $a_1 = a_2$ and $b_1 \neq b_2$.

Consider the set of ordered pairs (relation) {(3 , 2), (5 , 7), (1 , 0), (10 , 3)}. Here no two ordered pairs have the same first element and different second element. It is very easy to check this concept by drawing a proper diagram *(fig. 7.3)*.

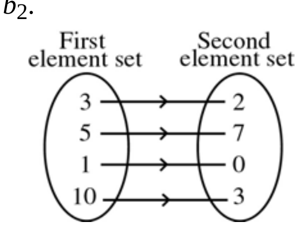


Fig 7. 3

∴ This relation is a function.

Consider another set of ordered pairs (relation) {(3, 5), (3, − 1), (2, 9)}. Here the ordered pairs (3, 5) and (3, − 1) have the same first element but different second element *(fig. 7.4)*.

This relation is not a function.

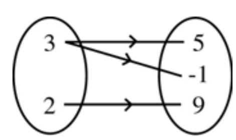


Fig 7. 4

Thus, a function f from a set A to B is a rule (relation) that assigns a unique element $f(x)$ in B to each element x in A.

Symbolically, f : A → B

i.e. $x \rightarrow f(x)$

To denote functions, we use the letters *f*, *g*, *h* etc. Thus for a function, each element of A is associated with exactly one element in B. The set A is called the **domain** of the function *f* and B is called **co-domain** of *f*. If *x* is in A, the element of B associated with *x* is

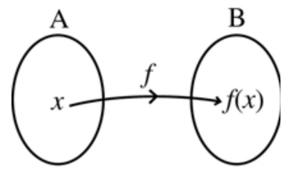


Fig 7. 5

called the **image** of *x* under *f*. i.e. $f(x)$. The set of all images of the elements of A is called the **range** of the function *f*. Note that range is a subset of the co-domain. The range of the function *f* need not be equal to the co-domain B. **Functions** are also known as **mappings**.

Example 7.3 : Let A = {1, 2, 3}, B ={3, 5, 7, 8} and *f* from A to B is defined by $f : x \to 2x + 1$ i.e. $f(x) = 2x + 1$.

(a) Find $f(1)$, $f(2)$, $f(3)$

(b) Show that *f* is a function from A to B

(c) Identify domain, co-domain, images of each element in A and range of *f*

(d) Verify that whether the range is equal to codomain

Solution:

(a) $f(x) = 2x + 1$

$f(1) = 2 + 1 = 3$, $f(2) = 4 + 1 = 5$, $f(3) = 6 + 1 = 7$

(b) The relation is {(1,3), (2, 5), (3, 7)}

Clearly each element of A has a unique image in B. Thus *f* is a function.

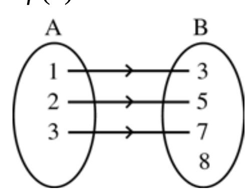


Fig 7. 6

(c) The domain set is A = {1, 2, 3}

The co-domain set is B = {3, 5, 7, 8}

Image of 1 is 3 ; 2 is 5 ; 3 is 7

The range of *f* is {3, 5, 7}

(d) $\{3, 5, 7\} \neq \{3, 5, 7, 8\}$

$\therefore$ The range is not equal to the co-domain

Example 7.4:

A father '*d*' has three sons *a*, *b*, *c*. By assuming sons as a set A and father as a singleton set B, show that

(i) the relation 'is a son of' is a function from $A \to B$ and

(ii) the relation 'is a father of' from $B \to A$ is not a function.

Solution:

(i) A = $\{a, b, c\}$, B = $\{d\}$

a is son of d

b is son of d

c is son of d

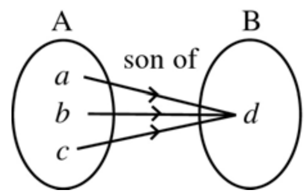


Fig 7. 7

The ordered pairs are (a, d), (b, d), (c, d). For each element in A there is a unique element in B. Clearly the relation 'is son of' from A to B is a function.

(ii) d is father of a

d is father of b

d is father of c

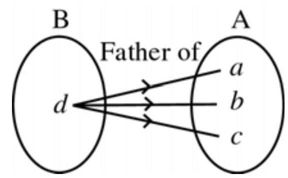


Fig 7. 8

The ordered pairs are (d, a), (d, b), (d, c). The first element d is associated with three different elements (not unique)

Clearly the relation'is father of' from B to A is not a function.

Example 7.5: A classroom consists of 7 benches. The strength of the class is 35. Capacity of each bench is 6. Show that the relation 'sitting' between the set of students and the set of benches is a function. If we interchange the sets, what will be happened?

Solution:

The domain set is the set of students and the co-domain set is the set of benches. Each student will occupy only one bench. Each student has seat also. By principle of function, ''each student occupies a single bench'. Therefore the relation 'sitting' is a function from set of Students to set of Benches.

If we interchange the sets, the set of benches becomes the domain set and the set of students becomes co-domain set. Here atleast one bench consists of more than one student. This is against the principle of function i.e. each element in the domain should have associated with only one element in the co-domain. Thus if we interchange the sets, it is not possible to define a function.

Note :

Consider the function $f : \text{A} \rightarrow \text{B}$

i.e. $x \rightarrow f(x)$ where $x \in \text{A}$, $f(x) \in \text{B}$.

Read '$f(x)$' as 'f of x'. The meaning of $f(x)$ is the value of the function f at x (which is the image of x under the function f). If we write $y = f(x)$, the symbol f represents the function name, x denotes the independent variable (argument) and y denotes the dependent variable.

Clearly, in $f(x)$, f is the name of the function and not $f(x)$. However we will often refer to the function as $f(x)$ in order to know the variable on which f depends.

Example 7.6: Identify the name of the function, the domain, co-domain, independent variable, dependent variable and range if $f : R \rightarrow R$ defined by $y = f(x) = x^2$

Solution:

Name of the function is a square function.

Domain set is R.

Co-domain set is R.

Independent variable is x.

Dependent variable is y.

x can take any real number as its value. But y can take only positive real number or zero as its value, since it is a square function.

$\therefore$ Range of f is set of non negative real numbers.

Example 7.7: Name the function and independent variable of the following function:

(i) $f(\theta) = \sin\theta$ (ii) $f(x) = \sqrt{x}$ (iii) $f(y) = e^y$ (iv) $f(t) = \log_e t$

Solution:

	Name of the function	independent variable
(i)	sine	θ
(ii)	square root	x
(iii)	exponential	y
(iv)	logarithmic	t

The domain conversion

If the domain is not stated explicitly for the function $y = f(x)$, the domain is assumed to be the largest set of x values for which the formula gives real y values. If we want to restrict the domain, we must specify the condition.

The following table illustrates the domain and range of certain functions.

Function	Domain (x)	Range (y or $f(x)$)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	R – {0} Non zero Real numbers	R – {0}
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = \sin x$	$(-\infty, \infty)$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ principal domain	$[-1, 1]$
$y = \cos x$	$(-\infty, \infty)$ $[0, \pi]$ principal domain	$[-1. 1]$
$y = \tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ principal domain	$(-\infty, \infty)$
$y = e^x$	$(-\infty, \infty)$	$(0, \infty)$
$y = \log_e^x$	$(0, \infty)$	$(-\infty, \infty)$

7.2.1 Graph of a function:

The graph of a function f is a graph of the equation $y = f(x)$

Example 7.8: Draw the graph of the function $f(x) = x^2$

Solution:

Draw a table of some pairs (x, y) which satisfy $y = x^2$

x	0	1	2	3	– 1	– 2	– 3
y	0	1	4	9	1	4	9

Plot the points and draw a smooth curve passing through the plotted points.

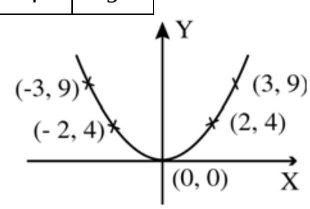


Fig 7. 9

Note:

Note that if we draw a vertical line to the above graph, it meets the curve at only one point i.e. for every x there is a unique y

Functions and their Graphs (Vertical line test)

Not every curve we draw is the graph of a function. A function f can have only one value $f(x)$ i.e. y for each x in its domain. Thus no vertical line can intersect the graph of a function more than once. Thus if 'a' is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$ only.

Consider the following graphs:

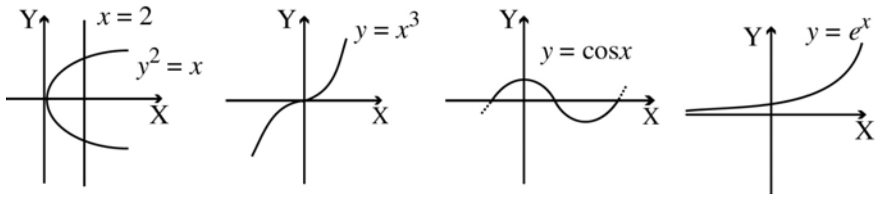


Fig 7. 10

Except the graph of $y^2 = x$, (or $y = \pm\sqrt{x}$) all other graphs are graphs of functions. But for $y^2 = x$, if we draw a vertical line $x = 2$, it meets the curve at two points $(2, \sqrt{2})$ and $(2, -\sqrt{2})$Therefore the graph of $y^2 = x$ is not a graph of a function.

Example 7.9: Show that the graph of $x^2 + y^2 = 4$ is not the graph of a function.

Solution:

Clearly the equation $x^2 + y^2 = 4$ represents a circle with radius 2 and centre at the origin.

Take
$$x = 1$$
$$y^2 = 4 - 1 = 3$$
$$y = \pm\sqrt{3}$$

For the same value $x = 1$, we have two y-values $\sqrt{3}$ and $-\sqrt{3}$. It violates the definition of a function. In the *fig* 7.11 the line $x = 1$ meets the curve at two places

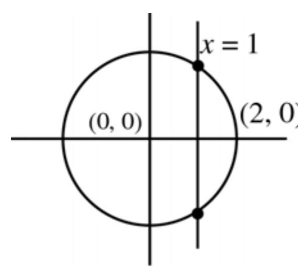


Fig 7. 11

$(1, \sqrt{3})$ and $(1, -\sqrt{3})$. Hence, the graph of $x^2 + y^2 = 4$ is not a graph of a function.

7.2.2 Types of functions:

1. Onto function

If the range of a function is equal to the co-domain then the function is called an onto function. Otherwise it is called an into function.

In f:A→B, the range of f or the image set f(A) is equal to the co-domain B i.e. f(A) = B then the function is onto.

Example 7.10

Let A = {1, 2, 3, 4}, B = {5, 6}. The function f is defined as follows: $f(1) = 5$, $f(2) = 5$, $f(3) = 6$, $f(4) = 6$. Show that f is an onto function.

Solution:

f = {(1, 5), (2, 5), (3, 6), (4, 6)}

The range of f, f(A) = {5, 6}

co-domain B = {5, 6}

i.e. f(A) = B

⇒ the given function is onto

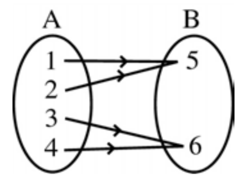


Fig 7. 12

Example 7.11: Let X = {a, b}, Y = {c, d, e} and f = {(a, c), (b, d)}. Show that f is not an onto function.

Solution:

Draw the diagram

The range of f is {c, d}

The co-domain is {c, d, e}

The range and the co-domain are not equal, and hence the given function is not onto

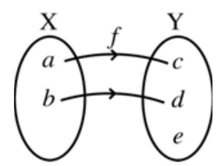


Fig 7. 13

Note :

(1) For an onto function for each element (image) in the co-domain, there must be a corresponding element or elements (pre-image) in the domain.

(2) Another name for onto function is surjective function.

Definition: A function f is onto if to each element b in the co-domain, there is atleast one element a in the domain such that $b = f(a)$

2. One-to-one function:

A function is said to be one-to-one if each element of the range is associated with exactly one element of the domain.

i.e. two different elements in the domain (A) have different images in the co-domain (B).

i.e. $a_1 \neq a_2 \quad \Rightarrow \quad f(a_1) \neq f(a_2) \quad a_1, a_2 \in$ A,

Equivalently $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

The function defined in 7.11 is one-to-one but the function defined in 7.10 is not one-to-one.

Example 7.12: Let A = {1, 2, 3}, B = {*a*, *b*, *c*}. Prove that the function *f* defined by $f = \{(1, a), (2, b), (3, c)\}$ is a one-to-one function.

Solution:

Here 1, 2 and 3 are associated with *a*, *b* and *c* respectively.

The different elements in A have different images in B under the function *f*. Therefore *f* is one-to-one.

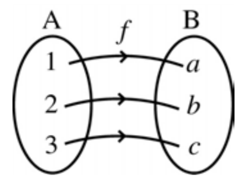


Fig 7. 14

Example 7.13: Show that the function $y = x^2$ is not one-to-one.

Solution:

For the different values of *x* (say 1, – 1) we have the same value of *y*. i.e. different elements in the domain have the same element in the co-domain. By definition of one-to-one, it is not one-to-one (OR)

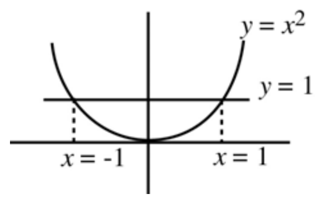


Fig 7. 15

$$y = f(x) = x^2$$

$$f(1) = 1^2 = 1$$

$$f(-1) = (-1)^2 = 1$$

$$\Rightarrow \quad f(1) = f(-1)$$

But $1 \neq -1$. Thus different objects in the domain have the same image.

$\therefore$ The function is not one-to-one.

Note: (1) A function is said to be injective if it is one-to-one.

(2) It is said to be bijective if it is both one-to-one and onto.

(3) The function given in example 7.12 is bijective while the functions given in 7.10, 7.11, 7.13 are not bijective.

Example 7.14. Show that the function $f : R \to R$ defined by $f(x) = x + 1$ is bijective.

Solution:

To prove that *f* is bijective, it is enough to prove that the function *f* is

(i) onto (ii) one-to-one

(i) Clearly the image set is R, which is same as the co-domain R. Therefore, it is onto. i.e. take $b \in R$. Then we can find $b - 1 \in R$ such that $f(b - 1) = (b - 1) + 1 = b$. So *f* is onto.

(ii) Further two different elements in the domain R have different images in the co-domain R. Therefore, it is one-to-one.

i.e. $f(a_1) = f(a_2) \Rightarrow a_1 + 1 = a_2 + 1 \Rightarrow a_1 = a_2$. So *f* is one-to-one.

Hence the function is bijective.

3. Identity function:

A function f from a set A to the same set A is said to be an identity function if $f(x) = x$ for all $x \in$ A i.e. f : A $\to$ A is defined by $f(x) = x$ for all $x \in$ A. Identity function is denoted by I_A or simply I. Therefore $I(x) = x$ always.

Graph of identity function:

The graph of the identity function $f(x) = x$ is the graph of the function $y = x$. It is nothing but the straight line $y = x$ as shown in the fig. (7.16)

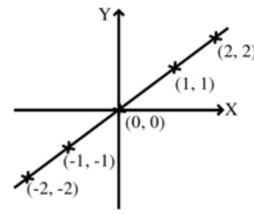


Fig 7. 16

4. Inverse of a function:

To define the inverse of a function f i.e. f^{-1} (read as 'f inverse'), the function f must be one-to-one and onto.

Let A = {1, 2, 3}, B = {a, b, c, d}. Consider a function f = {(1, a), (2, b), (3, c)}. Here the image set or the range is {a, b, c} which is not equal to the co-domain {a, b, c, d}. Therefore, it is not onto.

For the inverse function f^{-1} the co-domain of f becomes domain of f^{-1}.

i.e. If f : A $\to$ B then f^{-1} : B $\to$ A . According to the definition of domain, each element of the domain must have image in the co-domain. In f^{-1}, the element 'd' has no image in A. Therefore f^{-1} is not a function. This is because the function f is not onto.

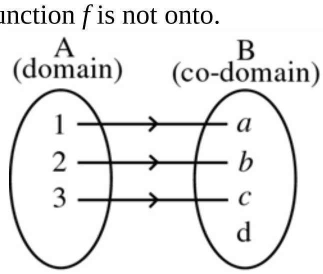


Fig 7. 17 a

$f(1) = a$
$f(2) = b$
$f(3) = c$
All the elements in A have images

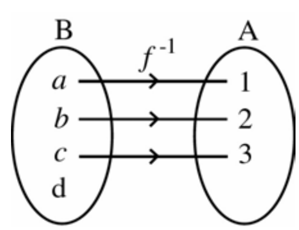


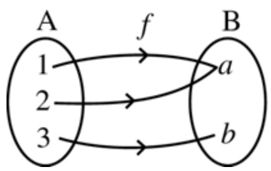
Fig 7.17 b

$f^{-1}(a) = 1$
$f^{-1}(b) = 2$
$f^{-1}(c) = 3$
$f^{-1}(d) = ?$
The element d has no image.

Again consider a function which is not one-to-one. i.e. consider
f = {(1, a), (2, a), (3, b)} where A = {1, 2, 3}, B = {a, b}

Here the two different elements '1' and '2' have the same image '*a*'. Therefore the function is not one-to-one.

The range = {*a*, *b*} = B. $\therefore$ The function is onto.



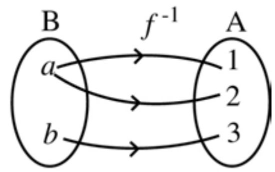


Fig 7. 18

$f(1) = a$

$f(2) = a$

$f(3) = b$

Here all the elements in A has unique image

$f^{-1}(a) = 1$

$f^{-1}(a) = 2$

$f^{-1}(b) = 3$

The element '*a*' has two images 1 and 2. It violates the principle of the function that each element has a unique image.

This is because the function is not one-to-one.

Thus, 'f^{-1} exists if and only if *f* is one-to-one and onto'.

Note:

(1) Since all the function are relations and inverse of a function is also a relation. We conclude that for a function which is not one-to-one and onto, the inverse f^{-1} does not exist

(2) To get the graph of the inverse function, interchange the co-ordinates and plot the points.

To define the mathematical definition of inverse of a function, we need the concept of composition of functions.

5. Composition of functions:

Let A, B and C be any three sets and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be any two functions. Note that the domain of *g* is the co-domain of *f*. Define a new function $(gof) : A \rightarrow C$ such that $(gof)(a) = g(f(a))$ for all $a \in A$. Here $f(a)$ is an element of B. $\therefore$ $g(f(a))$ is meaningful. The function *gof* is called the composition of two functions *f* and *g*.

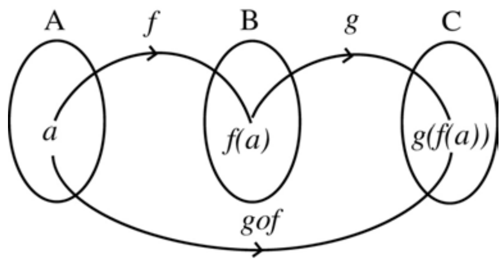


Fig 7. 19

Note:

The small circle *o* in *gof* denotes the composition of g and *f*

Example 7.15: Let A = {1, 2}, B = {3, 4} and C = {5, 6} and $f : A \to B$ and $g : B \to C$ such that $f(1) = 3$, $f(2) = 4$, $g(3) = 5$, $g(4) = 6$. Find *gof*.

Solution:

gof is a function from $A \to C$.

Identify the images of elements of A under the function *gof*.

$(gof)(1) = g(f(1)) = g(3) = 5$

$(gof)(2) = g(f(2)) = g(4) = 6$

i.e. image of 1 is 5 and

image of 2 is 6 under *gof*

$\therefore$ $gof = \{(1, 5), (2, 6)\}$

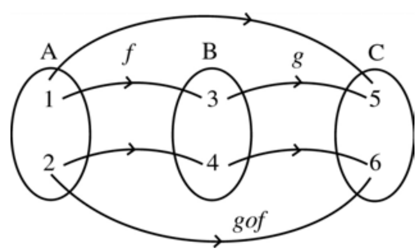


Fig 7. 20

Note:

For the above definition of *f* and *g*, we can't find *fog*. For some functions *f* and *g*, we can find both *fog* and *gof*. In certain cases *fog* and *gof* are equal. In general $fog \neq gof$ i.e. the composition of functions need not be commutative always.

Example 7.16: The two functions $f : R \to R$, $g : R \to R$ are defined by

$f(x) = x^2 + 1$, $g(x) = x - 1$. Find *fog* and *gof* and show that $fog \neq gof$.

Solution:

$$(fog)(x) = f(g(x)) = f(x-1) = (x-1)^2 + 1 = x^2 - 2x + 2$$

$$(gof)(x) = g(f(x)) = g(x^2+1) = (x^2+1) - 1 = x^2$$

Thus $(fog)(x) = x^2 - 2x + 2$

$$(gof)(x) = x^2$$

$$\Rightarrow \quad fog \neq gof$$

Example 7.17: Let $f, g : R \to R$ be defined by $f(x) = 2x + 1$, and $g(x) = \frac{x-1}{2}$.

Show that $(fog) = (gof)$.

Solution:

$$(fog)(x) = f(g(x)) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x - 1 + 1 = x$$

$$(gof)(x) = g(f(x)) = g(2x+1) = \frac{(2x+1)-1}{2} = x$$

Thus $(fog)(x) = (gof)(x)$

$\Rightarrow$ $fog = gof$

In this example f and g satisfy $(fog)(x) = x$ and $(gof)(x) = x$

Consider the example 7.17. For these f and g, $(fog)(x) = x$ and $(gof)(x) = x$. Thus by the definition of identity function $fog = I$ and $gof = I$ i.e. $fog = gof = I$

Now we can define the inverse of a function f.

Definition:

Let $f : A \to B$ be a function. If there exists a function $g : B \to A$ such that $(fog) = I_B$ and $(gof) = I_A$, then g is called the inverse of f. The inverse of f is denoted by f^{-1}

Note:

(1) The domain and the co-domain of both f and g are same then the above condition can be written as $fog = gof = I$.

(2) If f^{-1} exists then f is said to be invertible.

(3) $f\,o\,f^{-1} = f^{-1}o\,f = I$

Example 7.18: Let $f : R \to R$ be a function defined by $f(x) = 2x + 1$. Find f^{-1}

Solution:

Let $g = f^{-1}$

$(gof)(x) = x$ $\qquad \because gof = I$

$g(f(x)) = x \quad \Rightarrow \quad g(2x+1) = x$

Let $2x + 1 = y \quad \Rightarrow \quad x = \frac{y-1}{2}$

$\therefore \quad g(y) = \frac{y-1}{2}$ or $f^{-1}(y) = \frac{y-1}{2}$

Replace y by x

$$f^{-1}(x) = \frac{x-1}{2}$$

6. Sum, difference, product and quotient of two functions:

Just like numbers, we can add, subtract, multiply and divide the functions if both are having same domain and co-domain.

If f, g : A $\rightarrow$ B are any two functions then the following operations are true.

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\, g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ where } g(x) \neq 0$$

$$(cf)(x) = c.f(x) \text{ where } c \text{ is a constant}$$

Note: Product of two functions is different from composition of two functions.

Example 7.19: The two functions f, g : R$\rightarrow$R are defined by $f(x)=x + 1$, $g(x)=x^2$.

Find $f + g$, $f - g$, fg, $\frac{f}{g}$, $2f$, $3g$.

Solution:

Function	Definition
f	$f(x) = x + 1$
g	$g(x) = x^2$
$f + g$	$(f + g)(x) = f(x) + g(x) = x + 1 + x^2$
$f - g$	$(f - g)(x) = f(x) - g(x) = x + 1 - x^2$
fg	$(fg)(x) = f(x)\, g(x) = (x + 1)x^2$
$\frac{f}{g}$	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x + 1}{x^2}$, (it is defined for $x \neq 0$)
$2f$	$(2f)(x) = 2f(x) = 2(x + 1)$
$3g$	$(3g)(x) = 3g(x) = 3x^2$

7. Constant function:

If the range of a function is a singleton set then the function is called a constant function.

i.e. f : A $\rightarrow$ B is such that $f(a) = b$ for all $a \in$ A, then f is called a constant function.

Let A = {1, 2, 3}, B = {a, b}. If the function f is defined by $f(1) = a$, $f(2) = a$, $f(3) = a$ then f is a constant function.

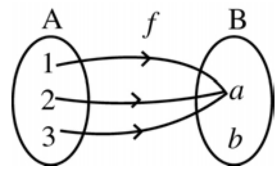


Fig 7. 21

Simply, $f : R \to R$, defined by $f(x) = k$ is a constant function and the graph of this constant function is given in fig. (7.22)

Note that 'is a son of' is a constant function between set of sons and the singleton set consisting of their father.

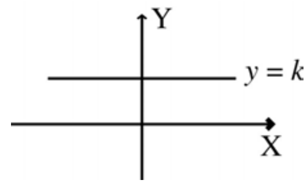


Fig 7. 22

8. Linear function:

If a function $f : R \to R$ is defined in the form $f(x) = ax + b$ then the function is called a linear function. Here a and b are constants.

Example 7.20: Draw the graph of the linear function $f : R \to R$ defined by $f(x) = 2x + 1$.

Solution:

Draw the table of some pairs $(x, f(x))$ which satisfy $f(x) = 2x + 1$.

x	0	1	-1	2
$f(x)$	1	3	-1	5

Plot the points and draw a curve passing through these points. Note that, the curve is a straight line.

Note:

(1) The graph of a linear function is a straight line.

(2) Inverse of a linear function always exists and also linear.

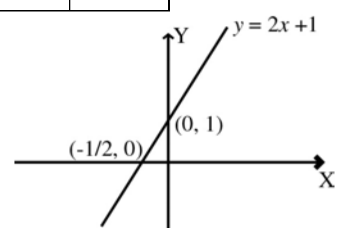


Fig 7. 23

9. Polynomial function:

If $f : R \to R$ is defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers, $a_n \neq 0$ then f is a polynomial function of degree n.

The function $f : R \to R$ defined by $f(x) = x^3 + 5x^2 + 3$ is a cubic polynomial function or a polynomial function of degree 3.

10. Rational function:

Let $p(x)$ and $q(x)$ be any two polynomial functions. Let S be a subset of R obtained after removing all values of x for which $q(x) = 0$ from R.

The function $f : S \to R$, defined by $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ is called a rational function.

Example 7.21: Find the domain of the rational function $f(x) = \frac{x^2 + x + 2}{x^2 - x}$.

Solution:

The domain S is obtained by removing all the points from R for which $g(x) = 0 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$

$\therefore$ S = R − {0, 1}

Thus this rational function is defined for all real numbers except 0 and 1.

11. Exponential functions:

For any number $a > 0$, $a \neq 1$, the function $f : R \to R$ defined by $f(x) = a^x$ is called an exponential function.

Note: For exponential function the range is always R^+ (the set of all positive real numbers)

Example 7.22: Draw the graphs of the exponential functions $f : R \to R^+$ defined by (1) $f(x) = 2^x$ (2) $f(x) = 3^x$ (3) $f(x) = 10^x$.

Solution:

For all these function $f(x) = 1$ when $x = 0$. Thus they cut the y axis at $y = 1$. For any real value of x, they never become zero. Hence the corresponding curves to the above functions do not meet the x-axis for real x. (or meet the x-axis at $-\infty$)

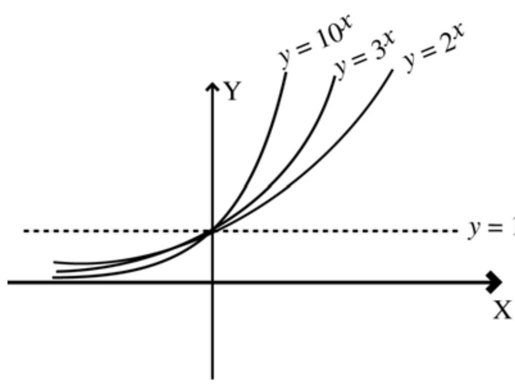


Fig 7. 24

In particular the curve corresponding to $f(x) = e^x$ lies between the curves corresponding to 2^x and 3^x, as $2 < e < 3$.

Example 7.23:

Draw the graph of the exponential function $f(x) = e^x$.

Solution:

For $x = 0$, $f(x)$ becomes 1 i.e. the curve cuts the y axis at $y = 1$. For no real value of x, $f(x)$ equals to 0. Thus it does not meet x-axis for real values of x.

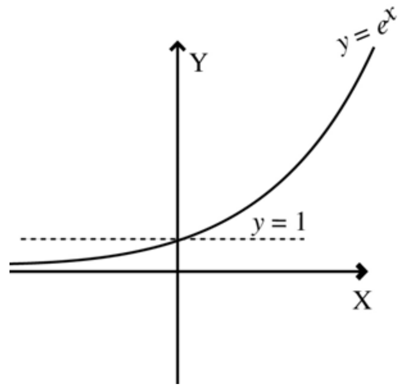


Fig 7. 25

Example 7.24:

Draw the graphs of the logarithmic functions

(1) $f(x) = \log_2 x$ (2) $f(x) = \log_e x$ (3) $f(x) = \log_3 x$

Solution:

The logarithmic function is defined only for positive real numbers. i.e. $(0, \infty)$

Domain : $(0, \infty)$

Range : $(-\infty, \infty)$

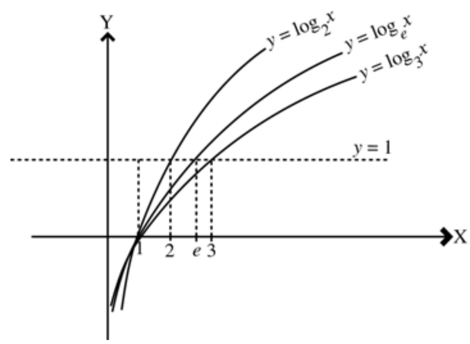


Fig 7. 26

Note:

The inverse of exponential function is a logarithmic function. The general form is $f(x) = \log_a x$, $a \neq 1$, a is any positive number. The domain $(0, \infty)$ of logarithmic function becomes the co-domain of exponential function and the co-domain $(-\infty, \infty)$ of logarithmic function becomes the domain of exponential function. This is due to inverse property.

11. Reciprocal of a function:

The function g : S→R, defined by $g(x) = \frac{1}{f(x)}$ is called reciprocal function of $f(x)$. Since this function is defined only for those x for which $f(x) \neq 0$, we see that the domain of the reciprocal function of $f(x)$ is R − {x : $f(x) = 0$}.

Example 7.25: Draw the graph of the reciprocal function of the function $f(x) = x$.

Solution:

The reciprocal function of $f(x)$ is $\frac{1}{f(x)}$

Thus $g(x) = \frac{1}{f(x)} = \frac{1}{x}$

Here the domain of

$g(x)$ = R − {set of points x for which $f(x) = 0$}

= R − {0}

The graph of $g(x) = \frac{1}{x}$ is as shown in fig 7.27.

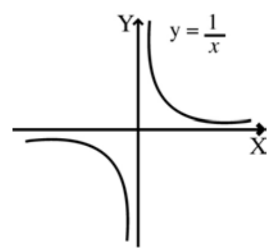


Fig 7. 27

Note:

(1) The graph of $g(x) = \frac{1}{x}$ does not meet either axes for finite real number. Note that the axes x and y meet the curve at infinity only. Thus x and y axes are the asymptotes of the curve $y = \frac{1}{x}$ or $g(x) = \frac{1}{x}$ [Asymptote is a tangent to a curve at infinity. Detailed study of asymptotes is included in XII Standard].

(2) Reciprocal functions are associated with product of two functions.
i.e. if f and g are reciprocals of each other then $f(x)\, g(x) = 1$.
Inverse functions are associated with composition of functions.
i.e.if f and g are inverses of each other then $fog = gof = I$

12. Absolute value function (or modulus function)

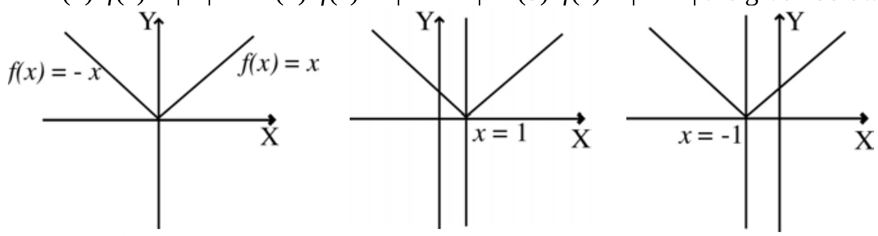
If f : R → R defined by $f(x) = |x|$ then the function is called absolute value function of x.

where $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

The domain is R and co-domain is set of all non-negative real numbers.

The graphs of the absolute functions

(1) $f(x) = |x|$ (2) $f(x) = |x - 1|$ (3) $f(x) = |x + 1|$ are given below.



$f(x) = |x|$ $f(x) = |x - 1|$ $f(x) = |x + 1|$

Fig 7. 28

13. Step functions:

(a) **Greatest integer function**

The function whose value at any real number x is the greatest integer less than or equal to x is called the greatest integer function. It is denoted by $\lfloor x \rfloor$

i.e. $f : R \to R$ defined by $f(x) = \lfloor x \rfloor$

Note that $\lfloor 2.5 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$, $\lfloor -2.1 \rfloor = -3$, $\lfloor .5 \rfloor = 0$, $\lfloor -.2 \rfloor = -1$, $\lfloor 4 \rfloor = 4$

The domain of the function is R and the range of the function is Z (the set of all integers).

(b) Least integer function

The function whose value at any real number x is the smallest integer greater than or equal to x is called the least integer function and is denoted by $\lceil x \rceil$

i.e. $f : R \to R$ defined by $f(x) = \lceil x \rceil$.

Note that $\lceil 2.5 \rceil = 3$, $\lceil 1.09 \rceil = 2$, $\lceil -2.9 \rceil = -2$, $\lceil 3 \rceil = 3$

The domain of the function is R and the range of the function is Z.

Graph of $f(x) = \lfloor x \rfloor$ Graph of $f(x) = \lceil x \rceil$

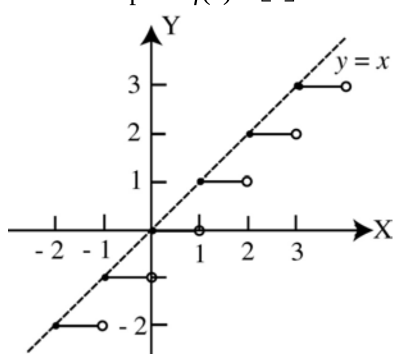


Fig 7. 29

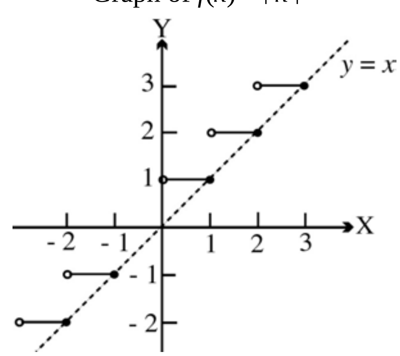


Fig 7. 30

14. Signum function:

If f:R→R is defined by $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then f is called signum function.

The domain of the function is R and the range is {– 1, 0, 1}.

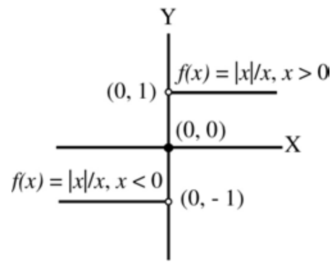


Fig 7. 31

15. Odd and even functions

If $f(x) = f(-x)$ for all x in the domain then the function is called an even function.

If $f(x) = -f(-x)$ for all x in the domain then the function is called an odd function.

For example, $f(x) = x^2$, $f(x) = x^2 + 2x^4$, $f(x) = \dfrac{1}{x^2}$, $f(x) = \cos x$ are some even functions.

and $f(x) = x^3$, $f(x) = x - 2x^3$, $f(x) = \dfrac{1}{x}$, $f(x) = \sin x$ are some odd functions.

Note that there are so many functions which are neither even nor odd. For even function, y axis divides the graph of the function into two exact pieces (symmetric). The graph of an even function is symmetric about y-axis. The graph of an odd function is symmetrical about origin.

Properties:

(1) Sum of two odd functions is again an odd function.
(2) Sum of two even functions is an even function.
(3) Sum of an odd and an even function is neither even nor odd.
(4) Product of two odd functions is an even function.
(5) Product of two even functions is an even function.
(6) Product of an odd and an even function is an odd function.
(7) Quotient of two even functions is an even function. (Denominator function ≠ O)
(8) Quotient of two odd functions is an even function. (Denominator function ≠ O)

(9) Quotient of a even and an odd function is an odd function. (Denominator function $\neq$ O)

16. Trigonometrical functions:

In Trigonometry, we have two types of functions.

(1) Circular functions (2)Hyperbolic functions.

We will discuss circular functions only. The circular functions are

(a) $f(x) = \sin x$ (b) $f(x) = \cos x$ (c) $f(x) = \tan x$

(d) $f(x) = \sec x$ (e) $f(x) = \operatorname{cosec} x$ (f) $f(x) = \cot x$

The following graphs illustrate the graphs of circular functions.

(a) $y = \sin x$ or $f(x) = \sin x$

Domain$(-\infty, \infty)$

Range $[-1, 1]$

Principal domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

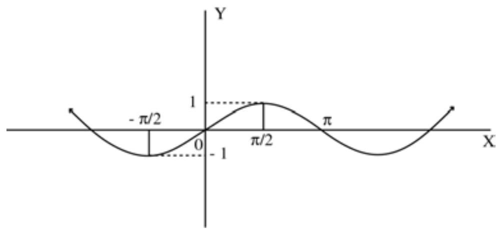


Fig 7. 32

(b) $y = \cos x$

Domain $(-\infty, \infty)$

Range $[-1, 1]$

Principal domain $[0 \;\; \pi]$

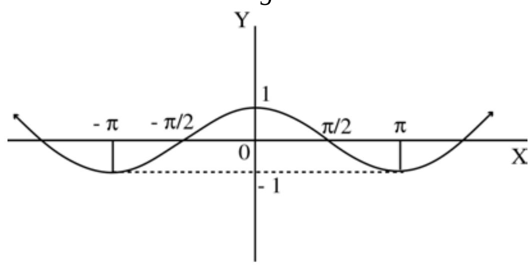


Fig 7. 33

(*c*) $y = \tan x$

Since $\tan x = \frac{\sin x}{\cos x}$, $\tan x$ is defined only for all the values of x for which $\cos x \neq 0$.

i.e. all real numbers except odd integer multiples of $\frac{\pi}{2}$ ($\tan x$ is not obtained for $\cos x = 0$ and hence not defined for x, an odd multiple of $\frac{\pi}{2}$)

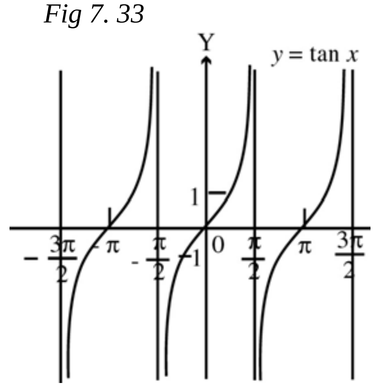


Fig 7. 34

Domain = R – $\left\{(2\,k + 1)\,\dfrac{\pi}{2}\right\}$, $k \in Z$

Range = $(-\infty, \infty)$

(d) y = cosec x

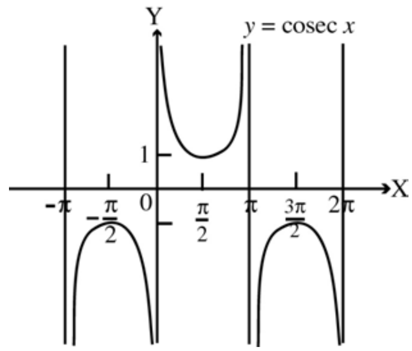


Fig 7. 35

Since cosec x is the reciprocal of sin x, the function cosec x is not defined for values of x for which sin $x = 0$.

$\therefore$ Domain is the set of all real numbers except multiples of π

Domain = R – $\{k\pi\}$, $k \in Z$

Range = $(-\infty, -1] \cup [1, \infty)$

(e) y = sec x

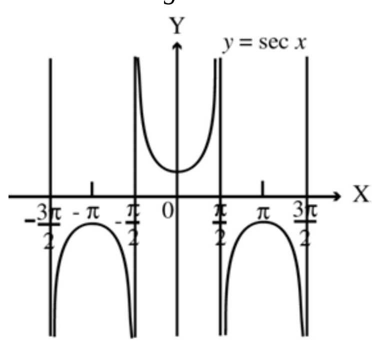


Fig 7. 36

Since sec x is reciprocal of cos$x$, the function sec$x$ is not defined for all values of x for which cos $x = 0$.

$\therefore$ Domain = R – $\left\{(2k + 1)\,\dfrac{\pi}{2}\right\}$, $k \in Z$

Range = $(-\infty, -1] \cup [1, \infty)$

(f) y = cot x

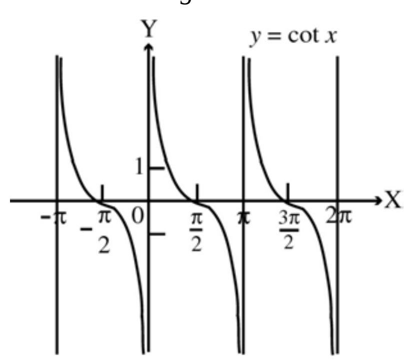


Fig 7. 37

since cot $x = \dfrac{\cos x}{\sin x}$, it is not defined for the values of x for which sin $x = 0$

$\therefore$ Domain = R – $\{k\,\pi\}$, $k \in Z$

Range = $(-\infty, \infty)$

17.Quadratic functions

It is a polynomial function of degree two.

A function $f : R \to R$ defined by $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$ is called a quadratic function. The graph of a quadratic function is always a parabola.

7.3 Quadratic Inequations:

Let $f(x) = ax^2 + bx + c$, be a quadratic function or expression. $a, b, c \in R$, $a \neq 0$

Then $f(x) \geq 0$, $f(x) > 0$, $f(x) \leq 0$ and $f(x) < 0$ are known as quadratic inequations.

The following general rules will be helpful to solve quadratic inequations.

General Rules:

1. If $a > b$, then we have the following rules:
   (i) $(a + c) > (b + c)$ for all $c \in R$
   (ii) $(a - c) > (b - c)$ for all $c \in R$
   (iii) $-a < -b$
   (iv) $ac > bc$, $\frac{a}{c} > \frac{b}{c}$ for any positive real number c
   (v) $ac < bc$, $\frac{a}{c} < \frac{b}{c}$ for any negative real number c.

The above properties, also holds good when the inequality < and > are replaced by ≤ and ≥ respectively.

2. (i) If $ab > 0$ then either $a > 0, b > 0$ (or) $a < 0, b < 0$
   (ii) If $ab \geq 0$ then either $a \geq 0, b \geq 0$ (or) $a \leq 0, b \leq 0$
   (iii) If $ab < 0$ then either $a > 0, b < 0$ (or) $a < 0, b > 0$
   (iv) If $ab \leq 0$ then either $a \geq 0, b \leq 0$ (or) $a \leq 0, b \geq 0$. $a, b, c \in R$

Domain and range of quadratic functions

Solving a quadratic inequation is same as finding the domain of the function $f(x)$ under the given inequality condition.

Different methods are available to solve a quadratic inequation. We can choose any one method which is suitable for the inequation.

Note : Eventhough the syllabus does not require the derivation, it has been derived for better understanding.

Method I: Factorisation method:

Let $ax^2 + bx + c \geq 0$ … (1)

be a quadratic inequation in x where $a, b, c \in R$ and $a \neq 0$.

The quadratic equation corresponding to this inequation is $ax^2 + bx + c = 0$. The discriminant of this equation is $b^2 - 4ac$.

Now three cases arises:

Case (i): $b^2 - 4ac > 0$

In this case, the roots of $ax^2 + bx + c = 0$ are real and distinct. Let the roots be α and β .

$\therefore\ ax^2 + bx + c = a(x - \alpha)(x - \beta)$

But $ax^2 + bx + c \geq 0$ from (1)

$\Rightarrow \quad a(x - \alpha)(x - \beta) \geq 0$

$\Rightarrow \quad (x - \alpha)(x - \beta) \geq 0$ if $a > 0$ (or)

$(x - \alpha)(x - \beta) \leq 0$ if $a < 0$

This inequality is solved by using the general rule (2).

Case (ii): $b^2 - 4ac = 0$

In this case, the roots of $ax^2 + bx + c = 0$ are real and equal. Let the roots be α and α

$\therefore\ ax^2 + bx + c = a(x - \alpha)^2$.

$\Rightarrow \quad a(x - \alpha)^2 \geq 0$

$\Rightarrow \quad (x - \alpha)^2 \geq 0$ if $a > 0$ (or) $(x - \alpha)^2 \leq 0$ if $a < 0$

This inequality is solved by using General rule-2

Case (iii): $b^2 - 4ac < 0$

In this case the roots of $ax^2 + bx + c = 0$ are non-real and distinct.

Here
$$ax^2 + bx + c = a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$\therefore$ The sign of $ax^2 + bx + c$ is same as that of a for all values of x because $\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$ is a positive real number for all values of x.

In the above discussion, we found the method of solving quadratic inequation of the type $ax^2 + bx + c \geq 0$.

Method: II

A quadratic inequality can be solved by factorising the corresponding polynomials.

1. Consider $ax^2 + bx + c > 0$

   Let $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

   Let $\alpha < \beta$

Case (i) : If $x < \alpha$ then $x - \alpha < 0$ & $x - \beta < 0$

$\therefore (x - \alpha)(x - \beta) > 0$

Case (ii): If $x > \beta$ then $x - \alpha > 0$ & $x - \beta > 0$

$\therefore (x - \alpha)(x - \beta) > 0$

Hence If $(x - \alpha)(x - \beta) > 0$ then the values of x lies outside α and β.

2. Consider $ax^2 + bx + c < 0$

   Let $ax^2 + bx + c = a(x - \alpha)(x - \beta)$; $\alpha, \beta \in R$

   Let $\alpha < \beta$ and also $\alpha < x < \beta$

   Then $x - \alpha > 0$ and $x - \beta < 0$

   $\therefore (x - \alpha)(x - \beta) < 0$

   Thus if $(x - \alpha)(x - \beta) < 0$, then the values of x lies between α and β

Method: III

Working Rules for solving quadratic inequation:

Step:1 If the coefficient of x^2 is not positive multiply the inequality by -1. Note that the sign of the inequality is reversed when it is multiplied by a negative quantity.

Step: 2 Factorise the quadratic expression and obtain its solution by equating the linear factors to zero.

Step: 3 Plot the roots obtained in step 2 on real line. The roots will divide the real line in three parts.

Step: 4 In the right most part, the quadratic expression will have positive sign and in the left most part, the expression will have positive sign and in the middle part, the expression will have negative sign.

Step: 5 Obtain the solution set of the given inequation by selecting the appropriate part in 4

Step: 6 If the inequation contains equality operator (i.e. $\leq$, $\geq$), include the roots in the solution set.

Example 7.26: Solve the inequality $x^2 - 7x + 6 > 0$

Method I:

Solution: $x^2 - 7x + 6 > 0$

$\Rightarrow (x - 1)(x - 6) > 0$ [Here $b^2 - 4ac = 25 > 0$]

Now use General rule-2 :

Either $x - 1 > 0, x - 6 > 0$ (or)	$(x - 1) < 0, (x - 6) < 0$
$\Rightarrow x > 1, x > 6$	$\Rightarrow x < 1, x < 6$
we can omit $x > 1$	we can omit $x < 6$
$\Rightarrow x > 6$	$\Rightarrow x < 1$

$\therefore x \in (-\infty, 1) \cup (6, \infty)$

Method II:

$x^2 - 7x + 6 > 0$

$\Rightarrow (x - 1)(x - 6) > 0$

(We know that if $(x - \alpha)(x - \beta) > 0$ then the values of x lies outside of (α, β)

(i.e.) x lies outside of (1, 6)

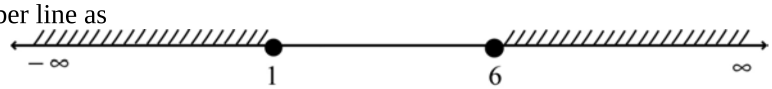
$\Rightarrow x \in (-\infty, 1) \cup (6, \infty)$

Method III:

$x^2 - 7x + 6 > 0$

$\Rightarrow (x - 1)(x - 6) > 0$

On equating the factors to zero, we see that $x = 1$, $x = 6$ are the roots of the quadratic equation. Plotting these roots on real line and marking positive and negative alternatively from the right most part we obtain the corresponding number line as



We have three intervals $(-\infty, 1)$, $(1, 6)$ and $(6, \infty)$. Since the sign of $(x - 1)(x - 6)$ is positive, select the intervals in which $(x - 1)(x - 6)$ is positive.

$\Rightarrow x < 1$ (or) $x > 6$

$\Rightarrow x \in (-\infty, 1) \cup (6, \infty)$

Note : Among the three methods, the third method, is highly useful.

Example 7.27: Solve the inequation $-x^2 + 3x - 2 > 0$

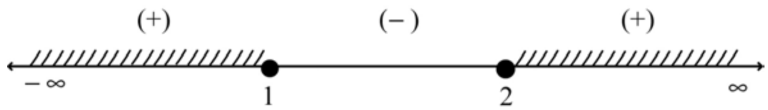
Solution :

$-x^2 + 3x - 2 > 0 \Rightarrow -(x^2 - 3x + 2) > 0$

$\Rightarrow x^2 - 3x + 2 < 0$

$\Rightarrow (x - 1)(x - 2) < 0$

On equating the factors to zero, we obtain $x = 1$, $x = 2$ are the roots of the quadratic equation. Plotting these roots on number line and making positive and negative alternatively from the right most part we obtain the corresponding numberline as given below.



The three intervals are $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$. Since the sign of $(x - 1)(x - 2)$ is negative, select the interval in which $(x - 1)(x - 2)$ is negative.

$\therefore\ x \in (1, 2)$

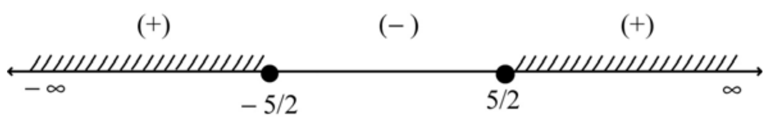
Note : We can solve this problem by the first two methods also.

Example 7.28: Solve : $4x^2 - 25 \geq 0$

Solution : $4x^2 - 25 \geq 0$

$\Rightarrow \quad (2x - 5)(2x + 5) \geq 0$

On equating the factors to zero, we obtain $x = \frac{5}{2}$, $x = -\frac{5}{2}$ are the roots of the quadratic equation. Plotting these roots on number line and making positive and negative alternatively from the right most part we obtain the corresponding number line as given below.



The three intervals are $\left(-\infty, -\frac{5}{2}\right), \left(-\frac{5}{2}, \frac{5}{2}\right) \left(\frac{5}{2}, \infty\right)$

Since the value of $(2x - 5)(2x + 5)$ is positive or zero. Select the intervals in which $f(x)$ is positive and include the roots also. The intervals are $\left(-\infty, -\frac{5}{2}\right)$ and $\left(\frac{5}{2}, \infty\right)$. But the inequality operator contains equality $(\geq)$ also. $\therefore$ The solution set or the domain set should contain the roots $-\frac{5}{2}$, $\frac{5}{2}$.

Thus the solution set is $(-\infty, \frac{-5}{2}\ ] \cup [\ \frac{5}{2}, \infty)$

Example 7.29: Solve the quadratic inequation $64x^2 + 48x + 9 < 0$

Solution:

$64x^2 + 48x + 9 = (8x + 3)^2$

$(8x + 3)^2$ is a perfect square. A perfect square cannot be negative for real x.

$\therefore$ The given quadratic inequation has no solution.

Example 7.30: Solve $f(x)=x^2+2x+6 > 0$ or find the domain of the function $f(x)$

$x^2 + 2x + 6 > 0$

$(x + 1)^2 + 5 > 0$

This is true for all values of x. $\therefore$ The solution set is R

Example 7.31: Solve $f(x) = 2x^2 - 12x + 50 \le 0$ or find the domain of the function $f(x)$.

Solution:

$2x^2 - 12x + 50 \le 0$

$2(x^2 - 6x + 25) \le 0$

$x^2 - 6x + 25 \le 0$

$(x^2 - 6x + 9) + 25 - 9 \le 0$

$(x - 3)^2 + 16 \le 0$

This is not true for any real value of x.

$\therefore$ Given inequation has no solution.

Some special problems (reduces to quadratic inequations)

Example 7.32: Solve: $\dfrac{x+1}{x-1} > 0,\ x \ne 1$

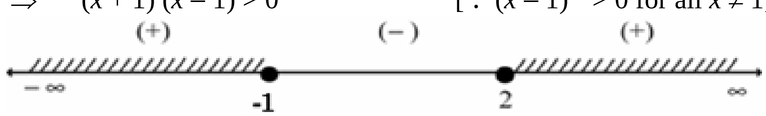
Solution:

$$\frac{x+1}{x-1} > 0$$

Multiply the numerator and denominator by $(x - 1)$

$\Rightarrow \quad \dfrac{(x+1)(x-1)}{(x-1)^2}$

$\Rightarrow \quad (x + 1)(x - 1) > 0 \qquad [\because (x - 1)^2 > 0 \text{ for all } x \ne 1]$



Since the value of $(x + 1)(x - 1)$ is positive or zero select the intervals in which $(x + 1)(x - 1)$ is positive.

$\therefore \quad x \in (-\infty, -1) \cup (1, \infty)$

Example 7.33: Solve : $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3}$

Solution: $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3}$

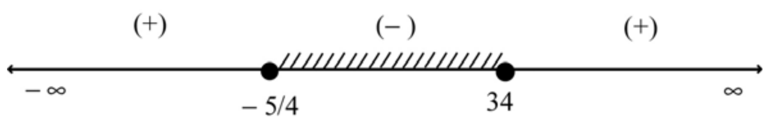
$\Rightarrow \quad \dfrac{x-1}{4x+5} - \dfrac{x-3}{4x-3} < 0$ (Here we cannot cross multiply)

$\Rightarrow \quad \dfrac{(x-1)(4x-3)-(x-3)(4x+5)}{(4x+5)(4x-3)} < 0$

$\Rightarrow \quad \dfrac{18}{(4x+5)(4x-3)} < 0$

$\Rightarrow \quad (4x+5)(4x-3) < 0$ since $18 > 0$

On equating the factors to zero, we obtain $x = \dfrac{-5}{4}$, $x = \dfrac{3}{4}$ are the roots of the quadratic equation. Plotting these roots on number line and making positive and negative alternatively from the right most part we obtain as shown in figure.



Since the value of $(4x + 5)(4x - 3)$ is negative, select the intervals in which $(4x + 5)(4x - 3)$ is negative. $\therefore\ x \in \left(\dfrac{-5}{4}, \dfrac{3}{4}\right)$

Example 7.34 : If $x \in R$, prove that the range of the function $f(x) = \dfrac{x^2-3x+4}{x^2+3x+4}$ is $\left[\dfrac{1}{7}, \ 7\right]$

Solution:

$$\text{Let } y = \frac{x^2-3x+4}{x^2+3x+4}$$

$$(x^2+3x+4)y = x^2-3x+4$$

$$\Rightarrow \quad x^2(y-1) + 3x(y+1) + 4(y-1) = 0$$

Clearly, this is a quadratic equation in x. It is given that x is real.

$\Rightarrow$ Discriminant ≥ 0

$\Rightarrow \quad 9(y+1)^2 - 16(y-1)^2 \geq 0$

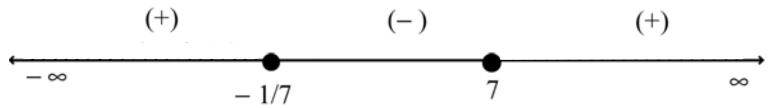
$\Rightarrow \quad [3(y+1)]^2 - [4(y-1)]^2 \geq 0$

$\Rightarrow \quad [3(y+1) + 4(y-1)]\ [3(y+1) - 4(y-1)] \geq 0$

$\Rightarrow \quad (7y-1)(-y+7) \geq 0$

$\Rightarrow \quad -(7y-1)(y-7) \geq 0$

$\Rightarrow \quad (7y-1)(y-7) \leq 0$



The intervals are $\left(-\infty, \frac{1}{7}\right)$, $\left(\frac{1}{7}, 7\right)$ and $(7, \infty)$. Since the value of $(7y-1)(y-7)$ is negative or zero, select the intervals in which $(7y-1)(y-1)$ is negative and include the roots $\frac{1}{7}$ and 7.

$\therefore y \in \left[\frac{1}{7}, 7\right]$ i.e. the value of $\frac{x^2-3x+4}{x^2+3x+4}$ lies between $\frac{1}{7}$ and 7

i.e. the range of $f(x)$ is $\left[\frac{1}{7}, 7\right]$

EXERCISE 7.1

(1) If $f, g : R \rightarrow R$, defined by $f(x) = x + 1$ and $g(x) = x^2$, find (i) $(fog)(x)$ (ii) $(gof)(x)$ (iii) $(fof)(x)$ (iv) $(gog)(x)$ (v) $(fog)(3)$ (vi) $(gof)(3)$

(2) For the functions f, g as defined in (1) define

(i) $(f+g)(x)$ (ii) $\left(\frac{f}{g}\right)(x)$ (iii) $(fg)(x)$ (iv) $(f-g)(x)$ (v) $(gf)(x)$

(3) Let $f : R \rightarrow R$ be defined by $f(x) = 3x + 2$. Find f^{-1} and show that $fof^{-1} = f^{-1}of = I$

(4) Solve each of the following inequations:

(i) $x^2 \leq 9$ (ii) $x^2 - 3x - 18 > 0$ (iii) $4 - x^2 < 0$

(iv) $x^2 + x - 12 < 0$ (v) $7x^2 - 7x - 84 \geq 0$ (vi) $2x^2 - 3x + 5 < 0$

(vii) $\frac{3x-2}{x-1} < 2,\ x \neq 1$ (viii) $\frac{2x-1}{x} > -1,\ x \neq 0$ (ix) $\frac{x-2}{3x+1} > \frac{x-3}{3x-2}$

(5) If x is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ cannot have any value between 5 and 9.

(6) If x is real, prove that the range of $f(x) = \frac{x^2-2x+4}{x^2+2x+4}$ is between $\left[\frac{1}{3}, 3\right]$

(7) If x is real, prove that $\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11}$ and 1.

8. DIFFERENTIAL CALCULUS

Calculus is the mathematics of motion and change. When increasing or decreasing quantities are made the subject of mathematical investigation, it frequently becomes necessary to estimate their rates of growth or decay. Calculus was invented for the purpose of solving problems that deal with continuously changing quantities. Hence, the primary objective of the Differential Calculus is to describe an instrument for the measurement of such rates and to frame rules for its formation and use.

Calculus is used in calculating the rate of change of velocity of a vehicle with respect to time, the rate of change of growth of population with respect to time, etc. Calculus also helps us to maximise profits or minimise losses.

Isacc Newton of England and Gottfried Wilhelm Leibnitz of Germany invented calculus in the 17th century, independently. Leibnitz, a great mathematician of all times, approached the problem of settling tangents geometrically; but Newton approached calculus using physical concepts. Newton, one of the greatest mathematicians and physicists of all time, applied the calculus to formulate his laws of motion and gravitation.

8.1 Limit of a Function

The notion of limit is very intimately related to the intuitive idea of nearness or closeness. Degree of such closeness cannot be described in terms of basic algebraic operations of addition and multiplication and their inverse operations subtraction and division respectively. It comes into play in situations where one quantity depends on another varying quantity and we have to know the behaviour of the first when the second is very close to a fixed given value.

Let us look at some examples, which will help in clarifying the concept of a limit. Consider the function $f : R \to R$ given by

$$f(x) = x + 4.$$

Look at tables 8.1 and 8.2 These give values of $f(x)$ as x gets closer and closer to 2 through values less than 2 and through values greater than 2 respectively.

x	1	1.5	1.9	1.99	1.999
$f(x)$	5	5.5	5.9	5.99	5.999

Table 8.1

x	3	2.5	2.1	2.01	2.001
$f(x)$	7	6.5	6.1	6.01	6.001

Table 8.2

From the above tables we can see that as x approaches 2, $f(x)$ approaches 6. In fact, the nearer x is chosen to 2, the nearer $f(x)$ will be to 6. Thus 6 is the value of $(x + 4)$ as x approaches 2. We call such a value the limit of $f(x)$ as x tends to 2 and denote it by $\lim_{x \to 2} f(x)=6$. In this example the value $\lim_{x \to 2} f(x)$ coincides with the value $(x + 4)$ when $x = 2$, that is, $\lim_{x \to 2} f(x) = f(2)$.

Note that there is a difference between '$x \to 0$' and '$x = 0$'. $x \to 0$ means that x gets nearer and nearer to 0, but never becomes equal to 0. $x = 0$ means that x takes the value 0.

Now consider another function f given by $f(x) = \dfrac{x^2 - 4}{(x - 2)}$. This function is not defined at the point $x = 2$, since division by zero is undefined. But $f(x)$ is defined for values of x which approach 2. So it makes sense to evaluate $\lim_{x \to 2} \dfrac{x^2 - 4}{(x - 2)}$. Again we consider the following tables 8.3 and 8. 4 which give the values of $f(x)$ as x approaches 2 through values less than 2 and through values greater than 2, respectively.

x	1	1.5	1.9	1.99	1.999
$f(x)$	3	3.5	3.9	3.99	3.999

Table 8.3

x	3	2.5	2.1	2.01	2.001
$f(x)$	5	4.5	4.1	4.01	4.001

Table 8.4

We see that $f(x)$ approaches 4 as x approaches 2. Hence $\lim_{x \to 2} f(x) = 4$.

You may have noticed that $f(x) = \dfrac{x^2 - 4}{(x - 2)} = \dfrac{(x + 2)(x - 2)}{(x - 2)} = x + 2$, if $x \neq 2$.

In this case a simple way to calculate the limit above is to substitute the value $x = 2$ in the expression for $f(x)$, when $x \neq 2$, that is, put $x = 2$ in the expression x + 2.

Now take another example. Consider the function given by $f(x) = \dfrac{1}{x}$. We see that $f(0)$ is not defined. We try to calculate $\lim_{x \to 0} f(x)$. Look at tables 8.5 and 8.6

x	1/2	1/10	1/100	1/1000
$f(x)$	2	10	100	1000

Table 8.5

x	– 1/2	– 1/10	– 1/100	– 1/1000
$f(x)$	– 2	– 10	– 100	– 1000

Table 8.6

We see that $f(x)$ does not approach any fixed number as x approaches 0. In this case we say that $\lim_{x \to 0} f(x)$ does not exist. This example shows that there are cases when the limit may not exist. Note that the first two examples show that such a limit exists while the last example tells us that such a limit may not exist. These examples lead us to the following.

Definition

Let f be a function of a real variable x. Let c, l be two fixed numbers. If $f(x)$ approaches the value l as x approaches c, we say l is the limit of the function $f(x)$ as x tends to c. This is written as $\lim_{x \to c} f(x) = l$.

Left Hand and Right Hand Limits

While defining the limit of a function as x tends to c, we consider values of $f(x)$ when x is very close to c. The values of x may be greater or less than c. If we restrict x to values less than c, then we say that x tends to c from below or from the left and write it symbolically as $x \to c - 0$ or simply $x \to c\_$. The limit of f with this restriction on x, is called the left hand limit. This is written as

$$\mathrm{L}f(c) = \lim_{x \to c\_} f(x), \text{ provided the limit exists.}$$

Similarly if x takes only values greater than c, then x is said to tend to c from above or from right, and is denoted symbolically as $x \to c + 0$ or $x \to c_+$. The limit of f is then called the right hand limit. This is written as $\mathrm{R}f(c) = \lim_{x \to c_+} f(x)$.

It is important to note that for the existence of $\lim_{x \to c} f(x)$ it is necessary that both $\mathrm{L}f(c)$ and $\mathrm{R}f(c)$ exists and $\mathrm{L}f(c) = \mathrm{R}f(c) = \lim_{x \to c} f(x)$. These left and right hand limits are also known as one sided limits.

8.1.1 Fundamental results on limits

(1) If $f(x) = k$ for all x, then $\lim_{x \to c} f(x) = k$.

(2) If $f(x) = x$ for all x, then $\lim_{x \to c} f(x) = c$.

(3) If f and g are two functions possessing limits and k is a constant then

(i) $\lim_{x \to c} k\, f(x) = k \lim_{x \to c} f(x)$

(ii) $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

(iii) $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

(iv) $\lim_{x \to c} [f(x) \,.\, g(x)] = \lim_{x \to c} f(x) \,.\, \lim_{x \to c} g(x)$

(v) $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \lim_{x \to c} f(x) \Big/ \lim_{x \to c} g(x), \qquad g(x) \neq 0$

(vi) If $f(x) \leq g(x)$ then $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$.

Example 8.1 :

Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ if it exists.

Solution:

Let us evaluate the left hand and right hand limits.

When $x \to 1_-$, put $x = 1 - h$, $h > 0$.

Then $\lim_{x \to 1-} \frac{x^2 - 1}{x - 1} = \lim_{h \to 0} \frac{(1 - h)^2 - 1}{1 - h - 1} = \lim_{h \to 0} \frac{1 - 2h + h^2 - 1}{-h}$

$= \lim_{h \to 0} (2 - h) = \lim_{h \to 0} (2) - \lim_{h \to 0} (h) = 2 - 0 = 2$

When $x \to 1_+$ put $x = 1 + h$, $h > 0$

Then $\lim_{x \to 1+} \frac{x^2 - 1}{x - 1} = \lim_{h \to 0} \frac{(1 + h)^2 - 1}{1 + h - 1} = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h}$

$= \lim_{h \to 0} (2 + h) = \lim_{h \to 0} (2) + \lim_{h \to 0} (h)$

$= 2 + 0 = 2$, using (1) and (2) of 8.1.1

So that both, the left hand and the right hand, limits exist and are equal. Hence the limit of the function exists and equals 2.

(i.e.) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$

Note: Since $x \neq 1$, division by $(x - 1)$ is permissible.

$$\therefore \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2 .$$

Example 8.2: Find the right hand and the left hand limits of the function at $x = 4$

$$f(x) = \begin{cases} \frac{|x - 4|}{x - 4} & \text{for } x \neq 4 \\ 0, & \text{for } x = 4 \end{cases}$$

Solution:

Now, when $x > 4$, $|x - 4| = x - 4$

Therefore $\lim_{x \to 4+} f(x) = \lim_{x \to 4+} \frac{|x - 4|}{x - 4} = \lim_{x \to 4+} \frac{x - 4}{x - 4} = \lim_{x \to 4} (1) = 1$

Again when $x < 4$, $|x - 4| = -(x - 4)$

Therefore $\lim_{x \to 4-} f(x) = \lim_{x \to 4-} \frac{-(x - 4)}{(x - 4)} = \lim_{x \to 4-} (-1) = -1$

Note that both the left and right hand limits exist but they are not equal.

i.e. $Rf(4) = \lim_{x \to 4+} f(x) \neq \lim_{x \to 4-} f(x) = Lf(4)$.

Example 8.3

Find $\lim_{x \to 0} \frac{3x + |x|}{7x - 5|x|}$, if it exists.

Solution:

$$Rf(0) = \lim_{x \to 0+} \frac{3x + |x|}{7x - 5|x|} = \lim_{x \to 0+} \frac{3x + x}{7x - 5x} \quad (\text{since } x > 0, |x| = x)$$

$$= \lim_{x \to 0+} \frac{4x}{2x} = \lim_{x \to 0+} 2 = 2 .$$

$$L f(0) = \lim_{x \to 0-} \frac{3x + |x|}{7x - 5|x|} = \lim_{x \to 0-} \frac{3x - x}{7x - 5(-x)} \quad (\text{since } x < 0, |x| = -x)$$

$$= \lim_{x \to 0-} \frac{2x}{12x} = \lim_{x \to 0-} \left(\frac{1}{6}\right) = \frac{1}{6} .$$

Since $Rf(0) \neq Lf(0)$, the limit does not exist.

Note: Let $f(x) = g(x) / h(x)$.

Suppose at $x = c$, $g(c) \neq 0$ and $h(c) = 0$, then $f(c) = \frac{g(c)}{0}$.

In this case $\lim_{x \to c} f(x)$ does not exist.

Example 8.4 : Evaluate $\lim_{x \to 3} \frac{x^2 + 7x + 11}{x^2 - 9}$.

Solution:

Let $f(x) = \frac{x^2 + 7x + 11}{x^2 - 9}$. This is of the form $f(x) = \frac{g(x)}{h(x)}$,

where $g(x) = x^2 + 7x + 11$ and $h(x) = x^2 - 9$. Clearly $g(3) = 41 \neq 0$ and $h(3) = 0$.

Therefore $f(3) = \frac{g(3)}{h(3)} = \frac{41}{0}$. Hence $\lim_{x \to 3} \frac{x^2 + 7x + 11}{x^2 - 9}$ does not exist.

Example 8.5: Evaluate $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \lim_{x \to 0} \frac{\left(\sqrt{1 + x} - 1\right)\left(\sqrt{1 + x} + 1\right)}{x\left(\sqrt{1 + x} + 1\right)}$$

$$= \lim_{x \to 0} \frac{(1 + x) - 1}{x\left(\sqrt{1 + x} + 1\right)} = \lim_{x \to 0} \frac{1}{\left(\sqrt{1 + x} + 1\right)}$$

$$= \frac{\lim_{x \to 0} (1)}{\lim_{x \to 0} \left(\sqrt{1 + x} + 1\right)} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2} .$$

8.1.2 Some important Limits

Example 8.6 :

For $\left|\frac{\Delta x}{a}\right| < 1$ and for any rational index n,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \qquad (a \neq 0)$$

Solution:

Put $\Delta x = x - a$ so that $\Delta x \to 0$ as $x \to a$ and $\left|\frac{\Delta x}{a}\right| < 1$.

Therefore $\quad \frac{x^n - a^n}{x - a} = \frac{(a + \Delta x)^n - a^n}{\Delta x} = \frac{a^n\left(1 + \frac{\Delta x}{a}\right)^n - a^n}{\Delta x}$

Applying Newton's Binomial Theorem for rational index we have

$$\left(1 + \frac{\Delta x}{a}\right)^n = 1 + \binom{n}{1}\left(\frac{\Delta x}{a}\right) + \binom{n}{2}\left(\frac{\Delta x}{a}\right)^2 + \binom{n}{3}\left(\frac{\Delta x}{a}\right)^3 + \ldots + \binom{n}{r}\left(\frac{\Delta x}{a}\right)^r + \ldots$$

$$\therefore \frac{x^n - a^n}{x - a} = \frac{a^n\left[1 + \binom{n}{1}\left(\frac{\Delta x}{a}\right) + \binom{n}{2}\left(\frac{\Delta x}{a}\right)^2 + \ldots + \binom{n}{r}\left(\frac{\Delta x}{a}\right)^r + \ldots\right] - a^n}{\Delta x}$$

$$= \frac{\left[\binom{n}{1}a^{n-1}\,\Delta x + \binom{n}{2}a^{n-2}(\Delta x)^2 + \ldots + \binom{n}{r}a^{n-r}(\Delta x)^r + \ldots\right]}{\Delta x}$$

$$= \binom{n}{1}a^{n-1} + \binom{n}{2}a^{n-2}(\Delta x) + \ldots + \binom{n}{r}a^{n-r}(\Delta x)^{r-1} + \ldots$$

$$= \binom{n}{1}a^{n-1} + \text{terms containing } \Delta x \text{ and higher powers of } \Delta x .$$

Since $\Delta x = x - a$, $x \to a$ means $\Delta x \to 0$ and therefore

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{\Delta x \to 0} \binom{n}{1}a^{n-1} + \lim_{\Delta x \to 0} (\text{terms containing } \Delta x \text{ and higher powers of } \Delta x)$$

$$= \binom{n}{1}a^{n-1} + 0 + 0 + \ldots = na^{n-1} \qquad \text{since } \binom{n}{1} = n .$$

As an illustration of this result, we have the following examples.

Example 8.7: Evaluate $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Solution: $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3(1)^{3-1} = 3(1)^2 = 3$

Example 8.8: Find $\lim_{x \to 0} \frac{(1 + x)^4 - 1}{x}$

Solution: Put $1 + x = t$ so that $t \to 1$ as $x \to 0$

$$\therefore \lim_{x \to 0} \frac{(1 + x)^4 - 1}{x} = \lim_{t \to 1} \frac{t^4 - 1^4}{t - 1} = 4(1)^3 = 4$$

Example 8.9: Find the positive integer n so that $\lim_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = 32$

Solution: We have $\lim_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = n2^{n-1}$

$$\therefore\ n2^{n-1} = 32 = 4 \times 8 = 4 \times 2^3 = 4 \times 2^{4-1}$$

Comparing on both sides we get $n = 4$

Example 8.10: $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

Solution:

We take $y = \dfrac{\sin\theta}{\theta}$. This function is defined for all θ, other than θ = 0, for which both numerator and denominator become zero. When θ is replaced by – θ , the magnitude of the fraction $\dfrac{\sin\theta}{\theta}$ does not change since $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$. Therefore it is enough to find the limit of the fraction as θ tends to 0 through positive values. i.e. in the first quadrant. We consider a circle with centre at O radius unity. A, B are two points on this circle so OA = OB = 1. Let θ be the angle subtended at the centre by the arc AE. Measuring angle in radians, we have sinθ = AC, C being a point on AB such that OD passes through C.

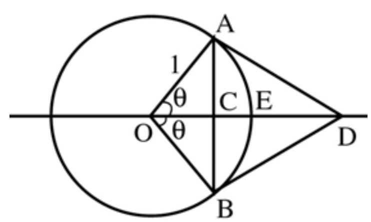


Fig. 8.1

$\cos\theta = OC$, $\theta = \dfrac{1}{2}$ arc AB, $\angle OAD = 90°$

In triangle OAD, AD = tanθ.

Now length of arc AB = 2θ and length of the chord AB = 2 sinθ

sum of the tangents = AD + BD = 2 tanθ

Since the length of the arc is intermediate between the length of chord and the sum of the tangents we can write $2\sin\theta < 2\theta < 2\tan\theta$.

Dividing by 2 sinθ , we have $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$ or $1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

But as $\theta \to 0$, cos θ, given by the distance OC, tends to 1

That is, $\lim_{\theta \to 0} \cos\theta = 1$.

Therefore $1 > \lim_{\theta \to 0} \frac{\sin \theta}{\theta} > 1$, by 3(vi) of 8.1.1

That is, the variable $y = \frac{\sin \theta}{\theta}$ always lies between unity and a magnitude tending to unity, and hence $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

The graph of the function $y = \frac{\sin \theta}{\theta}$ is shown in fig. 8.2

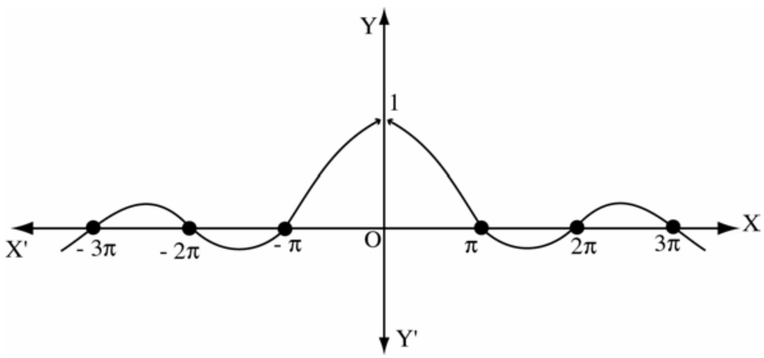


Fig. 8.2

Example 8.11: Evaluate $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta^2}$.

Solution:

$$\frac{1 - \cos \theta}{\theta^2} = \frac{2 \sin^2 \frac{\theta}{2}}{\theta^2} = \frac{1}{2} \frac{\sin^2 \left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)^2} = \frac{1}{2} \left(\frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}} \right)^2$$

If $\theta \to 0$, $\alpha = \frac{\theta}{2}$ also tends to 0 and $\lim_{\theta \to 0} \frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}} = \lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$ and

hence $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} = \lim_{\theta \to 0} \frac{1}{2} \left(\frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}} \right)^2 = \frac{1}{2} \left(\lim_{\theta \to 0} \frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}} \right)^2 = \frac{1}{2} \times 1^2 = \frac{1}{2}$

Example 8.12: Evaluate $\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}}$

Solution:

$$\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right)\sqrt{x}$$

$$= \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right) \cdot \lim_{x \to 0^+} (\sqrt{x}) = 1 \times 0 = 0 .$$

Note: For the above problem left hand limit does not exist since $\sqrt{x}$ is not real for $x < 0$.

Example 8.13: Compute $\lim_{x \to 0} \frac{\sin \beta x}{\sin \alpha x}$, $\alpha \neq 0$.

Solution:

$$\lim_{x \to 0} \frac{\sin \beta x}{\sin \alpha x} = \lim_{x \to 0} \frac{\beta \cdot \frac{\sin \beta x}{\beta x}}{\alpha \cdot \frac{\sin \alpha x}{\alpha x}} = \frac{\beta \lim_{x \to 0}\left(\frac{\sin \beta x}{\beta x}\right)}{\alpha \lim_{x \to 0}\left(\frac{\sin \alpha x}{\alpha x}\right)}$$

$$= \frac{\beta \lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right)}{\alpha \lim_{y \to 0}\left(\frac{\sin y}{y}\right)} = \frac{\beta \times 1}{\alpha \times 1} = \frac{\beta}{\alpha} .$$

since $\theta = \beta x \to 0$ as $x \to 0$ and $y = \alpha x \to 0$ as $x \to 0$

Example 8.14: Compute $\lim_{x \to \pi/6} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1}$

Solution:

We have

$$2 \sin^2 x + \sin x - 1 = (2 \sin x - 1)(\sin x + 1)$$

$$2 \sin^2 x - 3 \sin x + 1 = (2 \sin x - 1)(\sin x - 1)$$

Now $$\lim_{x \to \pi/6} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1} = \lim_{x \to \pi/6} \frac{(2 \sin x - 1)(\sin x + 1)}{(2 \sin x - 1)(\sin x - 1)}$$

$$= \lim_{x \to \pi/6} \frac{\sin x + 1}{\sin x - 1} \left(2 \sin x - 1 \neq 0 \text{ for } x \to \frac{\pi}{6}\right)$$

$$= \frac{\sin \pi/6 + 1}{\sin \pi/6 - 1} = \frac{1/2 + 1}{1/2 - 1} = -3 .$$

Example 8.15: $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

Solution: We know that $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^n}{\lfloor n} + \ldots$

and so $e^x - 1 = \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^n}{\lfloor n} + \ldots$

i.e. $\frac{e^x - 1}{x} = \frac{1}{\lfloor 1} + \frac{x}{\lfloor 2} + \ldots + \frac{x^{n-1}}{\lfloor n} + \ldots$

($\because x \neq 0$, division by x is permissible)

$$\therefore \lim_{x \to 0} \frac{e^x - 1}{x} = \frac{1}{\lfloor 1} = 1 .$$

Example 8.16: Evaluate $\lim_{x \to 3} \frac{e^x - e^3}{x - 3}$.

Solution: Consider $\frac{e^x - e^3}{x - 3}$. Put $y = x - 3$. Then $y \to 0$ as $x \to 3$.

Therefore $\lim_{x \to 3} \frac{e^x - e^3}{x - 3} = \lim_{y \to 0} \frac{e^{y+3} - e^3}{y} = \lim_{y \to 0} \frac{e^3 . e^y - e^3}{y}$

$$= e^3 \lim_{y \to 0} \frac{e^y - 1}{y} = e^3 \times 1 = e^3 .$$

Example 8.17: Evaluate $\lim_{x \to 0} \frac{e^x - \sin x - 1}{x}$.

Solution:

Now $\frac{e^x - \sin x - 1}{x} = \left(\frac{e^x - 1}{x}\right) - \left(\frac{\sin x}{x}\right)$

and so $\lim_{x \to 0} \frac{e^x - \sin x - 1}{x} = \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) - \lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1 - 1 = 0$

Example 8.18: Evaluate $\lim_{x \to 0} \frac{e^{\tan x} - 1}{\tan x}$

Solution: Put $\tan x = y$. Then $y \to 0$ as $x \to 0$

Therefore $\lim_{x \to 0} \frac{e^{\tan x} - 1}{\tan x} = \lim_{y \to 0} \frac{e^y - 1}{y} = 1$

Example 8.19: $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

Solution: We know that $\log_e(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$\frac{\log_e(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots$$

Therefore $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1.$

Note: log*x* means the natural logarithm $\log_e x$.

Example 8.20: Evaluate $\lim_{x \to 1} \frac{\log x}{x-1}$.

Solution: Put $x - 1 = y$. Then $y \to 0$ as $x \to 1$.

Therefore $\lim_{x \to 1} \frac{\log x}{x-1} = \lim_{y \to 0} \frac{\log(1+y)}{y}$

$= 1$ (by example 8.19)

Example 8.21: $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a, \quad a > 0$

Solution: We know that $f(x) = e^{\log f(x)}$ and so $a^x = e^{\log a^x} = e^{x \log a}$.

Therefore $\frac{a^x - 1}{x} = \frac{e^{x \log a} - 1}{x \log a} \times \log a$

Now as $x \to 0$, $y = x \log a \to 0$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{e^y - 1}{y} \times \log a = \log a \lim_{y \to 0} \left(\frac{e^y - 1}{y}\right)$$

$= \log a$. (since $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$)

Example 8.22: Evaluate $\lim_{x \to 0} \frac{5^x - 6^x}{x}$

Solution:

$$\lim_{x \to 0} \frac{5^x - 6^x}{x} = \lim_{x \to 0} \frac{(5^x - 1) - (6^x - 1)}{x}$$

$$= \lim_{x \to 0} \left(\frac{5^x - 1}{x}\right) - \lim_{x \to 0} \left(\frac{6^x - 1}{x}\right)$$

$$= \log 5 - \log 6 = \log\left(\frac{5}{6}\right).$$

Example 8.23: Evaluate $\lim_{x \to 0} \frac{3^x + 1 - \cos x - e^x}{x}$.

Solution:

$$\lim_{x \to 0} \frac{3^x + 1 - \cos x - e^x}{x} = \lim_{x \to 0} \frac{(3^x - 1) + (1 - \cos x) - (e^x - 1)}{x}$$

$$= \lim_{x \to 0} \left(\frac{3^x - 1}{x}\right) + \lim_{x \to 0} \left(\frac{1 - \cos x}{x}\right) - \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right)$$

$$= \log 3 + \lim_{x \to 0} \frac{2 \sin^2 x/2}{x} - 1$$

$$= \log 3 + \lim_{x \to 0} \frac{x}{2} \left(\frac{\sin x/2}{x/2}\right)^2 - 1$$

$$= \log 3 + \frac{1}{2} \lim_{x \to 0} (x) \lim_{x \to 0} \left(\frac{\sin x/2}{x/2}\right)^2 - 1$$

$$= \log 3 + \frac{1}{2} \times 0 \times 1 - 1 = \log 3 - 1.$$

Some important limits :

(1) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ exists and we denote this limit by e

(2) $\lim_{x \to 0} (1 + x)^{1/x} = e$ [by taking $x = \frac{1}{y}$ in (1)]

(3) $\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x = e^k$

Note : (1) The value of e lies between 2 & 3 i.e., $2 < e < 3$

(2) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$ is true for all real x

Thus $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$ for all real values of x.

Note that $e = e^1 = 1 + \frac{1}{1!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{r!} + \ldots$ This number e is also known as transcendental number in the sense that e never satisfies a polynomial (algebraic) equation of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$.

Example 8.24: Compute $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{3x}$.

Solution: Now $\left(1+\frac{1}{x}\right)^{3x} = \left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{x}$ and so

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.\left(1+\frac{1}{x}\right)^{x}.\left(1+\frac{1}{x}\right)^{x}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e.\ e.\ e. = e^3 .$$

Example 8.25: Evaluate $\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3}$.

Solution:

$$\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3} = \lim_{x\to\infty}\left(\frac{x-1+4}{x-1}\right)^{(x-1)+4}$$

$$= \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{(x-1)+4}$$

$$= \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y+4} \quad (\because y = x-1\to\infty \text{ as } x\to\infty)$$

$$= \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y}\left(1+\frac{4}{y}\right)^{4}$$

$$= \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y}.\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{4} = e^4.\ 1 = e^4$$

Example 8.26: Evaluate $\lim_{x\to\pi/2}(1+\cos x)^{3\sec x}$.

Solution: Put $\cos x = \frac{1}{y}$. Now $y\to\infty$ as $x\to\frac{\pi}{2}$.

$$\lim_{x\to\pi/2}(1+\cos x)^{3\sec x} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{3y} = \lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^{y}\right]^{3}$$

$$= \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^{3} = e^3.$$

Example 8.27. Evaluate $\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$

Solution :

$$\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1} = \lim_{x\to 0}\frac{2^x-1}{(1+x-1)}\left(\sqrt{1+x}+1\right)$$

$$= \lim_{x \to 0} \frac{2^x - 1}{x} \cdot \lim_{x \to 0} \left(\sqrt{1 + x} + 1\right)$$

$$= \log 2 \cdot \left(\sqrt{1} + 1\right) \qquad \left(\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right)$$

$$= 2 \log 2 = \log 4 .$$

Example 8.28: Evaluate $\lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{\sin^{-1} x}$

Solution:

Put $\sin^{-1} x = y$. Then $x = \sin y$ and $y \to 0$ as $x \to 0$.

$$\text{Now} \lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{\sin^{-1} x} = \lim_{x \to 0} \frac{(1 + x) - (1 - x)}{\sin^{-1} x} \left(\frac{1}{\sqrt{1 + x} + \sqrt{1 - x}}\right)$$

$$= \lim_{y \to 0} \frac{2 \sin y}{y} \cdot \lim_{y \to 0} \frac{1}{\sqrt{1 + \sin y} + \sqrt{1 - \sin y}}$$

$$= 2 \lim_{y \to 0} \left(\frac{\sin y}{y}\right) \left(\frac{1}{\sqrt{1 + 0} + \sqrt{1 - 0}}\right)$$

$$= 2 \times 1 \times \frac{1}{2} = 1$$

EXERCISE 8.1

Find the indicated limits.

(1) $\lim_{x \to 1} \frac{x^2 + 2x + 5}{x^2 + 1}$

(2) $\lim_{x \to 2^-} \frac{x - 2}{\sqrt{2 - x}}$

(3) $\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$

(4) $\lim_{x \to 1} \frac{x^m - 1}{x - 1}$

(5) $\lim_{x \to 4} \frac{\sqrt{2x + 1} - 3}{\sqrt{x - 2} - \sqrt{2}}$

(6) $\lim_{x \to 0} \frac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q}$

(7) $\lim_{x \to a} \frac{\sqrt[m]{x} - \sqrt[m]{a}}{x - a}$

(8) $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$

(9) $\lim_{x \to 0} \frac{\sqrt{1 + x + x^2} - 1}{x}$

(10) $\lim_{x \to 0} \frac{\sin^2 (x/3)}{x^2}$

(11) $\lim_{x \to 0} \frac{\sin (a + x) - \sin (a - x)}{x}$

(12) $\lim_{x \to 0} \frac{\log (1 + \alpha x)}{x}$

(13) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n + 5}$

(14) Evaluate the left and right limits of $f(x) = \frac{x^3 - 27}{x - 3}$ at $x = 3$. Does the limit of $f(x)$ as $x \to 3$ exist? Justify your answer.

(15) Find the positive integer n such that $\lim_{x \to 3} \frac{x^n - 3^n}{x - 3} = 108$.

(16) Evaluate $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$. Hint : Take e^x or $e^{\sin x}$ as common factor in numerator.

(17) If $f(x) = \frac{ax^2 + b}{x^2 - 1}$, $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to \infty} f(x) = 1$,

then prove that $f(-2) = f(2) = 1$.

(18) Evaluate $\lim_{x \to 0^-} \frac{|x|}{x}$ and $\lim_{x \to 0^+} \frac{|x|}{x}$.

What can you say about $\lim_{x \to 0} \frac{|x|}{x}$?

(19) Compute $\lim_{x \to 0} \frac{a^x - b^x}{x}$, $a, b > 0$. Hence evaluate $\lim_{x \to 0} \frac{5^x - 6^x}{x}$

(20) Without using the series expansion of $\log(1 + x)$,

prove that $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$

8.2 Continuity of a function

Let f be a function defined on an interval $I = [a, b]$. A continuous function on I is a function whose graph $y = f(x)$ can be described by the motion of a particle travelling along it from the point $(a, f(a))$ to the point $(b, f(b))$ without moving off the curve.

Continuity at a point

Definition: A function f is said to be continuous at a point c, $a < c < b$, if

$$\lim_{x \to c} f(x) = f(c)$$

A function f is said to be continuous from the left at c if $\lim_{x \to c^-} f(x) = f(c)$.

Also f is continuous from the right at c if $\lim_{x \to c^+} f(x) = f(c)$. Clearly a function is continuous at c if and only if it is continuous from the left as well as from the right.

Continuity at an end point

A function f defined on a closed interval $[a, b]$ is said to be continuous at the end point a if it is continuous from the right at a, that is,

$$\lim_{x \to a+} f(x) = f(a) .$$

Also the function is continuous at the end point b of $[a, b]$ if

$$\lim_{x \to b-} f(x) = f(b).$$

It is important to note that a function is continuous at a point c if

(i) f is well defined at $x = c$ i.e. $f(c)$ exists. (ii) $\lim_{x \to c} f(x)$ exists, and (iii) $\lim_{x \to c} f(x) = f(c)$.

Continuity in an interval

A function f is said to be continuous in an interval $[a, b]$ if it is continuous at each and every point of the interval.

Discontinuous functions

A function f is said to be discontinuous at a point c of its domain if it is not continuous at c. The point c is then called a point of discontinuity of the function.

Theorem 8.1: If f, g be continuous functions at a point c then the functions
$f + g$, $f - g$, fg are also continuous at c and if $g(c) \neq 0$ then f / g is also continuous at c.

Example 8.29: Every constant function is continuous.

Solution: Let $f(x) = k$ be the constant function.

Let c be a point in the domain of f.

Then $f(c) = k$.

Also $\lim_{x \to c} f(x) = \lim_{x \to c} (k) = k$,

Thus $\lim_{x \to c} f(x) = f(c)$.

Hence $f(x) = k$ is continuous at c.

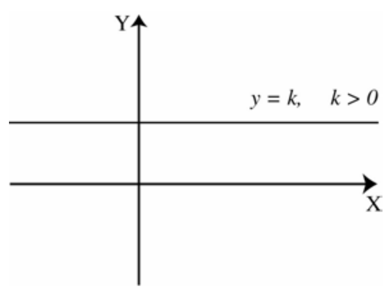


Fig. 8.3

Note : The graph of $y = f(x) = k$ is a straight line parallel to x-axis and which does not have any break. That is, continuous functions are functions, which do not admit any break in its graph.

Example 8.30: The function $f(x) = x^n$, $x \in$ R is continuous.

Solution. Let x_0 be a point of R.

$$\text{Then} \quad \lim_{x \to x_0} f(x) = \lim_{x \to x_0} (x^n) = \lim_{x \to x_0} (x.\,x \ldots n \text{ factors})$$

$$= \lim_{x \to x_0} (x).\lim_{x \to x_0} (x) \cdots \lim_{x \to x_0} (x) \ldots (n \text{ factors})$$

$$= x_0.x_0 \ldots x_0 \quad (n \text{ factors}) = x_0^{\,n}$$

$$\text{Also} \quad f(x_0) = x_0^{\,n}. \text{ Thus } \lim_{x \to x_0} f(x) = f(x_0) = x_0^{\,n}$$

$$\Rightarrow \quad f(x) = x^n \text{ is continuous at } x_0$$

Example 8.31: The function $f(x) = kx^n$ is continuous where $k \in$ R and $k \neq 0$.

Solution. Let $g(x) = k$ and $h(x) = x^n$.

By the example 8.29, g is continuous and by example 8.30, h is continuous and hence by Theorem 8.1, $f(x) = g(x) \cdot h(x) = kx^n$ is continuous.

Example 8.32: Every polynomial function of degree n is continuous.

Solution. Let $f(x) = a_0x^n + a_1 x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1} x + a_n$, $a_0 \neq 0$ be a polynomial function of degree n.

Now by example 8.31 a_ix^i, $i = 0, 1, 2, \ldots n$ are continuous. By theorem 8.1 sum of continuous functions is continuous and hence the function $f(x)$ is continuous.

Example 8.33: Every rational function of the form $p(x) / q(x)$ where $p(x)$ and $q(x)$ are polynomials, is continuous ($q(x) \neq 0$).

Solution. Let $r(x) = p(x) / q(x)$, $q(x) \neq 0$ be a rational function of x. Then we know that $p(x)$ and $q(x) \neq 0$ are polynomials. Also, $p(x)$ and $q(x)$ are continuous, being polynomials. Hence by theorem 8.1 the quotient $p(x) / q(x)$ is continuous. i.e. the rational function $r(x)$ is continuous.

Results without proof :

(1) The exponential function is continuous at all points of *R*.
In particular the exponential function $f(x) = e^x$ is continuous.

(2) The function $f(x) = \log x$, $x > 0$ is continuous at all points of R^+, where R^+ is the set of positive real numbers.

(3) The sine function $f(x) = \sin x$ is continuous at all points of R.

(4) The cosine function $f(x) = \cos x$ is continuous at all points of R.

Note : One may refer the SOLUTION BOOK for proof.

Example 8.34: Is the function $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 1 . & \text{when } x = 0 \end{cases}$ continuous at $x = 0$?

Justify your answer.

Solution. Note that $f(0) = 1$.

$$\text{Now } \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x} \quad \left(\because \text{for } x \neq 0,\ f(x) = \frac{\sin 2x}{x}\right)$$

$$= \lim_{x \to 0} 2\left(\frac{\sin 2x}{2x}\right) = 2 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)$$

$$= 2 \lim_{2x \to 0} \left(\frac{\sin 2x}{2x}\right) = 2.1 = 2 .$$

Since $\lim_{x \to 0} f(x) = 2 \neq 1 = f(0)$, the function is not continuous at $x = 0$. That is, the function is discontinuous at $x = 0$.

Note that the discontinuity of the above function can be removed if we define

$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ so that for this function $\lim_{x \to 0} f(x) = f(0)$.

Such points of discontinuity are called removable discontinuities.

Example 8.35:. Investigate the continuity at the indicated point:

$f(x) = \begin{cases} \dfrac{\sin (x - c)}{x - c} & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases}$ at $x = c$

Solution. We have $f(c) = 0$.

$$\text{Now } \lim_{x \to c} f(x) = \lim_{x \to c} \frac{\sin (x - c)}{x - c} = \lim_{h \to 0} \frac{\sin h}{h} \quad (\because h = x - c \to 0 \text{ as } x \to c)$$

$$= 1 .$$

Since $f(c) = 0 \neq 1 = \lim_{x \to c} f(x)$, the function $f(x)$ is discontinuous at $x = c$.

Note: This discontinuity can be removed by re-defining the function as

$$f(x) = \begin{cases} \dfrac{\sin(x - c)}{x - c} & \text{if } x \neq c \\ 1 & \text{if } x = c \end{cases}$$

Thus the point $x = c$ is a removable discontinuity.

Example 8.36: A function f is defined on by $f(x) = \begin{cases} -x^2 & \text{if } x \le 0 \\ 5x - 4 & \text{if } 0 < x \le 1 \\ 4x^2 - 3x & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \ge 2 \end{cases}$

Examine f for continuity at $x = 0, 1, 2$.

Solution.

(i) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x^2) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5x - 4) = (5.0 - 4) = -4$

Since $\lim_{x \to 0^-} f(x) \ne \lim_{x \to 0^+} f(x)$, $f(x)$ is discontinuous at $x = 0$

(ii) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5x - 4) = 5 \times 1 - 4 = 1.$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x^2 - 3x) = 4 \times 1^2 - 3 \times 1 = 1$

Also $f(1) = 5 \times 1 - 4 = 5 - 4 = 1$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$, $f(x)$ is continuous at $x = 1$.

(iii) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x^2 - 3x)$

$= 4 \times 2^2 - 3 \times 2 = 16 - 6 = 10$.

and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x + 4) = 3 \times 2 + 4 = 6 + 4 = 10$.

Also $f(2) = 3 \times 2 + 4 = 10$.

Since $f(2) = \lim_{x \to 2} f(x)$, the function $f(x)$ is continuous at $x = 2$.

Example 8.37: Let $\lfloor x \rfloor$ denote the greatest integer function. Discuss the continuity at $x = 3$ for the function $f(x) = x - \lfloor x \rfloor$, $x \ge 0$.

Solution. Now $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x - \lfloor x \rfloor = 3 - 2 = 1,$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x - \lfloor x \rfloor = 3 - 3 = 0,$

and $f(3) = 0$.

Note that $f(3) = \lim_{x \to 3^+} f(x) \ne \lim_{x \to 3^-} f(x)$.

Hence $f(x) = x - \lfloor x \rfloor$ is discontinuous at $x = 3$.

EXERCISE 8.2

Examine the continuity at the indicated points

(1) $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$ at $x = 2$

(2) $f(x) = x - |x|$ at $x = 0$

(3) $f(x) = \begin{cases} 2x \text{ when } 0 \leq x < 1 \\ 3 \text{ when } x = 1 \\ 4x \text{ when } 1 < x \leq 2 \end{cases}$ at $x = 1$

(4) $f(x) = \begin{cases} 2x - 1, & \text{if } x < 0 \\ 2x + 6, & \text{if } x \geq 0 \end{cases}$ at $x = 0$

(5) Find the values of a and b so that the function f given by

$f(x) = \begin{cases} 1, \text{ if } x \leq 3 \\ ax + b, \text{ if } 3 < x < 5 \\ 7, \text{ if } x \geq 5 \end{cases}$ is continuous at $x = 3$ and $x = 5$

(6) Let f be defined by $f(x) = \begin{cases} \dfrac{x^2}{2}, \text{ if } 0 \leq x \leq 1 \\ 2x^2 - 3x + \dfrac{3}{2}, \text{ if } 1 < x \leq 2 \end{cases}$

Show that f is continuous at $x = 1$.

(7) Discuss continuity of the function f, given by $f(x) = |x - 1| + |x - 2|$, at $x = 1$ and $x = 2$.

8.3 Concept of Differentiation

Having defined and studied limits, let us now try and find the exact rates of change at a point. Let us first define and understand what are increments?

Consider a function $y = f(x)$ of a variable x. Suppose x changes from an initial value x_0 to a final value x_1 . Then the increment in x is defined to be the amount of change in x. It is denoted by Δx (read as delta x).That is $\Delta x = x_1 - x_0$.

Thus $x_1 = x_0 + \Delta x$

If x increases then $\Delta x > 0$, since $x_1 > x_0$.

If x decreases then $\Delta x < 0$, since $x_1 < x_0$.

As x changes from x_0 to $x_1 = x_0 + \Delta x$, y changes from $f(x_0)$ to $f(x_0 + \Delta x)$. We put $f(x_0) = y_0$ and $f(x_0 + \Delta x) = y_0 + \Delta y$. The increment in y namely Δy depends on the values of x_0 and Δx. Note that Δy may be either positive, negative or zero (depending on whether y has increased, decreased or remained constant when x changes from x_0 to x_1).

If the increment Δy is divided by Δx, the quotient $\frac{\Delta y}{\Delta x}$ is called the average rate of change of y with respect to x, as x changes from x_0 to $x_0 + \Delta x$. The quotient is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

This fraction is also called a difference quotient.

Example 8.38: A worker is getting a salary of Rs. 1000/- p.m. She gets an increment of Rs. 100/- per year. If her house rent is half her salary, what is the annual increment in her house rent? What is the average rate of change of the house rent with respect to the salary?

Solution:

Let the salary be given by x and the house rent by y. Then $y = \frac{1}{2}x$. Also $\Delta x = 100$. Therefore, $\Delta y = \frac{1}{2}(x + \Delta x) - \frac{1}{2}x = \frac{\Delta x}{2} = \frac{100}{2} = 50$.

Thus, the annual increment in the house rent is Rs. 50/-.

Then the required average rate of change is $\frac{\Delta y}{\Delta x} = \frac{50}{100} = \frac{1}{2}$.

Example 8.39: If $y = f(x) = \frac{1}{x}$, find the average rate of change of y with respect to x as x changes from x_1 to $x_1 + \Delta x$.

Solution: $\Delta y = f(x_1 + \Delta x) - f(x_1) = \frac{1}{x_1 + \Delta x} - \frac{1}{x_1}$

$$= \frac{-\Delta x}{x_1(x_1 + \Delta x)}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{-1}{x_1(x_1 + \Delta x)}.$$

8.3.1 The concept of derivative

We consider a point moving in a straight line. The path s traversed by the point, measured from some definite point of the line, is evidently a function of time,

$$s = f(t) .$$

A corresponding value of s is defined for every definite value of t. If t receives an increment Δt, the path $s + \Delta s$ will then correspond to the new instant $t + \Delta t$, where Δs is the path traversed in the interval Δt.

In the case of uniform motion, the increment of the path is proportional to the increment of time, and the ratio $\frac{\Delta s}{\Delta t}$ represents the constant velocity of the motion. This ratio is in general dependent both on the choice of the instant t and on the increment Δt, and represents the average velocity of the motion during the interval from t to $t + \Delta t$.

The limit of the ratio $\frac{\Delta s}{\Delta t}$, if it exists with Δt tending to zero, defines the velocity v at the given instant : $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$. That is $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ is the instantaneous velocity v.

The velocity v, like the path s, is a function of time t; this function is called the derivative of function $f(t)$ with respect to t, thus, the velocity is the derivative of the path with respect to time.

Suppose that a substance takes part in certain chemical reaction. The quantity x of this substance, taking part in the reaction at the instant t, is a function of t. There is a corresponding increment Δx of magnitude x for an increment of time Δt, and the ratio $\frac{\Delta x}{\Delta t}$ gives the average speed of the reaction in the interval Δt while the limit of this ratio as Δt tends to zero gives the speed of the chemical reaction of the given instant t.

The above examples lead us to the following concept of the derivative of a function.

Definition

The derivative of a given function $y = f(x)$ is defined as the limit of the ratio of the increment Δy of the function to the corresponding increment Δx of the independent variable, when the latter tends to zero.

The symbols y' or $f'(x)$ or $\frac{dy}{dx}$ are used to denote derivative:

$$\frac{dy}{dx} = y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

It is possible for the above limit, not to exist in which case the derivative does not exist. We say that the function $y = f(x)$ is differentiable if it has a derivative.

Note.

(1) The operation of finding the derivative is called differentiation. Further it should be noted, the notation $\frac{dy}{dx}$ does not mean $dy \div dx$. It simply means $\frac{d(y)}{dx}$ or $\frac{d}{dx} f(x)$, the symbol $\frac{d}{dx}$ is an operator meaning that differentiation with respect to x whereas the fraction $\frac{\Delta y}{\Delta x}$ stands for $\Delta y \div \Delta x$. Although the notation $\frac{dy}{dx}$ suggests the ratio of two numbers dy and dx (denoting infinitesimal changes in y and x), it is really a single number, the limit of a ratio $\frac{\Delta y}{\Delta x}$ as both the terms approach 0.

(2) The differential coefficient of a given function $f(x)$ for any particular value of x say x_0 is denoted by $f'(x_0)$ or $\left(\frac{dy}{dx}\right)_{x = x_0}$ and stands for $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ provided this limit exists.

(3) If the limit of $\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists when $\Delta x \to 0$ from the right hand side i.e. $\Delta x \to 0$ through positive values alone, it is known as right or progressive differential coefficient and is denoted by $f'(x_0+) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = Rf'(x_0)$. Similarly the limit of $\frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x}$ as $\Delta x \to 0$ from the left hand side i.e. from negative values alone is known as the left or regressive differential coefficient and is denoted by

$$f'(x_0-) = \lim_{\Delta x \to 0} \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} = Lf'(x_0) .$$

If $Rf'(x_0) = Lf'(x_0)$, then f is said to be differentiable at $x = x_0$ and the common value is denoted by $f'(x_0)$. If $R f'(x_0)$ and $Lf'(x_0)$ exist but are unequal, then $f(x)$ is not differentiable at x_0. If none of them exists then also $f(x)$ is not differentiable at x_0.

Geometrically this means that the graph of the function has a corner and hence no tangent at the point $(x_0, f(x_0))$.

8.3.2 Slope or gradient of a curve $\left(\textbf{Geometrical meaning of } \frac{dy}{dx}\right)$

In this section we shall define what we mean by the slope of a curve at a point P on the curve.

Let P be any fixed point on a curve $y = f(x)$, and let Q be any other point on the same curve. Let PQ be the corresponding secant. If we let Q move along the curve and approach P, the secant PQ will in general rotate about the point P and may approach a limiting position PT. (Fig 8.4).

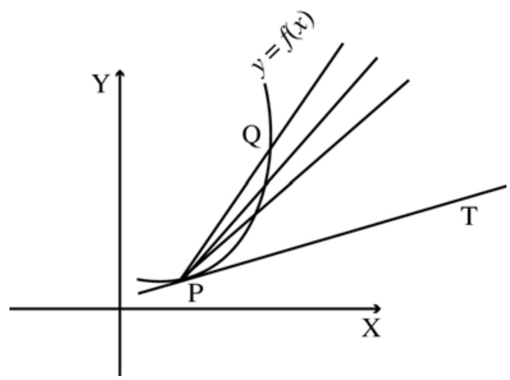


Fig. 8.4

Definition

The tangent to a curve at a point P on the curve is the limiting position PT of a secant PQ as the point Q approaches P by moving along the curve, if this limiting position exists and is unique.

If P_0 is (x_0, y_0) and P is $(x_0 + \Delta x, y_0 + \Delta y)$ are two points on a curve defined by $y = f(x)$, as in Fig. 8.5, then the slope of the secant through these two points is given by

$m' = \tan \alpha_0' = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{-\Delta x}$, where α_0' is the inclination of the secant.

As Δx approaches 0, P moves along the curve towards P_0 ; and if $f'(x_0)$ exists, the slope of the tangent at P_0 is the limit of the slope of the secant P_0 P, or

$$m_0 = \tan\alpha_0 = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0) = \left(\frac{dy}{dx}\right)_{x = x_0}$$ where α_0 is the inclination of the tangent P_0T and m_0 is its slope. The slope of the tangent to a curve at a point P_0 is often called the slope of the curve at that point.

Thus, geometrically we conclude that the difference ratio (or the difference coefficient) $\frac{\Delta y}{\Delta x}$ is the slope of the secant through the point $P_0(x_0,y_0)$ whereas the differential coefficient or the derivative of $y = f(x)$ at $x = x_0$ is the slope or gradient of the tangent to the curve at $P_0(x_0,y_0)$.

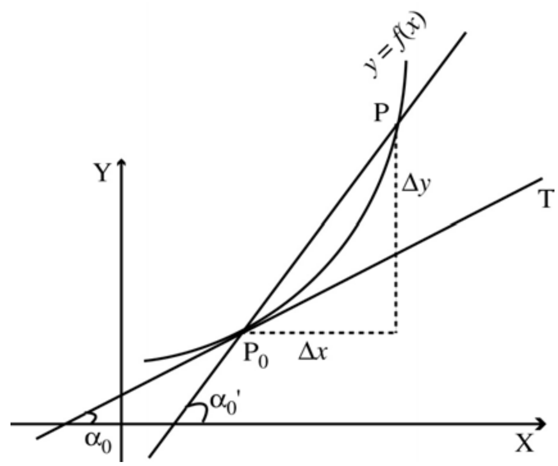


Fig. 8.5

Definition

If $f(x)$ is defined in the interval $x_0 \le x < b$, its right hand derivative at x_0 is defined as $f'(x_0+) = \lim_{x \to x_0+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ provided this limit exists; if $f(x)$ is defined in the interval $a < x \le x_0$ its left hand derivative at x_0 is defined as

$$f'(x_{0-}) = \lim_{x \to x_0-} \frac{f(x_0 - \Delta x) - f(x_0)}{\Delta x}$$ provided this limit exists.

If $f(x)$ is defined in the interval $a \le x \le b$, then we can write $f'(a)$ for $f'(a+)$, and we write $f'(b)$ for $f'(b\_)$

Relationship between differentiability and continuity.

Theorem 8.2 Every differentiable function is continuous.

Proof. Let a function f be differentiable at $x = c$. Then $f'(c)$ exists and $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

Now $f(x) - f(c) = (x - c)\,\frac{[f(x) - f(c)]}{(x - c)}$, $x \neq c$

Taking limit as $x \to c$, we have

$$\lim_{x \to c} \{f(x) - f(c)\} = \lim_{x \to c} (x - c) \,.\, \frac{[f(x) - f(c)]}{(x - c)}$$

$$= \lim_{x \to c} (x - c) \,.\, \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c} (x - c) \,.\, f'(c) = 0.\, f'(c) = 0.$$

Now $f(x) = f(c) + [f(x) - f(c)]$ and $\lim_{x \to c} f(x) = f(c) + 0 = f(c)$

and therefore f is continuous at $x = c$.

The converse need not be true. i.e. a function which is continuous at a point need not be differentiable at that point. We illustrate this by the following example.

Example 8.40: A function $f(x)$ is defined in an interval [0, 2] as follows :

$$f(x) = x \text{ when } 0 \leq x \leq 1$$
$$= 2x - 1 \text{ when } 1 < x \leq 2$$

Show that $f(x)$ is continuous at 1 but not differentiable at that point.

The graph of this function is as shown in fig. 8.6

This function is continuous at $x = 1$.

For, $\lim_{x \to 1-} f(x) = \lim_{h \to 0} f(1 - h)$

$= \lim_{h \to 0} (1 - h)$

$= 1 - 0 = 1$

$\lim_{x \to 1_+} f(x) = \lim_{h \to 0} f(1 + h)$

$= \lim_{h \to 0} (2(1 + h) - 1)$

$= \lim_{h \to 0} (2h + 1)$

$= 1$.

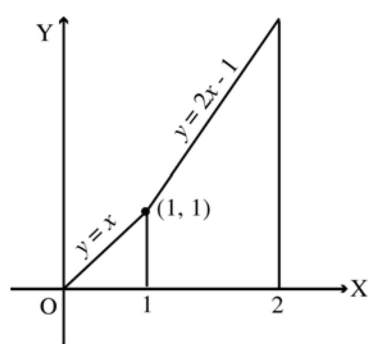


Fig. 8.6

Thus $f(x)$ is continuous at $x = 1$

$$\text{Now} \quad Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{[2(1+h) - 1] - [2(1) - 1]}{h} = \lim_{h \to 0} \frac{2h}{h} = 2 \text{ and}$$

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{(1-h) - 1} = \lim_{h \to 0} \frac{(1-h) - 1}{-h}$$

$$= \lim_{h \to 0} \frac{-h}{-h} = 1.$$

Since $Rf'(1) \neq Lf'(1)$, the given function is not differentiable at $x = 1$. Geometrically this means that the curve does not have a tangent line at the point (1, 1).

Example 8.41:

Show that the function $y = x^{1/3} = f(x)$ is not differentiable at $x = 0$.

[This function is defined and continuous for all values of the independent variable x. The graph of this function is shown in fig. 8.7]

Solution:

This function does not have derivative at $x = 0$

For, we have $y + \Delta y = \sqrt[3]{x + \Delta x}$

$\Delta y = \sqrt[3]{x + \Delta x} - \sqrt[3]{x}$

At $x = 0$, $y = 0$ and $\Delta y = \sqrt[3]{\Delta x}$.

$$\text{Now} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x}$$

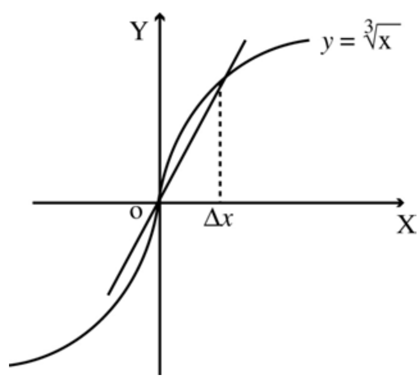


Fig. 8.7

$$= \lim_{\Delta x \to 0} \frac{\sqrt[3]{\Delta x} - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt[3]{(\Delta x)^2}} = +\infty.$$

Consequently this function is not differentiable at the point $x = 0$. The tangent to the curve at this point forms with the x-axis, an angle $\frac{\pi}{2}$, which means that it coincides with the y-axis.

Example 8.42: Show that the function $f(x) = x^2$ is differentiable on [0, 1].

Solution. Let c be any point such that $0 < c < 1$.

Then $$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c .$$

At the end points we have

$$f'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x^2}{x} = \lim_{x \to 0} (x) = 0$$

and $$f'(1) = \lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1-} (x + 1) = 2 .$$

Since the function is differentiable at each and every point of [0, 1], $f(x) = x^2$ is differentiable on [0, 1].

EXERCISE 8.3

(1) A function f is defined on R^+ by $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$.

Show that $f'(1)$ does not exist.

(2) Is the function $f(x) = |x|$ differentiable at the origin. Justify your answer.

(3) Check the continuity of the function $f(x) = |x| + |x - 1|$ for all $x \in R$. What can you say its differentiability at $x = 0$, and $x = 1$?

(4) Discuss the differentiability of the functions

(i) $f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ x, & x > 1 \end{cases}$ at $x = 1$ (ii) $f(x) = \begin{cases} 2x - 3, & 0 \leq x \leq 2 \\ x^2 - 3, & 2 < x \leq 4 \end{cases}$ at $x = 2, x = 4$

(5) Compute $Lf'(0)$ and $Rf'(0)$ for the function $f(x) = \begin{cases} \dfrac{x(e^{1/x} - 1)}{(e^{1/x} + 1)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

8.4. Differentiation Techniques

In this section we discuss different techniques to obtain the derivatives of given functions. In order to find the derivative of a function $y = f(x)$ from first principles (on the basis of the general definition of a derivative) it is necessary to carry out the following operations :

1) increase the argument x by Δx, calculate the increased value of the function

   $y + \Delta y = f(x + \Delta x)$.

2) find the corresponding increment of the function $\Delta y = f(x + \Delta x) - f(x)$;

3) form the ratio of the increment of the function to the increment of the argument $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$;

4) find the limit of this ratio as $\Delta x \to 0$;

   $$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

We shall apply this general method for evaluating the derivatives of certain elementary (standard) functions. As a matter of convenience we denote $\dfrac{dy}{dx} = f'(x)$ by y'.

8.4.1 Derivatives of elementary functions from first principles

I. The derivative of a constant function is zero.

That is, $\dfrac{d}{dx}(c) = 0$, where c is a constant … (1)

Proof. Let $f(x) = c$ Then $f(x + \Delta x) = c$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\therefore \frac{d}{dx}(c) = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0 .$$

II. The derivative of x^n is nx^{n-1}, where n is a rational number

i.e. $\dfrac{d}{dx}(x^n) = nx^{n-1}$. … (2)

Proof: Let $f(x) = x^n$. Then $f(x + \Delta x) = (x + \Delta x)^n$

Now $$\frac{d\,f(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\therefore \frac{d(x^n)}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^n\left(1+\frac{\Delta x}{x}\right)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} x^n \left[\frac{\left(1+\frac{\Delta x}{x}\right)^n - 1}{\Delta x}\right]$$

$$= x^{n-1} \lim_{\Delta x \to 0} \left[\frac{\left(1+\frac{\Delta x}{x}\right)^n - 1}{\frac{\Delta x}{x}}\right].$$

Put $y = 1 + \frac{\Delta x}{x}$ As $\Delta x \to 0$, $y \to 1$.

$$\therefore \frac{d(x^n)}{dx} = x^{n-1} \lim_{y \to 1} \left(\frac{y^n - 1}{y-1}\right)$$

$$= n\, x^{n-1}$$

$$= nx^{n-1}. \left[\because \lim_{y \to a} \frac{y^n - a^n}{y-a} = na^{n-1}\right]$$

Note. This result is also true for any real number n.

Example 8.43: If y = x^5 , find $\frac{dy}{dx}$

Solution : $\frac{dy}{dx} = 5x^{5-1} = 5x^4.$

Exampl 8.44: If $y = x$ find $\frac{dy}{dx}$

Solution : $\frac{dy}{dx} = 1.x^{1-1} = 1x^{\circ} = 1$.

Example 8.45: If $y = \sqrt{x}$ find $\frac{dy}{dx}$.

Solution:

Let us represent this function in the form of a power: $y = x^{\frac{1}{2}}$;

Then by formula (II) we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Example 8.46: If $y = \frac{1}{x\sqrt{x}}$, find $\frac{dy}{dx}$.

Solution:

Represent y in the form of a power. i.e. $y = x^{-\frac{3}{2}}$.

Then $\frac{dy}{dx} = -\frac{3}{2} x^{-\frac{3}{2}-1} = -\frac{3}{2} x^{-\frac{5}{2}}$

III. The derivative of sin*x* is cos*x*

i.e. if $y = \sin x$ then $\frac{dy}{dx} = \cos x$ … (3)

Proof:

Let $y = \sin x$. Increase the argument x by Δx, then

$y + \Delta y = \sin(x + \Delta x)$

$$\Delta y = \sin(x + \Delta x) - \sin x = 2\sin\frac{(x + \Delta x - x)}{2} \cos\frac{(x + \Delta x + x)}{2}$$

$$= 2\sin\frac{\Delta x}{2} . \cos\left(x + \frac{\Delta x}{2}\right)$$

$$\frac{\Delta y}{\Delta x} = \frac{2\sin\frac{\Delta x}{2}\cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x} = \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cos\left(x + \frac{\Delta x}{2}\right)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} . \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right)$$

$$= 1. \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) .$$

Since $f(x) = \cos x$ is continuous

$$= 1. \cos x \qquad \lim_{\Delta x \to 0} f(x + \Delta x) = \lim_{\Delta x \to 0} \cos(x + \Delta x)$$

$$= \cos x . \qquad = \cos x$$

IV. The derivative of cos*x* is – sin*x*

ie. if $y = \cos x$, then $\frac{dy}{dx} = -\sin x$. … (4)

Proof: Let $y = \cos x$ Increase the argument x by the increment Δx.

Then $y + \Delta y = \cos(x + \Delta x)$;

$$\Delta y = \cos(x + \Delta x) - \cos x$$

$$= -2\sin\frac{x + \Delta x - x}{2}\ \sin\frac{x + \Delta x + x}{2}$$

$$= -2\sin\frac{\Delta x}{2}\ \sin\left(x + \frac{\Delta x}{2}\right)$$

$$\frac{\Delta y}{\Delta x} = -\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \sin\left(x + \frac{\Delta x}{2}\right);$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \sin\left(x + \frac{\Delta x}{2}\right)$$

$$= -\lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right)$$

Since sin x is continuous, $\lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right) = \sin x$ and $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$$\therefore \frac{dy}{dx} = -\sin x.$$

Theorem 8.3

If f and g are differentiable functions of x and c is any constant, then the following are true.

(i) $$\frac{d(cf(x))}{dx} = c\,\frac{d(f(x))}{dx} \quad \ldots (5)$$

(ii) $$\frac{d(f(x) \pm g(x))}{dx} = \frac{d(f(x))}{dx} \pm \frac{d(g(x))}{dx}. \quad \ldots (6)$$

Example 8.47: If $y = \frac{3}{\sqrt{x}}$, find $\frac{dy}{dx}$

Solution: $$y = 3x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 3\left(-\frac{1}{2}\right) x^{-\frac{1}{2} - 1} = -\frac{3}{2}\, x^{-\frac{3}{2}}$$

Example 8.48: If $y = 3x^4 - 1/\sqrt[3]{x}$, find $\frac{dy}{dx}$

Solution:

$$y = 3x^4 - x^{-1/3}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^4 - x^{-1/3}\right) = 3\,\frac{d(x^4)}{dx} - \frac{d}{dx}\,(x^{-1/3})$$

$$= 3 \times 4x^{4-1} - \left(-\frac{1}{3}\right) x^{-\frac{1}{3}-1}$$

$$= 12x^3 + \frac{1}{3}\,x^{-\frac{4}{3}}$$

V. If $y = \log_a x$ then $\frac{dy}{dx} = \frac{1}{x}\ \log_a e$ … (7)

Corollary : If $y = \log_e x$ then $\frac{dy}{dx} = \frac{1}{x}$ … (8)

Proof: In the previous result take $a = e$. Then $\frac{d}{dx}\ (\log_e x) = \frac{1}{x}\ \log_e e = \frac{1}{x}\ .\ 1 = \frac{1}{x}$.

Example 8.49: Find y' if $y = x^2 + \cos x$.

Solution: We have $y = x^2 + \cos x$.

Therefore $y' = \frac{dy}{dx} = \frac{d}{dx}\ (x^2 + \cos x)$

$$= \frac{d(x^2)}{dx} + \frac{d(\cos x)}{dx}$$

$$= 2x^{2-1} + (-\sin x)$$

$$= 2x - \sin x$$

Example 8.50:

Differentiate $1/\sqrt[3]{x}\ + \log_5 x + 8$ with respect to x.

Solution: Let

$$y = x^{-1/3} + \log_5 x + 8$$

$$y' = \frac{dy}{dx} = \frac{d}{dx}\left(x^{-\frac{1}{3}} + \log_5 x + 8\right)$$

$$= \frac{d\left(x^{-\frac{1}{3}}\right)}{dx} + \frac{d(\log_5 x)}{dx} + \frac{d(8)}{dx}$$

$$= -\frac{1}{3}\, x^{-\frac{1}{3}-1} + \frac{1}{x}\log_5{}^{e} + 0,$$

$$= -\frac{1}{3}\, x^{-\frac{4}{3}} + \frac{1}{x}\log_5 e$$

Example 8.51 : Find the derivative of $x^5 + 4x^4 + 7x^3 + 6x^2 + 2$ w.r.t. to x .

Solution: Let $y = x^5 + 4x^4 + 7x^3 + 6x^2 + 8x + 2$

$$y' = \frac{d}{dx}\,(x^5 + 4x^4 + 7x^3 + 6x^2 + 8x + 2)$$

$$= \frac{d(x^5)}{dx} + \frac{d(4x^4)}{dx} + \frac{d(7x^3)}{dx} + \frac{d(6x^2)}{dx} + \frac{d\,(8x)}{dx} + \frac{d(2)}{dx}$$

$$= 5x^4 + 4 \times 4x^3 + 7 \times 3x^2 + 6 \times 2x + 8 \times 1 + 0$$

$$= 5x^4 + 16x^3 + 21x^2 + 12x + 8\ .$$

Example 8.52: Find the derivative of $y = e^{7x}$ from first principle.

Solution: We have $y = \mathrm{e}^{7x}$

$$y + \Delta y = e^{7\,(x + \Delta x)}$$

$$\frac{\Delta y}{\Delta x} = \frac{e^{7x}\,.\,\mathrm{e}^{7\Delta x} - \mathrm{e}^{7x}}{\Delta x}$$

$$= \mathrm{e}^{7x}\left(\frac{e^{7\Delta x} - 1}{\Delta x}\right)$$

$$y' = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} e^{7x}\left(\frac{e^{7\Delta x} - 1}{\Delta x}\right) = e^{7x}\lim_{\Delta x \to 0} 7\left(\frac{e^{7\Delta x} - 1}{7\Delta x}\right)$$

$$= 7\,e^{7x}\lim_{t \to 0}\left(\frac{e^{t} - 1}{t}\right) \qquad (\because\ t = 7\Delta x \to 0 \text{ as } \Delta x \to 0)$$

$$= 7\,e^{7x} \times 1 = 7e^{7x}. \qquad (\because \lim_{t \to 0}\frac{e^{t} - 1}{t} = 1)$$

In particular, if $y = e^x$, then $\frac{d}{dx}\,(e^x) = e^x$ … (9)

Similarly we can prove

VI. The derivative of $y = \tan x$ w.r.t. to x is $y' = \sec^2 x$. … (10)

VII. The derivative of $y = \sec x$ w.r.t. to x is $y' = \sec x \tan x$ … (11)

VIII. The derivative of $y = \operatorname{cosec} x$ as $y' = -\operatorname{cosec} x \cot x$ … (12)

IX. The derivative of $y = \cot x$ as $y' = -\operatorname{cosec}^2 x$ … (13)

Note : One may refer the SOLUTION BOOK for the proof.

EXERCISE 8.4

1. Find $\frac{dy}{dx}$ if $y = x^3 - 6x^2 + 7x + 6$.
2. If $f(x) = x^3 - 8x + 10$, find $f'(x)$ and hence find $f'(2)$ and $f'(10)$.
3. If for $f(x) = ax^2 + bx + 12$, $f'(2) = 11$, $f'(4) = 15$ find a and b.
4. Differentiate the following with respect to x:

   (i) $x^7 + e^x$ (ii) $\log_7 x + 200$

   (iii) $3 \sin x + 4 \cos x - e^x$ (iv) $e^x + 3\tan x + \log x^6$

   (v) $\sin 5 + \log_{10} x + 2\sec x$ (vi) $x^{-3/2} + 8e + 7 \tan x$

   (vii) $\left(x + \frac{1}{x}\right)^3$ (viii) $\frac{(x-3)(2x^2-4)}{x}$

Theorem 8.4: (Product rule for differentiation)

Let u and v be differentiable functions of x. Then the product function

$$y = u(x)\ v(x) \text{ is differentiable and}$$

$$y' = u(x)\ v'(x) + v(x)\ u'(x) \qquad \dots (14)$$

Proof: We have

$$y = u(x)\ v(x)$$

$$y + \Delta y = u(x + \Delta x)\ v(x + \Delta x)$$

$$\Delta y = u(x + \Delta x)\ v(x + \Delta x) - u(x)\ v(x)$$

$$\therefore\ \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)\ v(x + \Delta x) - u(x)\ v(x)}{\Delta x}.$$

Adding and subtracting $u(x + \Delta x)\ v(x)$ in the numerator and then re-arranging we get:

$$y' = \lim_{\Delta x \to 0} \frac{u(x + \Delta x)\, v(x + \Delta x) - u(x + \Delta x)\ v(x) + u(x + \Delta x)\ v(x) - u(x)\, v(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)\left[v(x + \Delta x) - v(x)\right] + v(x)\left[u(x + \Delta x) - u(x)\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} u(x+\Delta x) . \lim_{\Delta x \to 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} + v(x) \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

Now, since u is differentiable, it is continuous and hence

$$\lim_{\Delta x \to 0} u(x + \Delta x) = u(x)$$

Since u and v are differentiable we have

$$u'(x) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

and $$v'(x) = \lim_{\Delta x \to 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} .$$

Therefore $y' = u(x)\, v'(x) + v(x)\; u'(x)$.

Similarly, if u, v and w are differentiable and if $y = u(x)\; v(x)\; w(x)$ then

$$y' = u(x)\; v(x)\; w'(x) + u(x)\; v'(x)\; w(x) + u'(x)\; v(x)\; w(x)$$

Note (1). The above product rule for differentiation can be remembered as :

Derivative of the product of two functions

= (1st funct.) (derivative of 2nd funct.)+(2nd funct.) (derivative of 1st funct.).

Note (2). The product rule can be rewritten as follows :

$$\big(u(x)\,.\,v(x)\big)' = u(x)\,.\,v'(x) + v(x)\,.\,u'(x)$$

$$\frac{(u(x)\,.\,v(x))'}{u(x)\,.\,v(x)} = \frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)} . \qquad \ldots (15)$$

It can be generalised as follows:

If u_1, u_2, … ,u_n are differentiable functions with derivatives u_1', u_2', …, u_n' then

$$\frac{(u_1\,.\,u_2 \ldots u_n)'}{u_1\,.\,u_2 \ldots u_n} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \frac{u_3'}{u_3} + \ldots + \frac{u_n'}{u_n} . \qquad \ldots (16)$$

Example 8.53: Differentiate $e^x \tan x$ w.r. to x.

Solution: Let $y = e^x\,.\,\tan x$.

Then $$y' = \frac{d}{dx}(e^x\,.\,\tan x) = e^x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(e^x)$$

$$= e^x.\,\sec^2 x + \tan x\,.\,e^x$$

$$= e^x(\sec^2 x + \tan x)\,.$$

Example 8.54: If $y = 3x^4\, e^x + 2\sin x + 7$ find y'.

Solution: $$y' = \frac{dy}{dx} = \frac{d(3x^4\, e^x + 2\sin x + 7)}{dx}$$

$$= \frac{d(3x^4 e^x)}{dx} + \frac{d(2 \sin x)}{dx} + \frac{d(7)}{dx}$$

$$= 3 \frac{d(x^4 e^x)}{dx} + 2 \frac{d(\sin x)}{dx} + 0$$

$$= 3\left[x^4 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^4)\right] + 2 \cos x$$

$$= 3 [x^4 . e^x + e^x . 4x^3] + 2 \cos x$$

$$= 3x^3 e^x (x + 4) + 2 \cos x .$$

Example 8.55: Differentiate $(x^2 + 7x + 2)(e^x - \log x)$ with respect to x.

Solution: Let $y = (x^2 + 7x + 2)(e^x - \log x)$

$$y' = \frac{d}{dx}\left[(x^2 + 7x + 2)(e^x - \log x)\right]$$

$$= (x^2 + 7x + 2) \frac{d}{dx}(e^x - \log x) + (e^x - \log x) \frac{d}{dx}(x^2 + 7x + 2)$$

$$= (x^2 + 7x + 2)\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(\log x)\right]$$

$$+ (e^x - \log x)\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(7x) + \frac{d}{dx}(2)\right]$$

$$= (x^2 + 7x + 2)\left(e^x - \frac{1}{x}\right) + (e^x - \log x)(2x + 7 + 0)$$

$$= (x^2 + 7x + 2)\left(e^x - \frac{1}{x}\right) + (e^x - \log x)(2x + 7) .$$

Example 8.56: Differentiate $(x^2 - 1)(x^2 + 2)$ w.r. to x using product rule. Differentiate the same after expanding as a polynomial. Verify that the two answers are the same.

Solution: Let $y = (x^2 - 1)(x^2 + 2)$

$$\text{Now } y' = \frac{d}{dx}[(x^2 - 1)(x^2 + 2)]$$

$$= (x^2 - 1) \frac{d}{dx}(x^2 + 2) + (x^2 + 2) \frac{d}{dx}(x^2 - 1)$$

$$= (x^2 - 1)\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(2)\right] + (x^2 + 2)\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(-1)\right]$$

$$= (x^2 - 1)(2x + 0) + (x^2 + 2)(2x + 0)$$

$$= 2x(x^2 - 1) + 2x(x^2 + 2)$$

$$= 2x\,(x^2 - 1 + x^2 + 2) = 2x\,(2x^2 + 1)\,.$$

Another method

$$y = (x^2 - 1)\,(x^2 + 2) = x^4 + x^2 - 2$$

$$y' = \frac{d}{dx}\,(x^4 + x^2 - 2) = 4x^3 + 2x = 2x\,(2x^2 + 1)$$

We observe that both the methods give the same answer.

Example 8.57: Differentiate $e^x \log x \; \cot x$

Solution: Let $y = e^x \log x \cot x$

$= u_1 \,.\, u_2 \,.\, u_3$ (say)

where $u_1 = e^x \,;\, u_2 = \log x,\, u_3 = \cot x.$

$$y' = u_1\, u_2\, u_3' + u_1\, u_3\, u_2' + u_2\, u_3\, u_1'$$

$$= e^x \log x\,(-\operatorname{cosec}^2 x) + e^x \cot x \,.\, \frac{1}{x} + \log x \,.\, \cot x \,.\, e^x$$

$$= e^x \left[\cot x \,.\, \log x + \frac{1}{x} \cot x - \log x \,.\, \operatorname{cosec}^2 x \right]$$

Note: Solve this problem by using Note 2.

EXERCISE 8.5

Differentiate the following functions with respect to x.

(1) $e^x \cos x$	(2) $\sqrt[n]{x}\;\; \log \sqrt{x}\;,\; x > 0$
(3) $6 \sin x \log_{10} x + e$	(4) $(x^4 - 6x^3 + 7x^2 + 4x + 2)\,(x^3 - 1)$
(5) $(a - b \sin x)\,(1 - 2 \cos x)$	(6) $\operatorname{cosec} x \,.\, \cot x$
(7) $\sin^2 x$	(8) $\cos^2 x$
(9) $(3x^2 + 1)^2$	(10) $(4x^2 - 1)\,(2x + 3)$
(11) $(3 \sec x - 4 \operatorname{cosec} x)\,(2 \sin x + 5 \cos x)$	
(12) $x^2\, e^x \sin x$	(13) $\sqrt{x}\;\, e^x \log x.$

Theorem: 8.5 (Quotient rule for differentiation)

If u and v are differentiable function and if $v(x) \neq 0$, then

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \ldots (17)$$

$$\text{i.e. } \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} .$$

Exampl 8.58:

Differentiate $\frac{x^2 - 1}{x^2 + 1}$ with respect to x .

Solution:

$$\text{Let } y = \frac{x^2 - 1}{x^2 + 1} = \frac{u}{v},\ u = x^2 - 1 ;\ v = x^2 + 1$$

$$y' = \frac{d}{dx}\left(\frac{x^2 - 1}{x^2 + 1}\right) = \frac{(x^2+1)(x^2-1)' - (x^2-1)(x^2+1)'}{(x^2+1)^2} \quad \text{Using (17)}$$

$$= \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} \quad \frac{\left[(x^2 + 1) - (x^2 - 1)\right]2x}{(x^2 + 1)^2}$$

$$= 2x\frac{2}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2} .$$

Example 8.59: Find the derivative of $\frac{x^2 + e^x \sin x}{\cos x + \log x}$ with respect to x

Solution:

$$\text{Let } y = \frac{x^2 + e^x \sin x}{\cos x + \log x} = \frac{u}{v},\ u = x^2 + e^x \sin x,\ v = \cos x + \log x$$

$$\text{Now } y' = \frac{vu' - uv'}{v^2}$$

$$= \frac{(\cos x + \log x)(x^2 + e^x \sin x)' - (x^2 + e^x \sin x)(\cos x + \log x)'}{(\cos x + \log x)^2}$$

$$= \frac{(\cos x + \log x)\left[(x^2)' + (e^x \sin x)'\right] - (x^2 + e^x \sin x)\left[(\cos x)' + (\log x)'\right]}{(\cos x + \log x)^2}$$

$$= \frac{(\cos x + \log x)\left[2x + e^x \cos x + \sin x\, e^x\right] - (x^2 + e^x \sin x)\left(-\sin x + \frac{1}{x}\right)}{(\cos x + \log x)^2}$$

$$= \frac{(\cos x + \log x)\left[2x + e^x(\cos x + \sin x)\right] - (x^2 + e^x \sin x)\left(\frac{1}{x} - \sin x\right)}{(\cos x + \log x)^2}.$$

Example 8.60: Differentiate $\frac{\sin x + \cos x}{\sin x - \cos x}$ with respect to x.

Solution:

Let $y = \frac{\sin x + \cos x}{\sin x - \cos x} = \frac{u}{v}$, $u = \sin x + \cos x$, $v = \sin x - \cos x$

$$y' = \frac{vu' - uv'}{v^2} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-\left[(\sin x - \cos x)^2 + (\sin x + \cos x)\right]^2}{(\sin x - \cos x)^2}$$

$$= \frac{-\left(\sin^2 x + \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x\right)}{(\sin x - \cos x)^2}$$

$$= -\frac{2}{(\sin x - \cos x)^2}$$

EXERCISE 8.6

Differentiate the following functions using quotient rule.

(1) $\frac{5}{x^2}$ (2) $\frac{2x - 3}{4x + 5}$ (3) $\frac{x^7 - 4^7}{x - 4}$

(4) $\frac{\cos x + \log x}{x^2 + e^x}$ (5) $\frac{\log x - 2x^2}{\log x + 2x^2}$ (6) $\frac{\log x}{\sin x}$

(7) $\frac{1}{ax^2 + bx + c}$ (8) $\frac{\tan x + 1}{\tan x - 1}$ (9) $\frac{\sin x + x \cos x}{x \sin x - \cos x}$ (10) $\frac{\log x^2}{e^x}$

The derivative of a composite function (Chain rule)

If $u = f(x)$ and $y = F(u)$, then $y = F(f(x))$ is the composition of f and F.

In the expression $y = F(u)$, u is called the intermediate argument.

Theorem 8.6: If $u = f(x)$ has the derivative $f'(x)$ and $y = F(u)$ has the derivative $F'(u)$, then the function of a function $F(f(x))$ has the derivative equal to $F'(u)\, f'(x)$, where in place of u we must substitute $u = f(x)$.

Proof: We have $u = f(x)$, $y = F(u)$.

Now $u + \Delta u = f(x + \Delta x)$, $y + \Delta y = F(u + \Delta u)$

Therefore $\dfrac{\Delta u}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ and $\dfrac{\Delta y}{\Delta u} = \dfrac{F(u + \Delta u) - F(u)}{\Delta u}$

If $f'(x) = \dfrac{du}{dx} \neq 0$, then $\Delta u, \Delta x \neq 0$.

Since f is differentiable, it is continuous and hence when $\Delta x \to 0$, $x + \Delta x \to x$ and $f(x + \Delta x) \to f(x)$. That is, $\lim\limits_{\Delta x \to 0} (x+\Delta x) = x$ and $\lim\limits_{\Delta x \to 0} f(x+\Delta x) = f(x)$.

Therefore $\lim\limits_{\Delta x \to 0} (u + \Delta u) = \Delta u$

Since $\Delta u \neq 0$ as $\Delta x \to 0$, we may write $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$

Since both f and F are continuous functions

we have $\Delta u \to 0$ when $\Delta x \to 0$ and $\Delta y \to 0$ when $\Delta u \to 0$.

Therefore $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta u \to 0} \dfrac{\Delta y}{\Delta u} \cdot \lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x}$

$$= y'(u)\ u'(x) = F'(u)\ f'(x) = F'(f(x))\ f'(x) \qquad \ldots (18)$$

This chain rule can further be extended to

i.e. if $y = F(u)$, $u = f(t)$, $t = g(x)$ then

$$\frac{dy}{dx} = F'(u) \,.\, u'(t) \,.\, t'(x)$$

i.e. $$\frac{dy}{dx} = \frac{dF}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx} \,. \qquad \ldots (19)$$

Example 8.61: Differentiate $\log\sqrt{x}$ with respect to x.

Solution: Let $y = \log\sqrt{x}$

Take $u = \sqrt{x}$, and so $y = \log u$, Then by chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Now $\dfrac{dy}{du} = \dfrac{1}{u}$; $\dfrac{du}{dx}$; $\dfrac{1}{2\sqrt{x}}$

Therefore by chain rule $\frac{dy}{dx} = \frac{1}{u} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x} \,.\, 2\sqrt{x}} = \frac{1}{2x}$.

Example 8.62: Differentiate sin (log x)

Solution: Let $y = \sin u$, where $u = \log x$

Then by chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$,

$$\text{Now } \frac{dy}{du} = \cos u \;;\; \frac{du}{dx} = \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \cos u \,.\, \frac{1}{x} = \frac{\cos(\log x)}{x}.$$

Example 8.63:

Differentiate $e^{\sin x^2}$

Solution: Let $y = e^{\sin x^2}$; $u = \sin x^2$; $t = x^2$

Then $y = e^u$, $u = \sin t$, $t = x^2$

∴ By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx} = e^u \,.\, \cos t \,.\, 2x$$

$$= e^{\sin x^2} \,.\, \cos(x^2) \,.\, 2x = 2x\, e^{\sin(x^2)} \cos(x^2)$$

$$= 2x\, e^{\sin(x^2)} \cos(x^2) \,.$$

Example 8.64: Differentiate sin ($ax + b$) with respect to x

Solution: Let $y = \sin(ax + b) = \sin u$, $u = ax + b$

$$\frac{dy}{du} = \cos u \;;\; \frac{du}{dx} = a$$

$$\therefore \frac{dy}{dx} = \cos u \,.\, a = a \cos(ax + b).$$

EXERCISE 8.7

Differentiate the following functions with respect to *x*

(1) $\log(\sin x)$ (2) $e^{\sin x}$ (3) $\sqrt{1 + \cot x}$

(4) $\tan(\log x)$ (5) $\frac{e^{bx}}{\cos(ax + b)}$ (6) $\log \sec\left(\frac{\pi}{4} + \frac{x}{2}\right)$

(7) $\log \sin(e^x + 4x + 5)$ (8) $\sin\left(x^{\frac{3}{2}}\right)$ (9) $\cos\left(\sqrt{x}\right)$ (10) $e^{\sin(\log x)}$.

8.4.2 Derivatives of inverse functions

If for the function $y = f(x)$ there exists an inverse function $x = \phi(y)$ and if $\frac{dx}{dy} = \phi'(y) \neq 0$, then $y = f(x)$ has derivative $f'(x)$ equal to $\frac{1}{\phi'(y)}$; that is

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \ldots (20)$$

Proof. We have $x = \phi(y)$ Then $\frac{dx}{dx} = \frac{d(\phi(y))}{dx}$

That is, $1 = \phi'(y)\,\frac{dy}{dx}$ (by chain rule)

$1 = \frac{dx}{dy} \cdot \frac{dy}{dx}$. Hence, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Derivatives of inverse trigonometrical functions.

I. The derivative of $y = \sin^{-1}x$ is $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ … (21)

Proof: We have $y = \sin^{-1}x$ and $x = \sin y$

Then $\frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$$\frac{d(\sin^{-1}x)}{dx} = \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\sqrt{1-x^2}} .$$

II. The derivative of $y = \cos^{-1}x$ is $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ … (22)

Proof: We have $y = \cos^{-1}x$ and $x = \cos y$

$$\therefore \quad \frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$$

$$\frac{d(\cos^{-1}x)}{dx} = \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{-1}{\sqrt{1-x^2}} .$$

Aliter : We know that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

This implies $\frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}(\cos^{-1}x) = \frac{d}{dx}\left(\frac{\pi}{2}\right)$

$$\frac{1}{\sqrt{1-x^2}} + \frac{d(\cos^{-1}x)}{dx} = 0 \quad \therefore \frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}} .$$

III. The derivative of the function $y = \tan^{-1}x$ is $\frac{dy}{dx} = \frac{1}{1+x^2}$ … (23)

Proof: We have $y = \tan^{-1}x$ and $x = \tan y$

This implies $x' = \frac{d}{dy}(\tan y) = \sec^2 y = 1 + \tan^2 y = 1 + x^2$

$$y' = \frac{1}{x'} = \frac{1}{1+x^2}$$

IV. The derivative of $y = \cot^{-1}x$ is $y' = -\frac{1}{1+x^2}$. … (24)

Proof: We have $y = \cot^{-1}x$ and $x = \cot y$.

$$\frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1+\cot^2 y) = -(1+x^2)$$

$\therefore$ by (20), $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{1}{1+x^2}$.

Aliter : We know that $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

Differentiating with respect to x on both sides,

$$\frac{d(\tan^{-1}x)}{dx} + \frac{d(\cot^{-1}x)}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx}$$

$$\frac{1}{1+x^2} + \frac{d(\cot^{-1}x)}{dx} = 0$$

$$\therefore \frac{d(\cot^{-1}x)}{dx} = -\frac{1}{1+x^2} .$$

V. The derivative of $y = \sec^{-1}x$ is $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$ … (25)

Proof: We have $y = \sec^{-1}x$ and $x = \sec y$

$$\frac{dx}{dy} = \sec y \tan y = \sec y\sqrt{\sec^2 y - 1}$$

$\therefore$ by (20), $$\frac{d(\sec^{-1}x)}{dx} = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x\sqrt{x^2-1}}\ .$$

VI. The derivative of $y = \text{cosec}^{-1}x$ is $\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$ … (26)

Proof: We have $y = \text{cosec}^{-1}x$ and $x = \text{cosec}\, y$

$$\frac{dx}{dy} = \frac{d(\text{cosec}\, y)}{dy} = -\text{cosec}\, y \cot y$$

$$= -\text{cosec}\, y\sqrt{\text{cosec}^2 y - 1} = -x\sqrt{x^2-1}$$

Therefore by (20) $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{1}{x\sqrt{x^2-1}}\ .$$

Example 8.65: Differentiate $y = \sin^{-1}(x^2 + 2x)$ with respect to x.

Solution: We have $y = \sin^{-1}(x^2 + 2x)$

Take $u = x^2 + 2x$ Then $y = \sin^{-1}(u)$, a function of function.

Therefore by chain rule,

$$y' = \frac{dy}{du}\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\ \frac{d(x^2+2x)}{dx}, \text{ by (21)}$$

$$= \frac{1}{\sqrt{1-(x^2+2x)^2}}\ (2x+2) = \frac{2(x+1)}{\sqrt{1-x^2(x+2)^2}}\ .$$

Example 8.66: Find $\frac{dy}{dx}$ if $y = \cos^{-1}\left(\frac{1-x}{1+x}\right)$.

Solution: We have $y = \cos^{-1}\left(\frac{1-x}{1+x}\right)$.

Take $u = \frac{1-x}{1+x}$. Therefore $y = \cos^{-1}(u)$, a function of function.

By chain rule $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}\ .$$

$$\therefore\ \frac{dy}{dx} = -\frac{1}{\sqrt{1-u^2}}\cdot\frac{d\left(\frac{1-x}{1+x}\right)}{dx}$$

$$= -\frac{1}{\sqrt{1-u^2}}\left[\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}\right] = -\frac{1}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}}\cdot\frac{-2}{(1+x)^2}$$

$$= -\frac{1}{\frac{\sqrt{(1+x)^2-(1-x)^2}}{1+x}} \quad \frac{-2}{(1+x)^2} = \frac{(1+x)}{\sqrt{4x}} \quad \frac{2}{(1+x)^2} = \frac{1}{\sqrt{x}\,(1+x)}\ .$$

Example 8.67: Find y' if $y = \tan^{-1}(e^x)$

Solution: We have $y = \tan^{-1}(e^x)$. Take $u = e^x$ then $y = \tan^{-1}(u)$.

By chain rule, $$y' = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \quad \frac{d\,(e^x)}{dx} \quad = \frac{e^x}{1+e^{2x}}\ .$$

EXERCISE 8.8

Find the derivatives of the following functions:

(1) $\sin^{-1}\left(\frac{1-x}{1+x}\right)$ (2) $\cot^{-1}(e^{x^2})$

(3) $\tan^{-1}(\log x)$ (4) $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

8.4.3 Logarithmic Differentiation

We also consider the differentiation of a function of the form:

$y = u^v$ where u and v are functions of x.

We can write $y = e^{\log u^v} = e^{v\log u}$

Now y falls under the category of function of a function.

$$y' = e^{v\log u} \quad \frac{d\,(v\log u)}{dx}$$

$$= e^{v\log u}\left[v \cdot \frac{1}{u}u' + \log u . v'\right] = u^v\left[\frac{v}{u}u' + v'\log u\right]$$

$$= vu^{v-1}u' + u^v(\log u)\,v'. \qquad \ldots (27)$$

Another method:

$y = u^v$ Taking logarithm on both sides

$\log y = \log u^v \quad \Rightarrow \quad \log y = v\log u$

Diff. both sides with respect to x

$$\frac{1}{y}\ \frac{dy}{dx} = v\frac{1}{u}\ u' + v'\log u$$

$$\frac{dy}{dx} = y\left(\frac{v}{u}u' + v'\log u\right) = u^v\left(\frac{v}{u}u' + v'\log u\right)$$

Example 8.68: Find the derivative of $y = x^\alpha$, α is real .

Solution . We have $y = x^{\alpha}$

Then by (27) $y' = \alpha\, x^{\alpha - 1} . 1 + x^{\alpha} . (\log x) . 0$

$= \alpha x^{\alpha - 1} \quad (\because u = x, v = \alpha , v' = 0)$

Note: From example (8.74),we observe that the derivative of $x^n = nx^{n-1}$ is true for any real n.

Example 8.69: Find the derivative of $x^{\sin x}$ w.r. to x.

Solution: Let $y = x^{\sin x}$. Here $u = x$; $v = \sin x$; $u' = 1$; $v' = \cos x$.

Therefore by (27), $y' = \dfrac{dy}{dx} = \sin x . x^{\sin x - 1} . 1 + x^{\sin x} (\log x) \cos x$

$$= x^{\sin x} \left(\frac{\sin x}{x} + \cos x (\log x)\right) .$$

Example 8.70: Differentiate : $\dfrac{(1 - x)\sqrt{x^2 + 2}}{(x + 3)\sqrt{x - 1}}$

Solution: Let $y = \dfrac{(1 - x)\sqrt{x^2 + 2}}{(x + 3)\sqrt{x - 1}}$

In such cases we take logarithm on both sides and differentiate.

$$\log y = \log (1 - x)\sqrt{x^2 + 2} - \log (x + 3)\sqrt{x - 1}$$

$$= \log (1 - x) + \frac{1}{2} \log (x^2 + 2) - \log (x + 3) - \frac{1}{2} \log (x - 1).$$

Differentiating w.r. to x we get:

$$\therefore \frac{1}{y} \frac{dy}{dx} = \frac{-1}{1 - x} + \frac{2x}{2(x^2 + 2)} - \frac{1}{x + 3} - \frac{1}{2} . \frac{1}{x - 1}$$

$$= \frac{x}{x^2 + 2} + \frac{1}{2} . \frac{1}{x - 1} - \frac{1}{x + 3}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{x}{x^2 + 2} + \frac{1}{2(x - 1)} - \frac{1}{x + 3}\right] .$$

$$= \frac{(1 - x)\sqrt{x^2 + 2}}{(x + 3)\sqrt{x - 1}} \left[\frac{x}{x^2 + 2} + \frac{1}{2(x - 1)} - \frac{1}{x + 3}\right]$$

EXERCISE 8.9

Differentiate the following functions w.r. to *x*.

(1) $x^{\sqrt{2}}$ (2) x^{x^2} (3) $x^{\tan x}$ (4) $\sin x^{\sin x}$

(5) $(\tan^{-1}x)^{\log x}$ (6) $(\log x)^{\sin^{-1}x}$ (7) $\dfrac{(x^2+2)(x+\sqrt{2})}{(\sqrt{x+4})(x-7)}$

(8) $(x^2+2x+1)^{\sqrt{x-1}}$ (9) $\dfrac{\sin x \cos(e^x)}{e^x + \log x}$ (10) $x^{\sin x} + (\sin x)^x$

8.4.4 The method of substitution

Sometimes, a substitution facilitates differentiation. Following example will demonstrate this method.

Example 8.71: Differentiate the following w.r. to x

(i) $(ax+b)^n$ (ii) $\log(ax+b)^n$

(iii) $\sin^{-1}\dfrac{2x}{1+x^2}$ (iv) $\cos^{-1}\dfrac{1-x^2}{1+x^2}$ (v) $\sin^2(ax+b)$

Solution: (i) We have $y = (ax+b)^n$. Put $u = ax + b$. Then $y = u^n$.

Now y is a function of u and u is a function of x. By chain rule,

$$y' = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} \cdot \frac{d(ax+b)}{dx}$$

$$= n(ax+b)^{n-1} \cdot a = na(ax+b)^{n-1}.$$

(ii) Let $y = \log(ax+b)^n$. Put $ax + b = u$. Then as in (i) $y' = \dfrac{na}{ax+b}$.

(iii) Let $y = \sin^{-1}\dfrac{2x}{1+x^2}$. Put $x = \tan\theta$ so that $\theta = \tan^{-1}x$.

$$\therefore\ y = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}(\sin 2\theta) \qquad \left(\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= 2\theta \qquad (\because \sin^{-1}(\sin\theta) = \theta)$$

$$= 2\tan^{-1}x .$$

$$\therefore \frac{dy}{dx} = 2 \cdot \frac{d}{dx}(\tan^{-1}x) = \frac{2}{1+x^2} .$$

(iv) Let $y = \cos^{-1}\dfrac{1-x^2}{1+x^2}$. Put $x = \tan\theta$.

Then $\theta = \tan^{-1}x$ and $\dfrac{1-x^2}{1+x^2} = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$$\therefore\ y = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x$$

$$\frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2} .$$

(v) Let $y = \sin^2 (ax + b)$. Put $ax + b = u$ and $v = \sin u$

Then $y = v^2$, $v = \sin u$ and $u = ax + b$.

Therefore by chain rule,

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = 2\, v \cdot \cos u \cdot a$$

$$= 2\, a \sin u \cdot \cos u = a \sin 2u = a \sin 2\,(ax + b).$$

Example 8.72:

Differentiate (i) $\sin^{-1} (3x - 4x^3)$ (ii) $\cos^{-1} (4x^3 - 3x)$ (iii) $\tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2}\right)$.

Solution:

(i) Let $y = \sin^{-1} (3x - 4x^3)$

put $x = \sin \theta$, so that $\theta = \sin^{-1} x$.

Now $y = \sin^{-1} (3\sin\theta - 4 \sin^3\theta)$

$= \sin^{-1} (\sin 3\theta) = 3\theta = 3 \sin^{-1} x$. ($\because \sin 3\theta = 3 \sin \theta - 4 \sin^3\theta$)

$$\frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{1 - x^2}} = \frac{3}{\sqrt{1 - x^2}}$$

(ii) Let $y = \cos^{-1} (4x^3 - 3x)$

Put $x = \cos \theta$, so that $\theta = \cos^{-1} x$.

Now $y = \cos^{-1} (4 \cos^3\theta - 3 \cos \theta)$

$= \cos^{-1} (\cos 3\theta)$ ($\therefore \cos 3\theta = 4\cos^3\theta - 3 \cos \theta$)

$= 3\theta = 3 \cos^{-1} x$.

$$\therefore \frac{dy}{dx} = -\frac{3}{\sqrt{1 - x^2}} .$$

(iii) Let $y = tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2}\right)$

Put $x = \tan\theta$, so that $\theta = \tan^{-1} x$.

$$y = \tan^{-1} \left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1} (\tan 3\theta) = 3\theta = 3 \tan^{-1} x.$$

$$\therefore \frac{dy}{dx} = \frac{3}{1 + x^2} .$$

EXERCISE 8.10

Differentiate

(1) $\cos^{-1}\sqrt{\dfrac{1+\cos x}{2}}$ (2) $\sin^{-1}\sqrt{\dfrac{1-\cos 2x}{2}}$ (3) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

(4) $\tan^{-1}\left(\dfrac{\cos x+\sin x}{\cos x-\sin x}\right)$ (5) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ (6) $\tan^{-1}\dfrac{1+x^2}{1-x^2}$

(7) $\tan^{-1}\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}$ (8) $\tan^{-1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

(9) $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ Hint:$\sin^2 x/2+\cos^2 x/2=1$; $\sin x=2\sin x/2\cos x/2$)

8.4.5 Differentiation of parametric functions

Definition

If two variables, say, x and y are functions of a third variable, say, t, then the functions expressing x and y in terms of t are called a parametric functions. The variable 't' is called the parameter of the function.

Let $x = f(t)$, $y = g(t)$ be the parametric equations.

Let Δx, Δy be the increments in x and y respectively corresponding to an increment Δt in t.

Therefore $x + \Delta x = f(t + \Delta t)$ and $y + \Delta y = g(t + \Delta t)$

and $\Delta x = f(t + \Delta t) - f(t)$ $\Delta y = g(t + \Delta t) - g(t)$.

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left[\frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}}\right] = \frac{\lim\limits_{\Delta t \to 0}\frac{\Delta y}{\Delta x}}{\lim\limits_{\Delta t \to 0}\frac{\Delta x}{\Delta t}} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \qquad \ldots (28)$$

where $\dfrac{dx}{dt} \neq 0$. Note that $\Delta x \to 0 \Rightarrow f(t + \Delta t) \to f(t) \Rightarrow \Delta t \to 0$.

Example 8.73: Find $\dfrac{dy}{dx}$ when $x = a\cos^3 t$, $y = a\sin^3 t$.

Solution: We have $x = a\cos^3 t$, $y = a\sin^3 t$.

Now $\therefore \dfrac{dx}{dt} = -3a\cos^2 t\sin t$ and $\dfrac{dy}{dt} = 3a\sin^2 t\cos t$.

Therefore by (28) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t} = -\tan t.$$

Example 8.74: Find $\frac{dy}{dx}$, if $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

Solution: We have $\frac{dx}{d\theta} = a(1 + \cos\theta)$ $\quad \frac{dy}{d\theta} = a(0 + \sin\theta)$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}\ .$$

EXERCISE 8.11

Find $\frac{dy}{dx}$ if x and y are connected parametrically by the equations (without eliminating the parameter) .

(1) $x = a\cos\theta$, $y = b\sin\theta$

(2) $x = at^2$, $y = 2at$

(3) $x = a\sec^3\theta$, $y = b\tan^3\theta$

(4) $x = 4t$, $y = \frac{4}{t}$

(5) $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$

(6) $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$, $y = a\sin\theta$

(7) $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$

8.4.6 Differentiation of implicit functions

If the relation between x and y is given by an equation of the form $f(x, y) = 0$ and this equation is not easily solvable for y, then y is said to be an implicit function of x. In case y is given in terms of x, then y is said to be an explicit function of x. In case of implicit function also, it is possible to get $\frac{dy}{dx}$ by mere differentiation of the given relation, without solving it for y first. The following examples illustrate this method.

Example 8.75: Obtain $\frac{dy}{dx}$ when $x^3 + 8xy + y^3 = 64$.

Solution . We have $x^3 + 8xy + y^3 = 64$.

Differentiating with respect to x on both sides,

$$3x^2 + 8\left[x\frac{dy}{dx} + y\ .\ 1\right] + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 + 8y + 8x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(3x^2 + 8y) + (8x + 3y^2)\frac{dy}{dx} = 0$$

$$(8x + 3y^2)\frac{dy}{dx} = -(3x^2 + 8y) \qquad \therefore \frac{dy}{dx} = -\frac{(3x^2 + 8y)}{(8x + 3y^2)}$$

Example 8.76: Find $\frac{dy}{dx}$ when $\tan(x + y) + \tan(x - y) = 1$

Solution: We have $\tan(x + y) + \tan(x - y) = 1$.

Differentiating both sides w.r.t. to x,

$$\sec^2(x + y)\left(1 + \frac{dy}{dx}\right) + \sec^2(x - y)\left(1 - \frac{dy}{dx}\right) = 0$$

$$[\sec^2(x + y) + \sec^2(x - y)] + [\sec^2(x + y) - \sec^2(x - y)]\frac{dy}{dx} = 0$$

$$[\sec^2(x + y) - \sec^2(x - y)]\frac{dy}{dx} = -[\sec^2(x + y) + \sec^2(x - y)]$$

$$\therefore \frac{dy}{dx} = -\frac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x + y) - \sec^2(x - y)} = \frac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x - y) - \sec^2(x + y)}.$$

Example 8.77: Find $\frac{dy}{dx}$ if $xy + xe^{-y} + ye^x = x^2$.

Solution: We have $xy + xe^{-y} + ye^x = x^2$

Differentiating both sides w.r.t. to x,

$$x\frac{dy}{dx} + y.1 + xe^{-y}\left(-\frac{dy}{dx}\right) + e^{-y}.1 + y.e^x + e^x\frac{dy}{dx} = 2x$$

$$(y + e^{-y} + ye^x) + (x - xe^{-y} + e^x)\frac{dy}{dx} = 2x$$

$$(ye^x + y + e^{-y} - 2x) + (e^x - xe^{-y} + x)\frac{dy}{dx} = 0$$

$$(e^x - xe^{-y} + x)\frac{dy}{dx} = -(ye^x + y + e^{-y} - 2x)$$

$$\therefore \frac{dy}{dx} = -\frac{(ye^x + y + e^{-y} - 2x)}{(e^x - xe^{-y} + x)} = \frac{(ye^x + y + e^{-y} - 2x)}{(xe^{-y} - e^x - x)}.$$

EXERCISE 8.12

Find $\frac{dy}{dx}$ for the following implicit functions.

(1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (2) $y = x \sin y$ (3) $x^4 + y^4 = 4a^2x^3y^3$

(4) $y \tan x - y^2 \cos x + 2x = 0$ (5) $(1 + y^2)\sec x - y \cot x + 1 = x^2$

(6) $2y^2 + \frac{y}{1+x^2} + \tan^2 x + \sin y = 0$ (7) $xy = \tan(xy)$ (8) $x^m y^n = (x+y)^{m+n}$

(9) $e^x + e^y = e^{x+y}$ (10) $xy = 100(x+y)$ (11) $x^y = y^x$

(12) If $ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$, show that $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$

8.4.7 Higher order Derivatives.

Let $y = f(x)$ be a differentiable function of x.

Then we know its derivative $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ is called first order derivative of $y = f(x)$ with respect to x. This first order derivative $f'(x)$, a function of x may or may not be differentiable. If $f'(x)$ is differentiable then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$ is called second order derivative of $y = f(x)$ with respect to x. It is denoted by $\frac{d^2y}{dx^2}$.

Other symbols like y_2, y'', $\ddot{y}$ or D^2y where $D^2 = \frac{d^2}{dx^2}$ also used to denote the second order derivative. Similarly, we can define third order derivative of $y = f(x)$ as

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \lim_{\Delta x \to 0} \frac{f''(x+\Delta x) - f''(x)}{\Delta x}$$ provided $f''(x)$ is differentiable.

As before, y_3, y''', $\dddot{y}$ or D^3y is used to denote third order derivative.

Example 8.78: Find y_3, if $y = x^2$

Solution:

$$y_1 = \frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2x) = 2$$

$$y_3 = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(2) = 0.$$

Example 8.79:

Let $y = A\cos 4x + B\sin 4x$, A and B are constants. Show that $y_2 + 16y = 0$

Solution:

$$y_1 = \frac{dy}{dx} = (A\cos 4x + B\sin 4x)' = -4A\sin 4x + 4B\cos 4x$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dx}\,(-\,4\text{ A}\sin 4x + 4\text{B}\cos 4\,x)$$
$$= -\,16\text{ A}\cos 4x - 16\text{ B}\sin 4x$$
$$= -\,16\,(\text{A}\cos 4x + \text{B}\sin 4x) = -\,16y$$
$$\therefore\ y_2 + 16y = 0$$

Example 8.80: Find the second derivative of the function log (log x)

Solution: Let $y = \log\ (\log x)$

By chain rule, $\frac{dy}{dx} = \frac{1}{\log x}\cdot\frac{d\,(\log x)}{dx} = \frac{1}{\log x}\cdot\frac{1}{x}$

$$= \frac{1}{x\log x} = (x\log x)^{-1}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\,(x\log x)^{-1}}{dx} = -\,(x\log x)^{-2}\ \frac{d\,(x\log x)}{dx}$$
$$= -\frac{1}{(x\log x)^2}\left[x\,.\,\frac{1}{x} + \log x\,.\,1\right] = -\frac{1+\log x}{(x\log x)^2}\,.$$

Example 8.81: If $y = \log\ (\cos x)$, find y_3

Solution: We have $y = \log\ (\cos x)$

$$y_1 = \frac{d\,[\log\,(\cos x)]}{dx} = \frac{1}{\cos x}\ \frac{d\,(\cos x)}{dx},\ \text{by chain rule}$$
$$= \frac{1}{\cos x}\ .\,(-\sin x) = -\tan x$$
$$y_2 = \frac{d\,y_1}{dx} = \frac{d\,(-\tan x)}{dx} = -\sec^2 x$$
$$y_3 = \frac{d\,(y_2)}{dx} = \frac{d\,(-\sec^2 x)}{dx} = -\,2\sec x\,.\,\frac{d\,(\sec x)}{dx}$$
$$= -\,2\sec x\,.\,\sec x\,.\,\tan x = -\,2\sec^2 x\tan x.$$

Example 8.82: If $y = e^{ax}\sin bx$, prove that $\frac{d^2y}{dx^2} - 2a\,.\,\frac{dy}{dx} + (a^2 + b^2)\,y = 0$

Solution: We have $y = e^{ax}\sin bx$

$$\frac{dy}{dx} = e^{ax}\,.\,b\cos bx + a\,e^{ax}\sin bx$$
$$= e^{ax}\,(b\cos bx + a\sin bx)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{e^{ax}(b\cos bx + a\sin bx)\right\}$$

$$= e^{ax}\left\{-b^2\sin bx + ab\cos bx\right\} + (b\cos bx + a\sin bx)a\,e^{ax}$$

$$= -b^2(e^{ax}\sin bx) + a\;be^{ax}\cos bx + a.e^{ax}(b\cos bx + a\sin bx)$$

$$= -b^2\;y + a\left(\frac{dy}{dx} - ae^{ax}\sin bx\right) + a\frac{dy}{dx}$$

$$= -b^2\;y + a\left(\frac{dy}{dx} - a.y\right) + a\frac{dy}{dx}$$

$$= 2a\frac{dy}{dx} - (a^2 + b^2)y$$

Therefore, $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$.

Example 8.83: If $y = \sin(ax + b)$, prove that $y_3 = a^3\sin\left(ax + b + \frac{3\pi}{2}\right)$.

Solution: We have $y = \sin(ax + b)$

$$y_1 = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + 2.\frac{\pi}{2}\right)$$

$$y_3 = a^3\cos\left(ax + b + 2.\frac{\pi}{2}\right) = a^3\sin\left(ax + b + 2.\frac{\pi}{2} + \frac{\pi}{2}\right) = a^3\sin\left(ax + b + 3\frac{\pi}{2}\right)$$

Example 8.84: If $y = \cos(m\sin^{-1}x)$, prove that $(1 - x^2)y_3 - 3xy_2 + (m^2 - 1)y_1 = 0$

Solution: We have $y = \cos(m\sin^{-1}x)$

$$y_1 = -\sin(m\sin^{-1}x) . \frac{m}{\sqrt{1 - x^2}}$$

$$y_1^2 = \sin^2(m\sin^{-1}x)\frac{m^2}{(1 - x^2)}$$

This implies $(1 - x^2)y_1^2 = m^2\sin^2(m\sin^{-1}x) = m^2\left[1 - \cos^2(m\sin^{-1}x)\right]$

That is, $(1 - x^2)\,y_1^2 = m^2(1 - y^2)$.

Again differentiating,

$$(1 - x^2)2y_1\frac{dy_1}{dx} + y_1^2(-2x) = m^2\left(-2y\frac{dy}{dx}\right)$$

$$(1 - x^2)\,2y_1y_2 - 2xy_1^2 = -2m^2yy_1$$

$$(1-x^2)\,y_2 - xy_1 = -\,m^2y$$

Once again differentiating,

$$(1-x^2)\frac{d\,y_2}{dx} + y_2\,(-\,2x) - \left[x\,.\frac{d\,y_1}{dx} + y_1\,.\,1\right] = -\,m^2\frac{dy}{dx}$$

$$(1-x^2)\,y_3 - 2xy_2 - xy_2 - y_1 = -\,m^2y_1$$

$$(1-x^2)\,y_3 - 3xy_2 + (m^2-1)\,y_1 = 0.$$

EXERCISE 8.13

(1) Find $\frac{d^2y}{dx^2}$ if $y = x^3 + \tan x$.

(2) Find $\frac{d^3y}{dx^3}$ if $y = x^2 + \cot x$.

(3) Find the second order derivative of:

(i) $x^2 + 6x + 5$ (ii) $x \sin x$ (iii) $\cot^{-1}x$.

(4) Find the third order derivatives of:

(i) $e^{mx} + x^3$ (ii) $x \cos x$.

(5) If $y = 500\,e^{7x} + 600e^{-\,7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

(6) If $y = e^{\tan^{-1}x}$ prove that $(1+x^2)\,y_2 + (2x-1)y_1 = 0$.

(7) If $y = \log\,(x^2 - a^2)$, prove that $y_3 = 2\left[\frac{1}{(x+a)^3} + \frac{1}{(x-a)^3}\right]$.

(8) If $x = \sin t$; $y = \sin pt$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0.$

(9) If $x = a\,(\cos\theta + \theta\sin\theta)$, $y = a\,(\sin\theta - \theta\cos\theta)$,

show that $a\,\theta\frac{d^2y}{dx^2} = \sec^3\theta.$

(10) If $y = (x^3 - 1)$, prove that $x^2\,y_3 - 2xy_2 + 2y_1 = 0$.

TABLE OF DERIVATIVES

	Function	Derivative
1.	k ; (k is a cosntant)	$(k)' = 0$
2.	k$f(x)$	$(kf(x))' = kf'(x)$
3.	$u \pm v$	$(u \pm v)' = u' \pm v'$
4.	$u_1 + u_2 + \dots + u_n$	$(u_1 + u_2 + \dots u_n)' = u_1' + u_2' + \dots + u_n'$
5.	$u \cdot v$	$(uv)' = uv' + vu'$
		$\frac{(uv)'}{uv} = \frac{u'}{u} + \frac{v'}{v}$
6.	$u_1.u_2 \dots u_n$	$(u_1.u_2 .. u_n)' = u_1' u_2 u_3 .. u_n + u_1 u_2' .. u_n$
		$+ \dots + u_1 u_2 \dots u_{n-1} u_n'$
		$\frac{(u_1.u_2 .. u_n)'}{u_1.u_2 .. u_n} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \dots + \frac{u_n'}{u_n}$
7.	x^n ($n \in$ R)	$(x^n)' = \text{n}x^{n-1}$
8.	$\log_a^x$	$(\log_a^x)' = \frac{\log a^e}{x}$
9.	$\log_e^x$	$(\log x)' = \frac{1}{x}$
10.	$\sin x$	$(\sin x)' = \cos x$
11.	$\cos x$	$(\cos x)' = -\sin x$
12.	$\tan x$	$(\tan x)' = \sec^2 x$
13.	$\cot x$	$(\cot x)' = -\text{cosec}^2 x$
14.	$\sec x$	$(\sec x)' = \sec x \cdot \tan x$
15.	$\text{cosec } x$	$(\text{cosec } x)' = -\text{cosec } x \cdot \cot x$
	Function	**Derivative**
16.	$\sin^{-1} x$	$(\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}}$

17.	$\cos^{-1}x$	$(\cos^{-1}x)' = \dfrac{-1}{\sqrt{1-x^2}}$
18.	$\tan^{-1}x$	$(\tan^{-1}x)' = \dfrac{1}{1+x^2}$
19.	$\cot^{-1}x$	$(\cot^{-1}x)' = -\dfrac{1}{1+x^2}$
20.	$\sec^{-1}x$	$(\sec^{-1}x)' = \dfrac{1}{x\sqrt{x^2-1}}$
21.	$\text{cosec}^{-1}x$	$(\text{cosec}^{-1}x)' = -\dfrac{1}{x\sqrt{x^2-1}}$
22.	$\dfrac{u}{v}$	$\left(\dfrac{u}{v}\right)' = \dfrac{v.u' - u.v'}{v^2}$
23.	e^x	$(e^x)' = e^x$
24.	u^v	$(u^v)' = vu^{v-1}.\, u' + u^v (\log u) v'$
25.	a^x	$(a^x)' = a^x(\log a)$
26.	$\left.\begin{array}{r} y = f(x) \\ x = \varphi(y) \text{ (inverse of } f) \end{array}\right\}$	$\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$.
27.	$y = f(u),\ u = \varphi(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.
28.	$\left.\begin{array}{r} y = f(u) \\ u = g(t) \\ t = h(x) \end{array}\right\}$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dt} \times \dfrac{dt}{dx}$.
29.	$\left.\begin{array}{r} y = g(t) \\ x = f(t) \end{array}\right\}$	$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{y'(t)}{x'(t)}$
30.	$f(x, y) = k$	$\dfrac{dy}{dx} = \dfrac{f_1(x,y)}{f_2(x,y)}$, $f_2(x, y) \neq 0$

Note : In the above formulae from 1 to 25 $(\,.\,)' = \dfrac{d(\,.\,)}{dx}$.

9. INTEGRAL CALCULUS

9.1 Introduction:

Calculus deals principally with two geometric problems.

(i) The problem of finding SLOPE of the tangent line to the curve, is studied by the limiting process known as differentiation and

(ii) Problem of finding the AREA of a region under a curve is studied by another limiting process called Integration.

Actually integral calculus was developed into two different directions over a long period independently.

(i) Leibnitz and his school of thought approached it as the anti derivative of a differentiable function.

(ii) Archimedes, Eudoxus and others developed it as a numerical value equal to the area under the curve of a function for some interval. However as far back as the end of the 17^{th} century it became clear that a general method for solution of finding the area under the given curve could be developed in connection with definite problems of integral calculus.

In the first section of this chapter, we study integration, the process of obtaining a function from its derivative, and in the second we examine certain limit of sums that occur frequently in applications.

We are already familiar with inverse operations. (+, −) ; (×, ÷), $\left((\ )^n, \ \sqrt[n]{\ }\right)$ are some pairs of inverse operations. Similarly differentiation and integrations are also inverse operations. In this section we develop the inverse operation of differentiation called anti differentiation.

Definition

A function F(x) is called an anti derivative or integral of a function f(x) on an interval **I** if

$$F'(x) = f(x) \text{ for every value of } x \text{ in } \mathbf{I}$$

i.e. If the derivative of a function F(x) w.r.t. to x is f(x), then we say that the integral of f(x) w.r.t. to x is F(x).

i.e. $$\int f(x)\, dx = F(x)$$

For example we know that

$$\frac{d}{dx}(\sin x) = \cos x, \quad \text{then} \quad \int \cos x\, dx = \sin x.$$

Also $$\frac{d}{dx}(x^5) = 5x^4, \quad \text{gives} \int 5x^4\, dx = x^5$$

The symbol **'$\int$'** is the sign of integration. **'$\int$'** is elongated S, which is the first letter of the word sum.

The function ***f*(*x*)** is called **Integrand.**

The variable ***x*** in ***dx*** is called **variable of integration** or **integrator.**

The process of finding the integral is called **integration.**

Constant of integration:

Consider the following two examples.

Example 9.1:

$$\left.\begin{aligned} \frac{d}{dx}(2x+5) &= 2 \\ \frac{d}{dx}(2x) &= 2 \\ \frac{d}{dx}(2x-4) &= 2 \\ \frac{d}{dx}\left(2x-\sqrt{7}\right) &= 2 \end{aligned}\right\} \Rightarrow \int 2dx = 2x + ? = 2x + C$$

Where this 'C' may be 5, 0, – 4 or $-\sqrt{7}$ as shown in the above example. (See fig. 1(*a*)).

Example 9.2:

$$\left.\begin{aligned} \frac{d}{dx}(x^2+1) &= 2x \\ \frac{d}{dx}(x^2) &= 2x \\ \frac{d}{dx}(x^2-4) &= 2x \end{aligned}\right\} \Rightarrow \int 2xdx = x^2 + ? = x^2 + C$$

'C' is any constant (See fig 1(b))

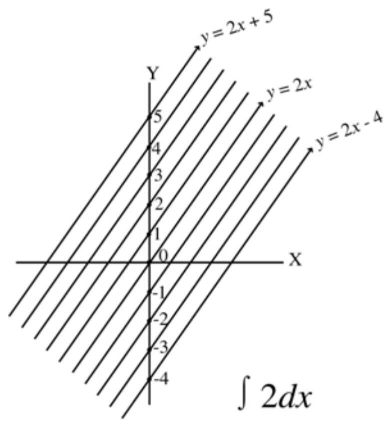


fig. 9.1.a

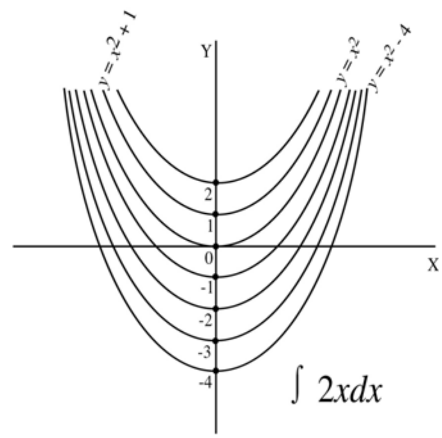


fig 9.1.b

By the way it is accepted to understand that the expression $\int f(x)\,dx$ is not a particular integral, but family of integrals of that function.

If F(x) is one such integral, it is customary to write $\int f(x)\,dx = F(x) + C$

Where 'C' is an arbitrary constant. 'C' is called **'the constant of integration'.** Since C is arbitrary, $\int f(x)\,dx$ is called the **"indefinite integral".**

Formulae

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x^n}\,dx = -\frac{1}{(n-1)\,x^{n-1}} + c \quad (n \neq 1]$$

$$\int \frac{1}{x}\,dx = \log x + c$$

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{a^x}{\log a} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

Example 9.3 – 9.7: Integrate the following with respect to x.

(3) x^6 (4) x^{-2} (5) $\frac{1}{x^{10}}$ (6) $\sqrt{x}$ (7) $\frac{1}{\sqrt{x}}$

Solution:

(3) $\int x^6\,dx = \frac{x^{6+1}}{6+1} = \frac{x^7}{7} + c$

(4) $\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x} + c$

(5) $\int \frac{1}{x^{10}}\,dx = \int x^{-10}\,dx$

$= \frac{x^{-10+1}}{-10+1} + c$

$= \frac{x^{-9}}{-9} + c$

$\int \frac{1}{x^{10}}\,dx = -\frac{1}{9x^9} + c$

[Here we can also use the formula

$\int \frac{1}{x^n}\,dx = -\frac{1}{(n-1)\,x^{n-1}}$ where $n \neq 1$]

(6) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$\int \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$

(7) $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx$

$= \frac{x^{-1/2+1}}{-1/2+1} + c$

$= \frac{x^{1/2}}{+1/2} + c$

$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c$

Example 9.8 – 9.10: Integrate:

(8) $\frac{\sin x}{\cos^2 x}$ (9) $\frac{\cot x}{\sin x}$ (10) $\frac{1}{\sin^2 x}$

Solution:

(8) $\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\,dx = \int \tan x \sec x\,dx = \sec x + c$

(9) $\int \frac{\cot x}{\sin x}\,dx = \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(10) $\int \frac{1}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x\,dx = -\cot x + c$

EXERCISE 9.1

Integrate the following with respect to x

(1) (i) x^{16} (ii) $x^{\frac{5}{2}}$ (iii) $\sqrt{x^7}$ (iv) $\sqrt[3]{x^4}$ (v) $(x^{10})^{\frac{1}{7}}$

(2) (i) $\frac{1}{x^5}$ (ii) x^{-1} (iii) $\frac{1}{x^{\frac{5}{2}}}$ (iv) $\frac{1}{\sqrt[3]{x^5}}$ (v) $\left(\frac{1}{x^3}\right)^{\frac{1}{4}}$

(3) (i) $\frac{1}{\text{cosec } x}$ (ii) $\frac{\tan x}{\cos x}$ (iii) $\frac{\cos x}{\sin^2 x}$ (iv) $\frac{1}{\cos^2 x}$ (v) $\frac{1}{e^{-x}}$

9.2 Integrals of function containing linear functions of *x* i.e. $\int f(ax + b)\, dx$

We know that

$$\frac{d}{dx}\left[\frac{(x-a)^{10}}{10}\right] = (x-a)^9 \quad \Rightarrow \int (x-a)^9 dx = \frac{(x-a)^{10}}{10}$$

$$\frac{d}{dx}[\sin(x+k)] = \cos(x+k) \quad \Rightarrow \int \cos(x+k)\, dx = \sin(x+k)$$

It is clear that whenever a constant is added to the independent variable x or subtracted from x the fundamental formulae remain the same.

But

$$\frac{d}{dx}\left[\frac{1}{\boldsymbol{l}}(e^{\boldsymbol{l}x+m})\right] = e^{\boldsymbol{l}x+m} \quad \Rightarrow \int e^{\boldsymbol{l}x+m}\, dx = \frac{1}{\boldsymbol{l}}\, e^{(lx+m)}$$

$$\frac{d}{dx}\left[\frac{1}{\boldsymbol{a}}\sin(\boldsymbol{a}x+b)\right] = \cos(\boldsymbol{a}x+b) \Rightarrow \int \cos(\boldsymbol{a}x+b)\, dx = \frac{1}{\boldsymbol{a}}\sin(ax+b)$$

Here, if any constant is multiplied with the independent variable x, then the same fundamental formula can be used after dividing it by the coefficient of x

i.e. if $\int f(x)\, dx = g(x) + c$, then $\int f(\boldsymbol{a}x+b)\, dx = \frac{1}{\boldsymbol{a}}\, g(\boldsymbol{a}x+b) + c$

The extended forms of fundamental formulae

$$\int (ax+b)^n dx = \frac{1}{a}\left[\frac{(ax+b)^{n+1}}{n+1}\right] + c \quad (n \neq -1)$$

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a}\log(ax+b) + c$$

$$\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + c$$

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$$

$$\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$$

$$\int \frac{1}{1+(ax)^2}\,dx = \frac{1}{a}\tan^{-1}(ax) + c$$

$$\int \frac{1}{\sqrt{1-(ax)^2}}\,dx = \frac{1}{a}\sin^{-1}(ax) + c$$

The above formulae can also be derived by using substitution method, which will be studied later.

Example 9.11 – 9.17: Integrate the following with respect to x.

(11) $(3-4x)^7$ (12) $\frac{1}{3+5x}$ (13) $\frac{1}{(lx+m)^n}$ (14) e^{8-4x}

(15) $\sin(lx+m)$ (16) $\sec^2(p-qx)$ (17) $\operatorname{cosec}(4x+3)\ \cot(4x+3)$

Solution:

(11) $$\int (3-4x)^7 dx = \left(-\frac{1}{4}\right)\frac{(3-4x)^8}{8} + c$$

$$\int (3-4x)^7 dx = -\frac{1}{32}(3-4x)^8 + c$$

(12) $$\int \frac{1}{3+5x}\,dx = \frac{1}{5}\log(3+5x) + c$$

(13) $$\int \frac{1}{(lx+m)^n}\,dx = \left(\frac{1}{l}\right)\left[\frac{(-1)}{(n-1)(lx+m)^{n-1}}\right] + c$$

$$\therefore \int \frac{1}{(lx+m)^n}\,dx = -\left(\frac{1}{l(n-1)}\right)\frac{1}{(lx+m)^{n-1}} + c$$

(14) $\int e^{8-4x}dx = \left(\frac{1}{-4}\right) e^{8-4x} + c$

$\int e^{8-4x}dx = -\frac{1}{4} e^{8-4x} + c$

(15) $\int \sin(lx+m)\,dx = \left(\frac{1}{l}\right)[-\cos(lx+m)] + c$

$= -\frac{1}{l}\cos(lx+m) + c$

(16) $\int \sec^2(p-qx)\,dx = \left(-\frac{1}{q}\right)[\tan(p-qx)] + c$

(17) $\int \operatorname{cosec}(4x+3)\cot(4x+3)\,dx = -\frac{1}{4}\operatorname{cosec}(4x+3) + c$

EXERCISE 9.2

Integrate the following with respect to *x*

(1) (i) x^4 (ii) $(x+3)^5$ (iii) $(3x+4)^6$ (iv) $(4-3x)^7$ (v) $(lx+m)^8$

(2) (i) $\frac{1}{x^6}$ (ii) $\frac{1}{(x+5)^4}$ (iii) $\frac{1}{(2x+3)^5}$ (iv) $\frac{1}{(4-5x)^7}$ (v) $\frac{1}{(ax+b)^8}$

(3) (i) $\frac{1}{x+2}$ (ii) $\frac{1}{3x+2}$ (iii) $\frac{1}{3-4x}$ (iv) $\frac{1}{p+qx}$ (v) $\frac{1}{(s-tx)}$

(4) (i) $\sin(x+3)$ (ii) $\sin(2x+4)$ (iii) $\sin(3-4x)$
(iv) $\cos(4x+5)$ (v) $\cos(5-2x)$

(5) (i) $\sec^2(2-x)$ (ii) $\operatorname{cosec}^2(5+2x)$ (iii) $\sec^2(3+4x)$
(iv) $\operatorname{cosec}^2(7-11x)$ (v) $\sec^2(p-qx)$

(6) (i) $\sec(3+x)\tan(3+x)$ (ii) $\sec(3x+4)\tan(3x+4)$
(iii) $\sec(4-x)\tan(4-x)$ (iv) $\sec(4-3x)\tan(4-3x)$
(v) $\sec(ax+b)\tan(ax+b)$

(7) (i) $\operatorname{cosec}(2-x)\cot(2-x)$ (ii) $\operatorname{cosec}(4x+2)\cot(4x+2)$
(iii) $\operatorname{cosec}(3-2x)\cot(3-2x)$ (iv) $\operatorname{cosec}(lx+m)\cot(lx+m)$
(v) $\cot(s-tx)\operatorname{cosec}(s-tx)$

(8) (i) e^{3x} (ii) e^{x+3} (iii) e^{3x+2} (iv) e^{5-4x} (v) e^{ax+b}

(9) (i) $\frac{1}{\cos^2(px+a)}$ (ii) $\frac{1}{\sin^2(l-mx)}$ (iii) $(ax+b)^{-8}$ (iv) $(3-2x)^{-1}$ (v) e^{-x}

(10) (i) $\dfrac{\tan(3-4x)}{\cos(3-4x)}$ (ii) $\dfrac{1}{e^{p+qx}}$ (iii) $\dfrac{1}{\tan(2x+3)\sin(2x+3)}$

(iv) $(lx+m)^{\frac{1}{2}}$ (v) $\sqrt{(4-5x)}$

Properties of integrals

> **(1) If k is any constant then $\int kf(x)\,dx = k\int f(x)\,dx$**
>
> **(2) If $f(x)$ and $g(x)$ are any two functions in x then**
> **$\int [f(x)+g(x)]dx = \int f(x)\,dx + \int g(x)\,dx$**

Example 9.18 – 9.21: Integrate the following with respect to x

(18) $10x^3 - \dfrac{4}{x^5} + \dfrac{2}{\sqrt{3x+5}}$ (19) $k\sec^2(ax+a) - \sqrt[3]{(4x+5)^2} + 2\sin(7x-2)$

(20) $a^x + x^a + 10 - \operatorname{cosec} 2x\ \cot 2x$ (21) $\dfrac{1}{5}\cos\left(\dfrac{x}{5}+7\right) + \dfrac{3}{(lx+m)} + e^{\frac{x}{2}+3}$

Solution:

(18) $\displaystyle\int\left(10x^3 - \frac{4}{x^5} + \frac{2}{\sqrt{3x+5}}\right)dx = 10\int x^3\,dx - 4\int\frac{dx}{x^5} + 2\int\frac{1}{\sqrt{3x+5}}\,dx$

$$= 10\left(\frac{x^4}{4}\right) - 4\left(-\frac{1}{4x^4}\right) + 2\,\frac{\left[2\sqrt{3x+5}\right]}{3}$$

$$= \frac{5}{2}x^4 + \frac{1}{x^4} + \frac{4}{3}\sqrt{3x+5} + c$$

(19) $\displaystyle\int [k\sec^2(ax+b) - \sqrt[3]{(4x+5)^2} + 2\sin(7x-2)]\,dx$

$$= k\int \sec^2(ax+b)\,dx - \int (4x+5)^{\frac{2}{3}}\,dx + 2\int \sin(7x-2)\,dx$$

$$= k\,\frac{\tan(ax+b)}{a} - \frac{1}{4}\,\frac{(4x+5)^{\frac{2}{3}+1}}{\left(\frac{2}{3}+1\right)} + (2)\left(\frac{1}{7}\right)(-\cos(7x-2)) + c$$

$$= \frac{k}{a}\tan(ax+b) - \frac{3}{20}(4x+5)^{\frac{5}{3}} - \frac{2}{7}\cos(7x-2) + c$$

(20) $\int \left(a^x + x^a + 10 - \text{cosec } 2x \ \cot 2x\right)dx$

$$= \int a^x dx + \int x^a dx + 10 \int dx - \int \text{cosec } 2x \ \cot 2x \, dx$$

$$= \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + 10x + \frac{\text{cosec } 2x}{2} + c$$

(21) $\int \left(\frac{1}{5}\cos\left(\frac{x}{5}+7\right) + \frac{3}{lx+m} + e^{\frac{x}{2}+3}\right) dx$

$$= \frac{1}{5}\int \cos\left(\frac{x}{5}+7\right) dx + 3\int \frac{1}{lx+m} dx + \int e^{\frac{x}{2}+3} dx$$

$$= \frac{1}{5} \cdot \frac{1}{(1/5)} \sin\left(\frac{x}{5}+7\right) + 3\left(\frac{1}{l}\right) \log(lx+m) + \frac{1}{(1/2)} e^{\frac{x}{2}+3} + c$$

$$= \sin\left(\frac{x}{5}+7\right) + \frac{3}{l}\log(lx+m) + 2e^{\frac{x}{2}+3} + c$$

EXERCISE 9.3

Integrate the following with respect to x

(1) $5x^4 + 3(2x+3)^4 - 6(4-3x)^5$ (2) $\frac{3}{x} + \frac{m}{4x+1} - 2(5-2x)^5$

(3) $4 - \frac{5}{x+2} + 3\cos 2x$ (4) $3e^{7x} - 4\sec(4x+3)\tan(4x+3) + \frac{11}{x^5}$

(5) $p \text{ cosec}^2 (px-q) - 6(1-x)^4 + 4e^{3-4x}$

(6) $\frac{4}{(3+4x)} + (10x+3)^9 - 3\text{cosec }(2x+3)\cot(2x+3)$

(7) $6\sin 5x - \frac{l}{(px+q)^m}$ (8) $a\sec^2(bx+c) + \frac{q}{e^{l-mx}}$

(9) $\frac{1}{\left(3+\frac{2}{3}x\right)} - \frac{2}{3}\cos\left(x-\frac{2}{3}\right) + 3\left(\frac{x}{3}+4\right)^6$

(10) $7\sin\frac{x}{7} - 8\sec^2\left(4-\frac{x}{4}\right) + 10\left(\frac{2x}{5}-4\right)^{\frac{3}{2}}$ (11) $2x^e + 3e^x + e^e$

(12) $(ae)^x - a^{-x} + b^x$

9.3 Methods of Integration

Integration is not as easy as differentiation. This is first due to its nature. Finding a derivative of a given function is facilitated by the fact that the differentiation itself has a constructive character. A derivative is simply defined as

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Suppose we are asked to find the derivative of **log*x*,** we know in all details how to proceed in order to obtain the result.

When we are asked to find the integral of **log*x*,** we have no constructive method to find integral or even how to start.

In the case of differentiation we use the laws of differentiation of several functions in order to find derivatives of their various combinations, e.g. their sum, product, quotient, composition of functions etc.

There are very few such rules available in the theory of integration and their application is rather restricted. But the significance of these methods of integration is very great.

In every case one must learn to select the most appropriate method and use it in the most convenient form. This skill can only be acquired after long practice.

Already we have seen two important properties of integration. The following are the four important methods of integrations.

(1) Integration by decomposition into sum or difference.

(2) Integration by substitution.

(3) Integration by parts

(4) Integration by successive reduction.

Here we discuss only the first three methods of integration and the other will be studied in higher classes.

9.3.1 Decomposition method

Sometimes it is impossible to integrate directly the given function. But it can be integrated after decomposing it into a sum or difference of number of functions whose integrals are already known.

For example $(1 + x^2)^3$, $\sin 5x \cos 2x$, $\frac{x^2 - 5x + 1}{x}$, $\sin^5 x$, $\frac{e^x + 1}{e^x}$, $(\tan x + \cot x)^2$

do not have direct formulae to integrate. But these functions can be decomposed into a sum or difference of functions whose individual integrals are known. In most of the cases the given integrand will be any one of the algebraic, trigonometric or exponential forms, and sometimes combinations of these functions.

Example 9.22 - Integrate

(22) $\int (1 + x^2)^3 dx = \int \left(1 + 3x^2 + 3x^4 + x^6\right) dx$

$$= x + \frac{3x^3}{3} + \frac{3x^5}{5} + \frac{x^7}{7} + c$$

$$= x + x^3 + \frac{3}{5}x^5 + \frac{x^7}{7} + c$$

(23) $\int \sin 5x \cos 2x \, dx = \int \frac{1}{2}\left[\sin(5x + 2x) + \sin(5x - 2x)\right] dx$

$[\because 2\sin A \cos B = \sin(A+B) + \sin(A - B)]$

$$= \frac{1}{2} \int [\sin 7x + \sin 3x] \, dx$$

$$= \frac{1}{2}\left[\frac{-\cos 7x}{7} - \frac{\cos 3x}{3}\right] + c$$

$\therefore \int \sin 5x \cos 2x \, dx = -\frac{1}{2}\left[\frac{\cos 7x}{7} + \frac{\cos 3x}{3}\right] + c$

(24) $\int \frac{x^2 - 5x + 1}{x} dx = \int \left(\frac{x^2}{x} - \frac{5x}{x} + \frac{1}{x}\right) dx = \int \left(x - 5 + \frac{1}{x}\right) dx$

$$= \int x dx - 5\int dx + \int \frac{1}{x} dx$$

$$= \frac{x^2}{2} - 5x + \log x + c$$

(25) $\int \cos^3 x \, dx = \int \frac{1}{4}[3\cos x + \cos 3x] \, dx$

$$= \frac{1}{4}\left[3\int \cos x \, dx + \int \cos 3x \, dx\right]$$

$$= \frac{1}{4}\left[3 \sin x + \frac{\sin 3x}{3}\right] + c$$

(26) $\int \frac{e^x + 1}{e^x}\, dx \quad = \quad \int \left(\frac{e^x}{e^x} + \frac{1}{e^x}\right) dx \; = \; \int 1dx + \int e^{-x}\, dx$

$\quad = \quad x - e^{-x} + c$

(27) $\int (\tan x + \cot x)^2 dx \quad = \quad \int\left(\tan^2 x + 2\tan x \cot x + \cot^2 x\right) dx$

$\quad = \quad \int [\left(\sec^2 x - 1\right) + 2 + (\operatorname{cosec}^2 x - 1)]\, dx$

$\quad = \quad \int\left(\sec^2 x + \operatorname{cosec}^2 x\right) dx$

$\quad = \quad \tan x + (-\cot x) + c$

$\quad = \quad \tan x - \cot x + c$

(28) $\int \frac{1}{1 + \cos x}\, dx = \int \frac{(1 - \cos x)}{(1 + \cos x)\,(1 - \cos x)}\, dx$

$= \int \frac{1 - \cos x}{1 - \cos^2 x}\, dx = \int \frac{1 - \cos x}{\sin^2 x}\, dx$

$= \int \left[\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right] dx = \int \left[\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x\right] dx$

$= \int \operatorname{cosec}^2 x\, dx - \int \operatorname{cosec} x \cot x\, dx$

$= -\cot x - (-\operatorname{cosec} x) + c$

$= \operatorname{cosec} x - \cot x + c$

Note: Another method

$$\left(\int \frac{1}{1 + \cos x}\, dx = \int \frac{1}{2\cos^2 \frac{x}{2}}\, dx = \frac{1}{2}\int \sec^2 \frac{x}{2}\, dx = \frac{1}{2}\, \frac{\tan \frac{x}{2}}{\frac{1}{2}} + c = \tan \frac{x}{2} + c\right)$$

(29) $\int \frac{1 - \cos x}{1 + \cos x}\, dx \quad = \quad \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}\, dx \; = \; \int \tan^2 \frac{x}{2}\, dx$

$$= \int \left(\sec^2\frac{x}{2} - 1\right) dx = \frac{\tan\frac{x}{2}}{\frac{1}{2}} - x + c$$

$$= 2\tan\frac{x}{2} - x + c \quad \dots \text{(i)}$$

Another method:

$$\int \frac{1-\cos x}{1+\cos x}\,dx = \int \frac{(1-\cos x)}{(1+\cos x)}\frac{(1-\cos x)}{(1-\cos x)}\,dx$$

$$= \int \frac{(1-\cos x)^2}{1-\cos^2 x}\,dx = \int \frac{1-2\cos x+\cos^2 x}{\sin^2 x}\,dx$$

$$= \int \left[\frac{1}{\sin^2 x} - \frac{2\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right] dx$$

$$= \int \left(\text{cosec}^2 x - 2\,\text{cosec}\,x\cot x + \cot^2 x\right) dx$$

$$= \int \left[\text{cosec}^2 x - 2\,\text{cosec}\,x\cot x + (\text{cosec}^2 x - 1)\right] dx$$

$$= \int \left[2\,\text{cosec}^2 x - 2\,\text{cosec}\,x\cot x - 1\right] dx$$

$$= 2\int \text{cosec}^2 x\,dx - 2\int \text{cosec}\,x\cot x\,dx - \int dx$$

$$= -2\cot x - 2(-\text{cosec}\,x) - x + c$$

$$\int \frac{1-\cos x}{1+\cos x}\,dx = 2\,\text{cosec}\,x - 2\cot x - x + c \quad \dots \text{(ii)}$$

Note: From (i) and (ii) both $2\tan\frac{x}{2} - x + c$ and $2\,\text{cosec}\,x - 2\cot x - x + c$ are trigonometrically equal.

(30) $\int \sqrt{1+\sin 2x}\,dx = \int \sqrt{(\cos^2 x + \sin^2 x) + (2\sin x\cos x)}\,dx$

$$= \int \sqrt{(\cos x + \sin x)^2}\,dx = \int (\cos x + \sin x)\,dx$$

$$= [\sin x - \cos x] + c$$

(31) $\int \frac{x^3+2}{x-1}\,dx = \int \frac{x^3-1+3}{x-1}\,dx = \int \left(\frac{x^3-1}{x-1} + \frac{3}{x-1}\right) dx$

$$= \int \left[\frac{(x-1)\,(x^2+x+1)}{x-1} + \frac{3}{x-1}\right] dx$$

$$= \int \left(x^2 + x + 1 + \frac{3}{x-1}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + 3\log(x-1) + c$$

(32) $\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\,dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x}\,dx$

$$= \int \left(\frac{\cos^2 x}{\sin^2 x \cos^2 x} - \frac{\sin^2 x}{\sin^2 x \cos^2 x}\right) dx$$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int (\operatorname{cosec}^2 x - \sec^2 x)\,dx$$

$$= -\cot x - \tan x + c$$

(33) $\int \frac{3^x - 2^{x+1}}{6^x}\,dx = \int \left(\frac{3^x}{6^x} - \frac{2^{x+1}}{6^x}\right) dx = \int \left[\left(\frac{3}{6}\right)^x - 2.\left(\frac{2}{6}\right)^x\right] dx$

$$= \int \left[\left(\frac{1}{2}\right)^x - 2\left(\frac{1}{3}\right)^x\right] dx = \int (2^{-x} - 2.3^{-x})\,dx$$

$$= \frac{-2^{-x}}{\log 2} - 2\,.\,\frac{(-3^{-x})}{\log 3} + c$$

$$= \frac{2}{\log 3}\,3^{-x} - \frac{1}{\log 2}\,2^{-x} + c$$

(34) $\int e^{x\log 2}\,.\,e^x\,dx = \int e^{\log 2^x}\,e^x\,dx = \int 2^x e^x\,dx$

$$= \int (2e)^x\,dx = \frac{(2e)^x}{\log 2e} + c$$

(35) $\int \frac{dx}{\sqrt{x+3}-\sqrt{x-4}} = \int \frac{\sqrt{x+3}+\sqrt{x-4}}{\left\{\sqrt{x+3}-\sqrt{x-4}\right\}\left\{\sqrt{x+3}+\sqrt{x-4}\right\}} dx$

$= \int \frac{\sqrt{x+3}+\sqrt{x-4}}{(x+3)-(x-4)} dx = \int \frac{\sqrt{x+3}+\sqrt{x-4}}{7} dx$

$= \frac{1}{7} \int \left[(x+3)^{1/2} + (x-4)^{1/2}\right] dx$

$\int \frac{dx}{\sqrt{x+3}-\sqrt{x-4}} = \frac{1}{7} \left[\frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x-4)^{3/2}\right] + c$

(36) $\int (x-1)\sqrt{x+1}\, dx = \int \{(x+1)-2\}\left(\sqrt{x+1}\right) dx$

$= \int \left[(x+1)^{3/2} - 2(x+1)^{1/2}\right] dx$

$= \frac{2}{5}(x+1)^{5/2} - 2.\frac{2}{3}(x+1)^{3/2} + c$

$\int (x-1)\sqrt{x+1}\, dx = \frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + c$

(37) $\int (3x+4)\sqrt{3x+7}\, dx = \int \{(3x+7)-3\}\sqrt{3x+7}\, dx$

$= \int ((3x+7)\sqrt{3x+7} - 3\sqrt{3x+7})\, dx$

$= \int ((3x+7)^{3/2} - 3(3x+7)^{1/2})\, dx$

$= \frac{1}{3}\frac{(3x+7)^{5/2}}{5/2} - 3 \cdot \frac{1}{3}\frac{(3x+7)^{3/2}}{3/2} + c$

$= \frac{2}{15}(3x+7)^{5/2} - \frac{2}{3}(3x+7)^{3/2} + c$

(37a) $\int \frac{9}{(x-1)(x+2)^2} dx = \int \left[\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\right] dx$ resolve into partial fraction

$= \int \left[\frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}\right] dx$

$= \log(x-1) - \log(x+2) + \frac{3}{(x+2)} + c$

EXERCISE 9.4

Integrate the following

(1) $(2x - 5)(36 + 4x)$ (2) $(1 + x^3)^2$ (3) $\frac{x^3 + 4x^2 - 3x + 2}{x^2}$

(4) $\frac{x^4 - x^2 + 2}{x + 1}$ (5) $\frac{(1 + x)^2}{\sqrt{x}}$ (6) $\frac{e^{2x} + e^{-2x} + 2}{e^x}$

(7) $\sin^2 3x + 4\cos 4x$ (8) $\cos^3 2x - \sin 6x$ (9) $\frac{1}{1 + \sin x}$

(10) $\frac{1}{1 - \cos x}$ (11) $\sqrt{1 - \sin 2x}$ (12) $\sqrt{1 + \cos 2x}$

(13) $\frac{1}{\sin^2 x \cos^2 x}$ (14) $\frac{\sin^2 x}{1 + \cos x}$ (15) $\sin 7x \cos 5x$

(16) $\cos 3x \cos x$ (17) $\cos 2x \sin 4x$ (18) $\sin 10x \sin 2x$

(19) $\frac{1 + \cos 2x}{\sin^2 2x}$ (20) $(e^x - 1)^2 e^{-4x}$ (21) $\frac{1 - \sin x}{1 + \sin x}$

(22) $\frac{2^{x+1} - 3^{x-1}}{6^x}$ (23) $e^{x\log a}e^x$ (24) $\frac{a^{x+1} - b^{x-1}}{c^x}$

(25) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ (26) $\sin mx \cos nx \ (m > n)$ (27) $\cos px \cos qx \ (p > q)$

(28) $\cos^2 5x \sin 10x$ (29) $\frac{1}{\sqrt{x + 1} - \sqrt{x - 2}}$ (30) $\frac{1}{\sqrt{ax + b} - \sqrt{ax + c}}$

(31) $(x + 1)\sqrt{x + 3})$ (32) $(x - 4)\sqrt{x + 7}$ (33) $(2x + 1)\sqrt{2x + 3}$

(34) $\frac{x + 1}{(x + 2)(x + 3)}$ (35) $\frac{x^2 + 1}{(x - 2)(x + 2)(x^2 + 9)}$

9.3.2 Method of substitution or change of variable

Sometimes the given functions may not be in an integrable form and the variable of integration (x in dx) can be suitably changed into a new variable by substitution so that the new function will be found integrable.

Suppose $\quad F(u) = \int f(u)\, du,$

then $\quad \frac{dF(u)}{du} = f(u)$

Put $u = \phi(x)$, then $\frac{du}{dx} = \phi'(x)$

Also we know that

$$\frac{dF(u)}{dx} = \frac{dF(u)}{du} \cdot \frac{du}{dx}$$

$$= f(u)\,\phi'(x)$$

i.e. $$\frac{dF(u)}{dx} = f[\phi(x)]\,\phi'(x)$$

$$\Rightarrow \quad F(u) = \int f[\phi(x)]\,\phi'(x)\,dx$$

$$\therefore \quad \int f(u)du = \int f[\phi(x)]\,\phi'(x)\,dx$$

$$\boxed{\int f[\phi(x)]\phi'(x)\,dx = \int f(u)\,du}$$

The success of the above method depends on the selection of suitable substitution either $x = \phi(u)$ or $u = g(x)$.

Example 9.38 – 9.41: Integrate

(38) $\int 5x^4 e^{x^5} dx$ (39) $\int \frac{\cos x}{1 + \sin x}\,dx$ (40) $\int \frac{1}{\sqrt{1 - x^2}}\,dx$ (41) $\int \frac{1}{1 + x^2}\,dx$

For the first two problems (38) and (39) the substitution in the form $u = \phi(x)$ and for (40) and (41) the substitution in the form $x = \phi(u)$.

(38) Let $I = \int 5x^4 e^{x^5} dx$

Put $x^5 = u$ … (i)

$\therefore$ $5x^4\,dx = du$ … (ii)

Since the variable of integration is changed from x to u, we have to convert entire integral in terms of the new variable u.

$\therefore$ We get $I = \int \left(e^{x^5}\right)\left(5x^4\,dx\right)$

$= \int e^u\,du$ (by (i) and (ii))

$= e^u + c$

$= e^{x^5} + c$ (replacing u by x^5, as the function of given variable)

(39) Let $I = \int \frac{\cos x}{1+\sin x}\, dx$

Put $(1+\sin x) = u$ … (i)

$\cos x\, dx = du$ … (ii)

$\therefore\ I = \int \frac{1}{(1+\sin x)} (\cos x\, dx)$

$= \int \frac{1}{u}\, du$ (by (i) and (ii))

$= \log u + c$

$\int \frac{\cos x}{1+\sin x}\, dx = \log(1+\sin x) + c$

(40) Let $I = \int \frac{1}{\sqrt{1-x^2}}\, dx$

Put $x = \sin u$ ….(i) $\Rightarrow$ $u = \sin^{-1} x$

$dx = \cos u\, du$ … (ii)

$\therefore$ $I = \int \frac{1}{\sqrt{1-x^2}}\, dx$

$= \int \frac{1}{\sqrt{1-\sin^2 u}} (\cos u\, du)$ by (i) and (ii)

$= \int \frac{1}{\sqrt{\cos^2 u}} (\cos u\, du)$

$= \int du = u + c$

$\therefore$ $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c$ ($\because$ $u = \sin^{-1} x$)

(41) Let $I = \int \frac{1}{1+x^2}\, dx$

put $x = \tan u$ $\Rightarrow$ $u = \tan^{-1} x$

$dx = \sec^2 u\, du$

$$\therefore \quad I = \int \frac{1}{1+\tan^2 u} \sec^2 u \, du$$

$$= \int \frac{1}{\sec^2 u} \sec^2 u \, du = \int du$$

$$I = u + c$$

$$\therefore \quad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + c$$

Some standard results of integrals

(i) $\displaystyle\int \frac{f'(x)}{f(x)} \, dx = \log [f(x)] + c$

(ii) $\displaystyle\int \frac{f'(x)}{\sqrt{f(x)}} \, dx = 2\sqrt{f(x)} + c$

(iii) $\displaystyle\int f'(x) \, [f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ **where** $n \neq -1$

Proof :

(i) Let $\displaystyle I = \int \frac{f'(x)}{f(x)} \, dx$

Put $f(x) = u$

$\therefore \quad f'(x)dx = du$

$\displaystyle\therefore \quad I = \int \frac{1}{u} \, du = \log u + c = \log [f(x)] + c$

i.e. $\displaystyle\int \frac{f'(x)}{f(x)} \, dx = \log [f(x)] + c$

(ii) Let $\displaystyle I = \int \frac{f'(x)}{\sqrt{f(x)}} \, dx$

$\displaystyle= \int \frac{1}{\sqrt{u}} \, du$ where $u = f(x)$ and $du = f'(x) \, dx$

$= 2\sqrt{u} + c = 2\sqrt{f(x)} + c$

$$\therefore \quad \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$$

(iii) Let $I = \int f'(x)\, [f(x)]^n\, dx$ where $n \neq -1$

Put $f(x) = u$

$\therefore \quad f'(x)\, dx = du$

$$\therefore \quad I = \int \{f(x)\}^n \left(f'(x)\, dx \right)$$

$$= \int u^n\, du = \frac{u^{n+1}}{n+1} + c \qquad (\because n \neq -1)$$

$$\therefore \quad \int f'(x)\, [f(x)]^n\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Examples 9.42 – 9.47: Integrate the following

(42) $\dfrac{2x + 1}{x^2 + x + 5}$ (43) $\dfrac{e^x}{5 + e^x}$ (44) $\dfrac{6x + 5}{\sqrt{3x^2 + 5x + 6}}$ (45) $\dfrac{\cos x}{\sqrt{\sin x}}$

(46) $(4x - 1)\, (2x^2 - x + 5)^4$ (47) $(3x^2 + 6x + 7)\, (x^3 + 3x^2 + 7x - 4)^{11}$

Solution

(42) Let $I = \int \frac{2x + 1}{x^2 + x + 5}\, dx = \int \frac{1}{(x^2 + x + 5)} \{(2x + 1)\, dx\}$

Put $x^2 + x + 5 = u$

$(2x + 1)\, dx = du$

$$\therefore \quad I = \int \frac{1}{u}\, du = \log u + c = \log (x^2 + x + 5) + c$$

$$\therefore \int \frac{2x + 1}{x^2 + x + 5}\, dx = \log (x^2 + x + 5) + c$$

(43) Let $I = \int \frac{e^x}{5 + e^x}\, dx$

put $5 + e^x = u$

$e^x\, dx = du$

$$\therefore \quad I = \int \frac{1}{5+e^x}(e^x\,dx)$$

$$\therefore \quad = \int \frac{1}{u}\,du$$

$$I = \log u + c = \log(5+e^x) + c$$

$$\text{i.e.} \int \frac{e^x}{5+e^x}\,dx = \log(5+e^x) + c$$

(44) Let $I = \int \frac{6x+5}{\sqrt{3x^2+5x+6}}\,dx$

put $3x^2 + 5x + 6 = t$

$(6x+5)\,dx = dt$

$$\therefore \quad I = \int \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t} + c = 2\sqrt{3x^2+5x+6} + c$$

$$\therefore \quad \int \frac{6x+5}{\sqrt{3x^2+5x+6}}\,dx = 2\sqrt{3x^2+5x+6} + c$$

(45) Let $I = \int \frac{\cos x}{\sqrt{\sin x}}\,dx$

put $\sin x = t$

$\cos x\,dx = dt$

$$\therefore \quad I = \int \frac{1}{\sqrt{t}}\,dt$$

i.e. $I = 2\sqrt{t} + c = 2\sqrt{\sin x} + c$

i.e. $\int \frac{\cos x}{\sqrt{\sin x}}\,dx = 2\sqrt{\sin x} + c$

(46) Let $I = \int (4x-1)(2x^2-x+5)^4\,dx$

put $2x^2 - x + 5 = u$

$(4x-1)\,dx = du$

$$\therefore \quad I = \int (2x^2-x+5)^4\big((4x-1)\,dx\big)$$

$$= \int u^4 du = \frac{u^5}{5} + c = \frac{(2x^2-x+5)^5}{5} + c$$

i.e. $\int (4x - 1)\,(2x^2 - x + 5)^4\,dx = \dfrac{(2x^2 - x + 5)^5}{5} + c$

(47) Let $\qquad I = \int (3x^2 + 6x + 7)\,(x^3 + 3x^2 + 7x - 4)^{11}\,dx$

put $x^3 + 3x^2 + 7x - 4 = u$

$\therefore \quad (3x^2 + 6x + 7)\,dx = du$

$\therefore \qquad I = \int (x^3 + 3x^2 + 7x - 4)^{11}\ \{(3x^2 + 6x + 7)dx\}$

$= \int u^{11} du$

$I = \dfrac{u^{12}}{12} + c = \dfrac{(x^3 + 3x^2 + 7x - 4)^{12}}{12} + c$

$\therefore \int (x^3 + 3x^2 + 7x - 4)^{11}\ (3x^2 + 6x + 7)\,dx = \dfrac{(x^3 + 3x^2 + 7x - 4)^{12}}{12} + c$

Example 9.48 – 9.67: Integrate the following

(48) $x^{16}\,(1 + x^{17})^4$ (49) $\dfrac{x^{24}}{(1 + x^{25})^{10}}$ (50) $\dfrac{x^{15}}{1 + x^{32}}$ (51) $x(a - x)^{17}$

(52) $\cot x$ (53) $\operatorname{cosec} x$ (54) $\dfrac{\log \tan x}{\sin 2x}$ (55) $\sin^{15} x \cos x$

(56) $\sin^7 x$ (57) $\tan x \sqrt{\sec x}$ (58) $\dfrac{e^{\tan x}}{\cos^2 x}$ (59) $\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$

(60) $\dfrac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}$ (61) $e^{2\log x}\, e^{x^3}$ (62) $\dfrac{\log x}{x}$ (63) $\dfrac{1}{x \log x}$

(64) $\dfrac{1}{x + \sqrt{x}}$ (65) $\dfrac{e^{x/2} - e^{-x/2}}{e^x - e^{-x}}$ (66) $\dfrac{x^{e-1} + e^{x-1}}{x^e + e^x}$

(67) $\alpha\beta\, x^{\alpha - 1}\, e^{-\beta x^{\alpha}}$ (68) $(2x - 3)\sqrt{4x + 1}$)

Solution:

(48) $\int x^{16}\,(1 + x^{17})^4\,dx$

Let $\qquad I = \int x^{16}\,(1 + x^{17})^4\,(dx)$

put $\qquad 1 + x^{17} = u \qquad \ldots$ (i)

$$17x^{16}dx = du$$

$$dx = \frac{1}{17x^{16}} du \qquad \ldots \text{(ii)}$$

$$\therefore\ \text{I} = \int x^{16}(u)^4 \left(\frac{1}{17x^{16}} dx\right) \qquad \text{by (i) and (ii)}$$

$$= \frac{1}{17} \int u^4 du \quad = \quad \frac{1}{17} \frac{u^5}{5} + c$$

$$\int x^{16} (1 + x^{17})^4 dx = \frac{1}{85} (1 + x^{17})^5 + c$$

(49) $\displaystyle\int \frac{x^{24}}{(1 + x^{25})^{10}} dx$

Let $\displaystyle \text{I} = \int \frac{x^{24}}{(1 + x^{25})^{10}} dx$

put $1 + x^{25} = u \qquad \ldots$ (i)

$$25x^4 dx = du$$

$$dx = \frac{1}{25x^{24}} du \qquad \ldots \text{(ii)}$$

$$\therefore \qquad \text{I} = \int \frac{x^{24}}{u^{10}} \left(\frac{1}{25x^{24}} du\right) \qquad \text{by (i) and (ii)}$$

$$= \frac{1}{25} \int \frac{1}{u^{10}} du \quad = \frac{1}{25} \left[-\frac{1}{9u^9}\right] + c$$

$$\therefore \quad \int \frac{x^{24}}{(1 + x^{25})^{10}} dx = -\frac{1}{225 (1 + x^{25})^9} + c$$

(50) $\displaystyle\int \frac{x^{15}}{1 + x^{32}} dx$

Let $\displaystyle \text{I} = \int \frac{x^{15}}{1 + x^{32}} dx$

$$\text{put} \quad x^{16} = u \qquad \ldots \text{(i)}$$

$$16x^{15}dx = du$$

$$dx = \frac{1}{16x^{15}}\, du \qquad \ldots \text{(ii)}$$

$$\therefore \quad = \int \frac{x^{15}}{1+u^2}\left(\frac{1}{16x^{15}}\, du\right) \qquad \text{by (i) and (ii)}$$

$$= \frac{1}{16} \int \frac{1}{1+u^2}\, du$$

$$I = \frac{1}{16} \tan^{-1} u + c$$

$$\int \frac{x^{15}}{1+x^{32}}\, dx = \frac{1}{16} \tan^{-1}(x^{16}) + c$$

(51) $\int x(a-x)^{17}\, dx$

$$\text{Let} \quad I = \int x(a-x)^{17}\, dx$$

$$\text{put} \quad (a-x) = u \quad \Rightarrow \quad x = a-u$$

$$dx = -\,du$$

$$\therefore \quad I = \int (a-u)u^{17}(-\,du)$$

$$= \int (u^{18} - au^{17})\, du$$

$$I = \frac{u^{19}}{19} - a\frac{u^{18}}{18} + c$$

$$\therefore \quad \int x(a-x)^{17}\, dx = \frac{(a-x)^{19}}{19} - \frac{a(a-x)^{18}}{18} + c$$

(52) $\int \cot x\, dx$

$$\text{Let} \quad I = \int \cot x\, dx$$

$$\text{put} \quad \sin x = u$$

$$\cos x\, dx = du$$

$$\therefore \quad I = \int \frac{\cos x}{\sin x}\,dx = \int \frac{1}{u}\,du = \log u + c$$

$$\therefore \quad \mathbf{\int \cot x\, dx = \log \sin x + c}$$

(53) $\int \operatorname{cosec} x\, dx$

$$\text{Let } I = \int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec}x\,[\operatorname{cosec}x - \cot x]}{[\operatorname{cosec}x - \cot x]}\,dx$$

$$\text{Put } \operatorname{cosec}x - \cot x = u \qquad \ldots (1)$$

$$(-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)dx = du$$

$$\operatorname{cosec}x\,(\operatorname{cosec}x - \cot x)\,dx = du \qquad \ldots (2)$$

$$\therefore \quad I = \int \frac{\operatorname{cosec}x\,[\operatorname{cosec}x - \cot x]}{[\operatorname{cosec}x - \cot x]}\,dx$$

$$= \int \frac{du}{u} = \log u + c$$

$$\therefore \quad \int \operatorname{cosec} x\, dx = \log(\operatorname{cosec}x - \cot x) + c$$

$$\mathbf{\int \operatorname{cosec} x\, dx = \log \tan \frac{x}{2} + c}$$

(54) $\int \frac{\log \tan x}{\sin 2x}\,dx$

$$\text{Let} \quad I = \int \frac{\log \tan x}{\sin 2x}\,dx$$

$$\text{Put} \quad \log \tan x = u \qquad \ldots (i)$$

$$\therefore \quad \frac{1}{\tan x}\sec^2 x dx = du \quad \Rightarrow \quad \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}\,dx = du$$

$$\text{i.e.} \quad \frac{2}{2\sin x \cos x}\,dx = du \quad \Rightarrow \quad \frac{2}{\sin 2x}\,dx = du$$

$$dx = \frac{\sin 2x}{2}\,du \qquad \ldots (ii)$$

$$\therefore \quad I = \int \frac{u}{\sin 2x} \cdot \left(\frac{\sin 2x}{2}\,du\right) \qquad \text{by (i) and (ii)}$$

$$= \frac{1}{2}\int u\,du = \frac{1}{2}\left[\frac{u^2}{2}\right] + c$$

$$\int \frac{\log \tan x}{\sin 2x}\,dx = \frac{1}{4}\,[\log \tan x]^2 + c$$

(55) $\int \sin^{15}x \cos x\, dx$

Let $I = \int \sin^{15}x \cos x\, dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore$ $I = \int t^{15}\, dt = \frac{t^{16}}{16} + c$

$\therefore$ $\int \sin^{15}x \cos x\, dx = \frac{\sin^{16}x}{16} + c$

(56) $\int \sin^{7}x\, dx$

Let $I = \int \sin^{7}x\, dx$

$\therefore$ $= \int \sin^6 x \sin x\, dx = \int (1 - \cos^2 x)^3 (\sin x\, dx)$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\sin x\, dx = (-dt)$

$\therefore$ $I = \int (1 - t^2)^3 (-dt)$

$= \int (1 - 3t^2 + 3t^4 - t^6)(-dt)$

$= \int (t^6 - 3t^4 + 3t^2 - 1)\, dt$

$= \frac{t^7}{7} - 3\frac{t^5}{5} + 3\frac{t^3}{3} - t + c$

$\therefore$ $\int \sin^7 x\, dx = \frac{\cos^7 x}{7} - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + c$

(Note : This method is applicable only when the power is odd).

(57) $\int \tan x \sqrt{\sec x}\, dx$

Let $I = \int \tan x \sqrt{\sec x}\, dx$

Put $\sec x = t$

$$\sec x \tan x \, dx = dt \qquad \therefore dx = \frac{dt}{\sec x \tan x}$$

Converting everything in terms of t.

$$\therefore I = \int \tan x \,(\sqrt{t}) \left(\frac{1}{\sec x \tan x} dt\right)$$

$$= \int \frac{\sqrt{t}}{\sec x} dt = \int \frac{\sqrt{t}}{t} dt = \int \frac{1}{\sqrt{t}} dt = 2\sqrt{t} + c$$

$$\therefore \int \tan x \sqrt{\sec x} \, dx = 2\sqrt{\sec x} + c$$

(When the integrand is with $e^{f(x)}$ and $f(x)$ is not a linear function in x, substitute $f(x) = u$.)

(58) $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x} dx$

$$\text{Let } I = \int \frac{e^{\tan x}}{\cos^2 x} dx$$

$$\text{Put } \tan x = t$$

$$\sec^2 x \, dx = dt \qquad \therefore dx = \cos^2 x \, dt$$

$$\therefore I = \int \frac{e^t}{\cos^2 x} . \cos^2 x \, dt = \int e^t \, dt = e^t + c$$

$$\therefore \int \frac{e^{\tan x}}{\cos^2 x} dx = e^{\tan x} + c$$

(59) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$\text{Let } I = \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$\text{Put } \sqrt{x} = t \qquad \therefore x = t^2 \Rightarrow dx = 2t\,dt$$

$$\therefore I = \int \frac{e^t}{t} . 2t \, dt = 2\int e^t \, dt = 2 e^t + c$$

$$\therefore \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 e^{\sqrt{x}} + c$$

(60) $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$

$$\text{Let I} = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$$

$$\text{put } \sin^{-1}x = t$$

$$\frac{1}{\sqrt{1-x^2}}\,dx = dt \quad \Rightarrow \quad dx = \sqrt{1-x^2}\;dt$$

$$\therefore\ \text{I} = \int \frac{e^t}{\sqrt{1-x^2}}\sqrt{1-x^2}\;dt$$

$$= \int e^t\,dt = e^t + c$$

$$\therefore\ \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx = e^{\sin^{-1}x} + c$$

(61) $\int e^{2\log x}\, e^{x^3}\,dx$

$$\text{Let I} = \int e^{2\log x}\, e^{x^3}\,dx$$

$$\text{put } x^3 = t \quad \Rightarrow \quad 3x^2\,dx = dt \qquad \therefore\ dx = \frac{1}{3x^2}\,dt$$

$$\therefore\ \text{I} = \int e^{\log x^2}\, e^{x^3}\,dx = \int x^2\, e^{x^3}\,dx$$

$$= \int x^2\, e^t\left(\frac{1}{3x^2}\,dt\right)$$

$$= \frac{1}{3}\int e^t\,dt = \frac{1}{3}\,e^t + c$$

$$\therefore\ \int e^{2\log x}\, e^{x^3}\,dx = \frac{1}{3}\,e^{x^3} + c$$

(62) $\int \frac{\log x}{x} dx$

$$\text{Let } I = \int \frac{\log x}{x} dx$$

$$\text{put } \log x = u \Rightarrow \frac{1}{x} dx = du \qquad \therefore \quad dx = x\, du$$

$$\therefore I = \int \frac{u}{x}(x\, du) = \int u\, du = \frac{u^2}{2} + c$$

$$\int \frac{\log x}{x} dx = \frac{1}{2}[\log x]^2 + c$$

(63) $\int \frac{1}{x \log x} dx$

$$\text{Let } I = \int \frac{1}{x \log x} dx$$

$$\text{put } \log x = u$$

$$\frac{1}{x} dx = du \qquad \therefore \quad dx = x\, du$$

$$\therefore I = \int \frac{1}{xu}(x\, du) = \int \frac{1}{u} du = \log u + c$$

$$\int \frac{1}{x \log x} dx = \log(\log x) + c$$

(64) $\int \frac{1}{x + \sqrt{x}} dx$

$$\text{Let } I = \int \frac{1}{x + \sqrt{x}} dx$$

$$\text{put } \sqrt{x} = t \Rightarrow x = t^2$$

$$dx = 2t\, dt$$

$$\therefore I = \int \frac{1}{t^2 + t} 2t\, dt = 2\int \frac{t}{t(t+1)} dt$$

$$= 2\int \left(\frac{1}{1+t}\right) dt = 2\log(1+t) + c$$

$$\therefore \int \frac{1}{x + \sqrt{x}} dx = 2\log\left(1 + \sqrt{x}\right) + c$$

(65) $\int \frac{e^{x/2} - e^{-x/2}}{e^x - e^{-x}}\, dx$

$$\text{Let I} = \int \frac{e^{x/2} - e^{-x/2}}{e^x - e^{-x}}\, dx$$

$$\text{put } e^{x\backslash 2} = t \quad \Rightarrow \quad \frac{1}{2}\, e^{x\backslash 2}\, dx = dt$$

$$dx = \frac{2}{e^{x/2}}\, dt = \frac{2}{t}\, dt$$

$$\therefore \quad I = \int \frac{t - 1/t}{t^2 - 1/t^2} \left(\frac{2dt}{t}\right)$$

$$= 2 \int \frac{\frac{(t^2 - 1)}{t}}{\frac{(t^4 - 1)}{t^2}} \frac{dt}{t} = 2 \int \frac{(t^2 - 1)}{t^4 - 1}\, dt$$

$$= 2 \int \frac{t^2 - 1}{(t^2 - 1)\,(t^2 + 1)}\, dt = 2 \int \frac{1}{1 + t^2}\, dt = 2 \tan^{-1} t + c$$

$$\therefore \int \frac{e^{x/2} - e^{-x/2}}{e^x - e^{-x}}\, dx = 2 \tan^{-1}\left(e^{x/2}\right) + c$$

(66) $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x}\, dx$

$$\text{Let I} = \int \frac{x^{e-1} + e^{x-1}}{x^e + e^x}\, dx$$

$$\text{Put } x^e + e^x = t \quad \dots \text{(i)}$$

$$(ex^{e-1} + e^x)\, dx = dt, \qquad e(x^{e-1} + e^{x-1})\, dx = dt$$

$$\therefore \quad dx = \frac{1}{e(x^{e-1} + e^{x-1})}\, dt \quad \dots \text{(ii)}$$

$$\therefore\ I = \int \frac{(x^{e-1}+e^{x-1})}{t}\left(\frac{1}{e\left(x^{e-1}+e^{x-1}\right)}\right) dt \quad \text{by (i) and (ii)}$$

$$= \frac{1}{e}\int \frac{1}{t}\, dt = \frac{1}{e}\log t + c$$

$$\therefore\ \int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}\, dx = \frac{1}{e}\log(x^e+e^x) + c$$

(67) $\int \alpha\beta\, x^{\alpha-1}\, e^{-\beta x^\alpha}\, dx$

$$\text{Let } I = \int \alpha\beta\, x^{\alpha-1}\, e^{-\beta x^\alpha}\, dx$$

$$\text{Put } -\beta x^\alpha = u \Rightarrow -\alpha\beta x^{\alpha-1} dx = du \quad \therefore\ dx = -\frac{1}{\alpha\beta\, x^{\alpha-1}}\, du$$

$$\therefore\ I = \int \alpha\beta x^{\alpha-1}\, e^u \left(\frac{-1}{\alpha\beta\, x^{\alpha-1}}\right) du = -\int e^u\, du = -e^u + c$$

$$\therefore\ \int \alpha\beta\, x^{\alpha-1}\, e^{-\beta x^\alpha}\, dx = -e^{-\beta x^\alpha} + c$$

(68) $\int (2x-3)\sqrt{4x+1}\, dx$

$$\text{Let } I = \int (2x-3)\sqrt{4x+1}\, dx$$

$$\text{Put } (4x+1) = t^2 \Rightarrow x = \frac{1}{4}(t^2-1) \qquad \therefore\ dx = \frac{t}{2}\, dt$$

$$\therefore\ I = \int \left\{2 \cdot \frac{1}{4}(t^2-1)-3\right\}(t)\left(\frac{t}{2}\right) dt = \int \frac{1}{2}(t^2-1-6) \cdot \frac{t^2}{2}\, dt$$

$$= \frac{1}{4}\int (t^4-7t^2)\, dt = \frac{1}{4}\left(\frac{t^5}{5}-\frac{7}{3}t^3\right) + c$$

$$\int (2x-3)\sqrt{4x+1}\, dx = \frac{1}{20}(4x+1)^{5/2} - \frac{7}{12}(4x+1)^{3/2} + c$$

EXERCISE 9.5

Integrate the following

(1) $x^5(1 + x^6)^7$ (2) $\frac{(2lx + m)}{lx^2 + mx + n}$ (3) $\frac{4ax + 2b}{(ax^2 + bx + c)^{10}}$

(4) $\frac{x}{\sqrt{x^2 + 3}}$ (5) $(2x + 3)\sqrt{x^2 + 3x - 5}$ (6) $\tan x$

(7) $\sec x$ (8) $\cos^{14}x \sin x$ (9) $\sin^5 x$

(10) $\cos^7 x$ (11) $\frac{1 + \tan x}{x + \log \sec x}$ (12) $\frac{e^{m\tan^{-1}x}}{1 + x^2}$

(13) $\frac{x\sin^{-1}(x^2)}{\sqrt{1 - x^4}}$ (14) $\frac{5(x + 1)(x + \log x)^4}{x}$ (15) $\frac{\sin(\log x)}{x}$

(16) $\frac{\cot x}{\log \sin x}$ (17) $\sec^4 x \tan x$ (18) $\tan^3 x \sec x$

(19) $\frac{\sin x}{\sin(x + a)}$ (20) $\frac{\cos x}{\cos(x - a)}$ (21) $\frac{\sin 2x}{a\cos^2 x + b\sin^2 x}$

(22) $\frac{1 - \tan x}{1 + \tan x}$ (23) $\frac{\sqrt{\tan x}}{\sin x \cos x}$ (24) $\frac{(\log x)^2}{x}$

(25) $e^{3\log x} e^{x^4}$ (26) $\frac{x^{e-1} + e^{x-1}}{x^e + e^x + e^e}$ (27) $x(l - x)^{16}$

(28) $x(x - a)^m$ (29) $x^2(2 - x)^{15}$ (30) $\frac{\sin\sqrt{x}}{\sqrt{x}}$

(31) $(x + 1)\sqrt{2x + 3}$ (32) $(3x + 5)\sqrt{2x + 1}$ (33) $(x^2 + 1)\sqrt{x + 1}$

9.3.3 Integration by parts

Integration by parts method is generally used to find the integral when the integrand is a product of two different types of functions or a single logarithmic function or a single inverse trigonometric function or a function which is not integrable directly.

From the formula for derivative of product of two functions we obtain this useful method of integration.

If $f(x)$ and $g(x)$ are two differentiable functions then we have

$$\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$$

By definition of anitiderivative

$$f(x)\,g(x) = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

rearranging we get

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx \qquad \ldots (1)$$

For computational purpose a more convenient form of writing this formula is obtained by

letting $\qquad u = f(x) \quad \text{and } v = g(x)$

$$\therefore \quad du = f'(x)\,dx \quad \text{and} \quad dv = g'(x)\,dx$$

So that (1) becomes

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The above formula expresses the integral.

$\int u\,dv$ interms of another integral $\int v\,du$ and does not give a final expression for the integral $\int u\,dv$. It only partially solves the problem of integrating the product uv'. Hence the term **'Partial Integration'** has been used in many European countries. The term **"Integration by Parts"** is established in many other languages as well as in our own.

The success of this method depends on the proper choice of u

(i) If integrand contains any non integrable functions directly from the formula, like $\log x$, $\tan^{-1}x$ etc., we have to take these unintegrable functions as u and other as dv.

(ii) If the integrand contains both the integrable function, and one of these is x^n (where n is a positive integer). Then take $u = x^n$.

(iii) For other cases choice of u is ours.

Examples **Suitable substitution for *u***

No.	Given Integrals	u	dv	Reason for u
1.	$\int \log x\, dx$ $\int \tan^{-1} x\, dx$	$\log x$ $\tan^{-1}x$	dx dx	$\log x$ and $\tan^{-1}x$ are not integrable directly from the formula.
2.	$\int x^n \log x\, dx$	$\log x$	$x^n\, dx$	
3.	$\int x^n \tan^{-1} x\, dx$	$\tan^{-1}x$	$x^n\, dx$	
4.	$\int x^n e^{ax}\, dx$ (n is a positive integer)	x^n	$e^{ax}\, dx$	both are integrable and power of x will be reduced by successive differentiation
5.	$\int x^n(\sin x \text{ or } \cos x)dx$	x^n	$\sin x\, dx$ or $\cos x dx$	both are integrable and power of x will be reduced by successive differentiation
6.	$\int e^{ax} \cos bx\, dx$ or $\int e^{ax} \sin bx\, dx$	e^{ax} or $\cos bx$ / $\sin bx$	Remains	–

Example 9.69 – 9.84: Integrate

(69) xe^x	(70) $x \sin x$	(71) $x \log x$	(72) $x \sec^2 x$
(73) $x \tan^{-1}x$	(74) $\log x$	(75) $\sin^{-1}x$	(76) $x \sin^2 x$
(77) $x \sin 3x \cos 2x$	(78) $x\, 5^x$	(79) $x^3 e^{x^2}$	(80) $e^{\sqrt{x}}$
(81) $\int \frac{x \sin^{-1}x}{\sqrt{1-x^2}}\, dx$	(82) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$	(83) $x^2 e^{3x}$	(84) $x^2 \cos 2x$

Solution:

(69) $\int xe^x\, dx = \int (x)\, (e^x dx)$

We apply integration by parts by taking

$$u = x \quad \text{and} \quad dv = e^x\, dx$$

Then $du = dx$ and $v = \int e^x\,dx = e^x$

$$\therefore \quad \int xe^x\,dx = xe^x - \int e^x dx = xe^x - e^x + c$$

(70) $\int x \sin x\,dx = \int (x)\,(\sin x\,dx)$

We use integration by parts with

$$u = x \quad \text{and} \quad dv = \sin dx$$

$$du = dx \text{ and } v = -\cos x$$

$$\therefore \int x \sin x\,dx = (x)\,(-\cos x) - \int (-\cos x)\,(dx)$$

$$= -x\cos x + \int \cos x\,dx$$

$$\therefore \quad \int x \sin x\,dx = -x\cos x + \sin x + c$$

(71) $\int x \log x = \int (\log x)\,(x\,dx)$

Since log x is not integrable take

$$u = \log x \quad \text{and} \quad dv = x\,dx$$

$$\therefore\ du = \frac{1}{x}\,dx \qquad v = \frac{x^2}{2}$$

$$\therefore \quad \int x \log x = (\log x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\,dx\right)$$

$$= \frac{x^2}{2}\ \log x - \frac{1}{2}\int x\,dx$$

$$\therefore \quad \int x \log x = \frac{x^2}{2}\ \log x - \frac{1}{4}x^2 + c$$

(72) $\int x \sec^2 x\,dx = \int (x)\,(\sec^2 x\,dx)$

Applying integration by parts, we get $\qquad dv = \sec^2 x\,dx$

$$\int x \sec^2 x\,dx = x\tan x - \int \tan x\,dx \qquad v = \tan x$$

$$= x\tan x - \log \sec x + c \qquad u = x$$

$$\therefore \quad \int x \sec^2 x\,dx = x\tan x + \log \cos x + c \qquad du = dx$$

(73) $\int x \tan^{-1} x\,dx = \int (\tan^{-1} x)\,(x\,dx)$

Applying integration by parts, we get

$u = \tan^{-1}x \quad dv = x\,dx \quad du = \frac{1}{1+x^2}\,dx \quad v = \frac{x^2}{2}$

$$\int x \tan^{-1}x\,dx = (\tan^{-1}x)\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left[\frac{(x^2+1)-1}{1+x^2}\right]dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - (\tan^{-1}x)\right] + c$$

$$\therefore \int x \tan^{-1}x\,dx = \frac{1}{2}\,[x^2\tan^{-1}x + \tan^{-1}x - x] + c$$

(74) $\int \log x\,dx = \int (\log x)\,(dx)$

Applying integration by parts, we get

$u = \log x \quad dv = dx \quad du = \frac{1}{x}\,dx \quad v = x$

$$= (\log x)\,(x) - \int x \cdot \frac{1}{x}\,dx$$

$$= x\log x - \int dx$$

$$\therefore \quad \int \log x\,dx = x\log x - x + c$$

(75) $\int \sin^{-1}x\,dx = \int (\sin^{-1}x)\,(dx)$

Applying integration by parts, we get

$u = \sin^{-1}x \quad dv = dx \quad du = \frac{1}{\sqrt{1-x^2}}\,dx \quad v = x$

$$\int \sin^{-1}x\,dx = (\sin^{-1}x)\,(x) - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Applying substitution method by substituting

$$\sqrt{1-x^2} = t$$
$$1-x^2 = t^2$$
$$-2x\,dx = 2t\,d\,t$$
$$dx = \frac{2tdt}{-2x} = \frac{-t}{x}\,dt$$
$$\therefore \int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{t}\left(\frac{-t}{x}\,dt\right)$$
$$= x\sin^{-1}x + \int dt = x\sin^{-1}x + t + c$$
$$\therefore \int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + c$$

(76) $\int x\sin^2 x\,dx$

Let $I = \int x\sin^2 x\,dx$

[To eliminate power of $\sin x$, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$]

$$= \int x\left\{\frac{1}{2}(1-\cos 2x)\right\}dx$$
$$= \frac{1}{2}\int (x - x\cos 2x)\,dx$$
$$= \frac{1}{2}\left[\int x dx - \int x\cos 2x\,dx\right]$$
$$I = \frac{1}{2}\left[\frac{x^2}{2} - I_1\right] \quad \ldots (1)$$

where $I_1 = \int x\cos 2x\,dx$

Applying integration by parts for I_1

$u = x$, $dv = \cos 2x\,dx$, $v = \frac{\sin 2x}{2}$, $du = dx$

$$I_1 = \int (x)\ (\cos 2x\,dx)$$
$$= \left[\frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}\,dx\right]$$
$$= \frac{x}{2}\sin 2x - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$$
$$I_1 = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x$$

substituting I_1 in (1) we get

$$I = \frac{1}{2}\left[\frac{x^2}{2} - I_1\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \left(\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right)\right] + c$$

$$\therefore \int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{\cos 2x}{8} + c$$

(77) $\int x \sin 3x \cos 2x \, dx = \int x \frac{1}{2}[\sin(3x + 2x) + \sin(3x - 2x)]\, dx$

$$\left(\because \sin A \cos B = \frac{1}{2}\{\sin(A + B) + \sin(A - B)\}\right)$$

$$= \int x \frac{1}{2}[\sin(3x + 2x) + \sin(3x - 2x)]\, dx$$

Applying integration by parts, we get $\quad u = x \quad dv = (\sin 5x + \sin x)dx$

$$= \frac{1}{2}\int x(\sin 5x + \sin x)\, dx \qquad du = dx \quad v = \left(-\frac{\cos 5x}{5} - \cos x\right)$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 5x}{5} - \cos x\right) - \int\left(-\frac{\cos 5x}{5} - \cos x\right)dx\right]$$

$$= \frac{1}{2}\left[-x\left(\frac{\cos 5x}{5} + \cos x\right) + \int\left(\frac{\cos 5x}{5} + \cos x\right)dx\right]$$

$$= \frac{1}{2}\left[-x\left(\frac{\cos 5x}{5} + \cos x\right) + \left(\frac{\sin 5x}{5 \times 5} + \sin x\right)\right] + c$$

$$\therefore \int x \sin 3x \cos 2x \, dx = \frac{1}{2}\left[-x\left(\frac{\cos 5x}{5} + \cos x\right) + \frac{\sin 5x}{25} + \sin x\right] + c$$

(78) $\int x\, 5^x\, dx = \int (x)(5^x\, dx)$

Applying integration by parts, we get $\qquad dv = 5^x\, dx$

$$\int x\, 5^x\, dx = x\frac{5^x}{\log 5} - \int \frac{5^x}{\log 5}dx \qquad u = x \qquad v = \frac{5^x}{\log 5}$$

$$du = dx$$

$$= \frac{x5^x}{\log 5} - \frac{1}{\log 5}\cdot\frac{5^x}{\log 5} + c$$

$$\therefore \int x\, 5^x\, dx = \frac{x5^x}{\log 5} - \frac{5^x}{(\log 5)^2} + c$$

For the following problems (79) to (82), first we have to apply substitution method to convert the given problem into a convenient form to apply integration by parts.

(79) $\int x^3 e^{x^2}\, dx$

Let $\quad I = \int x^3 e^{x^2}\, dx$

put $x^2 = t$

$\therefore\ 2x\, dx = dt$

$\therefore\ dx = \dfrac{dt}{2x}$

$\therefore\ I = \int x^3 \,.\, e^t \,.\, \dfrac{dt}{2x}$

$= \dfrac{1}{2} \int x^2 e^t\, dt \quad = \quad \dfrac{1}{2} \int (t)\,(e^t\, dt)$

$dv = e^t\, dt$

$u = t \qquad v = e^t$

$du = dt$

Now let us use integration by parts method

$\therefore\ I = \dfrac{1}{2}\left(te^t - \int e^t\, dt\right)$

$= \dfrac{1}{2}\left(te^t - e^t + c\right) \ = \ \dfrac{1}{2}\left(x^2 e^{x^2} - e^{x^2} + c\right)$

$\therefore \int x^3 e^{x^2}\, dx \ = \dfrac{1}{2}\ (x^2 e^{x^2} - e^{x^2}) + c$

(80) $\int e^{\sqrt{x}}\, dx$

Let $\quad I = \int e^{\sqrt{x}}\, dx$

put $\quad \sqrt{x} = t$

$\therefore\ x = t^2 \quad \Rightarrow \quad dx = 2t\, dt$

$I = \int e^t\, 2t dt$

$= 2 \int (t)\,(e^t\, dt)$

$dv = e^t\, dt$

$u = t \qquad v = e^t$

$du = dt$

Now applying integration by parts, we get

$I = 2\left(te^t - \int e^t\, dt\right)$

$= 2\,(te^t - e^t) + c$

$(\because\ t = \sqrt{x}\,)$

$\therefore \quad \int e^{\sqrt{x}}\, dx = 2\left(\sqrt{x}\, e^{\sqrt{x}} - e^{\sqrt{x}}\right) + c$

(81) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\, dx$

Let $\quad I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\, dx$

put $\sin^{-1} x = t \quad \Rightarrow \quad x = \sin t$

$\frac{1}{\sqrt{1-x^2}}\, dx = dt$

$dx = \sqrt{1-x^2}\; dt$

$\therefore \quad I = \int x \frac{t}{\sqrt{1-x^2}} \cdot \left(\sqrt{1-x^2}\, dt\right)$

$= \int xt\, dt$

$= \int (\sin t)\, (t)\, dt$

$I = \int (t)\ (\sin t\, dt)$

$u = t \qquad dv = \sin t\, dt$

$du = dt \qquad v = -\cos t$

Applying integration by parts, we get

$= t\,(-\cos t) - \int (-\cos t)\, dt$

$= -t \cos t + \int \cos t\, dt$

$= -t \cos t + \sin t + c$

$I = -(\sin^{-1} x)\left(\sqrt{1-x^2}\right) + x + c$

$\therefore \quad \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\, dx = x - \sqrt{1-x^2}\ \sin^{-1} x + c$

$\because t = \sin^{-1} x \Rightarrow \sin t = x$

$\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$

(82) $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Let $\quad I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

put $x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\ d\theta$

$$\therefore \qquad I = \int \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \sec^2\theta \, d\theta$$

$$= \int \tan^{-1} (\tan 2\theta) \sec^2\theta \, d\theta$$

$$= \int 2\theta \sec^2\theta \, d\theta$$

$$= 2 \int (\theta) (\sec^2\theta \, d\,\theta)$$

$dv = \sec^2\theta \, d\theta$

$u = \theta \qquad v = \tan\theta$

$du = d\theta$

Applying integration by parts

$$\therefore \; I = 2\left[\theta \tan\theta - \int \tan\theta \, d\,\theta\right]$$

$$= 2\theta \tan\theta - 2 \log \sec\theta + c$$

$$I = 2 (\tan^{-1}x) (x) - 2 \log \sqrt{1 + \tan^2\theta} + c$$

$$\therefore \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx = 2x \tan^{-1}x - 2 \log \sqrt{1+x^2} + c$$

For the following problems (83) and (84) we have to apply the integration by parts twice to find the solution.

(83) $\int x^2 e^{3x} \, dx = \int (x^2) (e^{3x} \, dx)$

Applying integration by parts, we get

$dv = e^{3x} \, dx$

$u = x^2 \qquad v = \frac{e^{3x}}{3}$

$du = 2x \, dx$

$$\int x^2 e^{3x} \, dx = \frac{x^2 e^{3x}}{3} - \int \frac{e^{3x}}{3} \, 2x \, dx$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3} \int (x) (e^{3x} \, dx)$$

again applying integration by parts, we get

$dv = e^{3x} \, dx$

$u = x \qquad v = \frac{e^{3x}}{3}$

$du = dx$

$$\int x^2 e^{3x} \, dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3} \left\{x.\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} \, dx\right\}$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{9} \int e^{3x} \, dx$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{27} e^{3x} + c$$

$$\therefore \int x^2 e^{3x} \, dx = \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + c$$

(84) $\int x^2\cos 2x\,dx = \int (x^2)\ (\cos 2x\,dx)$

Applying integration by part, we get

$u = x^2 \qquad dv = \cos 2x\,dx$

$du = 2x\,dx \qquad v = \dfrac{\sin 2x}{2}$

$$\int x^2\cos 2x\,dx = x^2\,\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}\,.\,2x\,dx$$

$$= x^2\,\frac{\sin 2x}{2} - \int (x)\,(\sin 2x\,dx)$$

again applying integration by parts we get

$u = x \qquad dv = \sin 2x\,dx$

$du = dx \qquad v = \dfrac{-\cos 2x}{2}$

$$\int x^2\cos 2x\,dx = x^2\,\frac{\sin 2x}{2} - \left\{\frac{x(-\cos 2x)}{2} - \int\left(\frac{-\cos 2x}{2}\,dx\right)\right\}$$

$$= \frac{x^2\,\sin 2x}{2} + \frac{x\cos 2x}{2} - \frac{1}{2}\int \cos 2x\,dx$$

$$\text{I} = \frac{x^2\,\sin 2x}{2} + \frac{x\cos 2x}{2} - \frac{1}{4}\,\sin 2x + c$$

$$\therefore \int x^2\cos 2x\,dx = \frac{1}{2}\,x^2\,\sin 2x + \frac{1}{2}\,x\cos 2x - \frac{1}{4}\,\sin 2x + c$$

The following examples illustrate that there are some integrals whose integration continues forever.

Example 9.85 – 9.87: Evaluate the following

(85) $\int e^x\cos x\,dx$ (86) $\int e^{ax}\sin bx\,dx$ (87) $\int \sec^3 x\,dx$

Solution:

(85) $\int e^x\cos x\,dx = \int (e^x)\ (\cos x\,dx)$

Here both the functions in the integrand are integrable directly from the formula. Hence the choice of u is ours.

Applying the integration by parts

$u = e^x \qquad dv = \cos x\,dx$

$du = e^x\,dx \qquad v = \sin x$

$$\int e^x\cos x\,dx = e^x\sin x - \int \sin x\,e^x\,dx$$

$$= e^x\sin x - \int (e^x)\,(\sin x\,dx) \quad \ldots (1)$$

Again applying integration by parts we get

$u = e^x \qquad dv = \sin x\,dx$

$du = e^x\,dx \qquad v = -\cos x$

$$\int e^x \cos x\, dx = e^x \sin x - \left[e^x (-\cos x) - \int (-\cos x)(e^x\, dx) \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

i.e. $$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx \quad \ldots (2)$$

Note that $\int e^x \cos x\, dx$ appears on both the sides.

$\therefore$ rearranging, we get

$$2 \int e^x \cos x\, dx = (e^x \sin x + e^x \cos x)$$

$$\therefore \int e^x \cos x\, dx = \frac{1}{2} [e^x \sin x + e^x \cos x] + c$$

$$\int e^x \cos x\, dx = \frac{e^x}{2} (\cos x + \sin x) + c$$

(86) $\int \boldsymbol{e^{ax}} \textbf{ sin } \boldsymbol{bx\, dx} = \int (\sin bx)(e^{ax} dx)$

since both functions are integrable,
we can take any one of them as u

$u = \sin bx$
$du = b \cos bx\, dx$
$dv = e^{ax}\, dx$
$v = \dfrac{e^{ax}}{a}$

$$\int e^{ax} \sin bx\, dx = (\sin bx) \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} (b \cos bx)\, dx$$

$$= \frac{1}{a} \quad e^{ax} \quad \sin \quad bx \quad - \quad \frac{b}{a}$$

$$\int \cos bx \,.\, e^{ax}\, dx$$

$dv = e^{ax}\, dx$
$u = \cos bx$
$du = -b \sin bx\, dx$
$v = \dfrac{e^{ax}}{a}$

Again applying integration by parts we get

$$\int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \left[(\cos bx)\left(\frac{e^{ax}}{a}\right) - \int \frac{e^{ax}}{a} (-b \sin bx\, dx) \right]$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx\, dx$$

$$\int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx\, dx$$

The integral on the right hand side is same as the integral on the left hand side.

$\therefore$ Rearranging we get

$$\int e^{ax} \sin bx\, dx + \frac{b^2}{a^2} \int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx$$

i.e. $$\left[1+\frac{b^2}{a^2}\right]\int e^{ax}\sin bx\,dx = \left[\frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx\right]$$

$$\left(\frac{a^2+b^2}{a^2}\right)\int e^{ax}\sin bx\,dx = e^{ax}\left(\frac{a\sin bx - b\cos bx}{a^2}\right)$$

$$\therefore \quad \int e^{ax}\sin bx\,dx = \left(\frac{a^2}{a^2+b^2}\right)\times\frac{e^{ax}}{a^2}(a\sin bx - b\cos bx)$$

$$\boxed{\therefore \int e^{ax}\sin bx\,dx = \left(\frac{e^{ax}}{a^2+b^2}\right)(a\sin bx - b\cos bx) + c}$$

Whenever we integrate function of the form $e^{ax}\cos bx$ or $e^{ax}\sin bx$, we have to apply the Integration by Parts rule twice to get the similar integral on both sides to solve.

Caution:

In applying integration by parts to specific integrals, once pair of choice for u and dv initially assumed should be maintained for the successive integrals on the right hand side. (See the above two examples). The pair of choice should not be interchanged.

Consider the example: $\int e^x \sin x\,dx$ **Initial assumption**

$$\int e^x\sin x\,dx = -e^x\cos x + \int \cos x\, e^x\,dx$$

$dv = \sin x\,dx$

$u = e^x \quad v = -\cos x$

$du = e^x\,dx$

Again applying integration by parts for R.H.S by interchanging the initial assumption we get

$$\int e^x\sin x\,dx = -e^x\cos x + \int \cos x\, e^x - \int e^x(-\sin x)\,dx$$

$dv = e^x\,dx$

$u = \cos x \quad v = e^x$

$du = -\sin x\,dx$

$$\int e^x\sin x\,dx = -e^x\cos x + \cos x\, e^x + \int e^x\sin x\,dx$$

$$\int e^x\sin x\,dx = \int e^x\sin x\,dx \ ?$$

Finally we have arrived at the same given problem on R.H.S!

87) $\int \sec^3 x\,dx = \int(\sec x)\ (\sec^2 x\,dx)$

Applying integration by parts, we get

$$\int \sec^3 x\,dx = \sec x\tan x - \int(\tan x)(\sec x\tan x\,dx)$$

$$= \sec x\tan x - \int \tan^2 x\sec x\,dx$$

$dv = \sec^2 x\,dx$

$u = \sec x \quad v = \tan x$

$du = \sec x\tan x\,dx$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \log(\sec x + \tan x)$$

Rearranging we get,

$$2 \int \sec^3 x \, dx = \sec x \tan x + \log(\sec x + \tan x)$$

$$\int \sec^3 x \, dx = \frac{1}{2} [\sec x \ \tan x + \log(\sec x + \tan x)] + c$$

EXERCISE 9.6

Integrate the followings with respect to *x*

(1) xe^{-x} (2) $x \cos x$ (3) $x \operatorname{cosec}^2 x$ (4) $x \sec x \tan x$

(5) $\tan^{-1} x$ (6) $x \tan^2 x$ (7) $x \cos^2 x$ (8) $x \cos 5x \cos 2x$

(9) $2x\, e^{3x}$ (10) $x^2 e^{2x}$ (11) $x^2 \cos 3x$ (12) $(\sin^{-1} x) \dfrac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}$

(13) $x^5 e^{x^2}$ (14) $\tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ (15) $x \sin^{-1}(x^2)$ (16) $\operatorname{cosec}^3 x$

(17) $e^{ax} \cos bx$ (18) $e^{2x} \sin 3x$ (19) $e^x \cos 2x$ (20) $e^{3x} \sin 2x$

(21) $\sec^3 2x$ (22) $e^{4x} \cos 5x \sin 2x$ (23) $e^{-3x} \cos^3 x$

Type I: 9.88 – 9.93: Standard integrals

(88) $\displaystyle\int \frac{dx}{a^2 - x^2}$ **(89)** $\displaystyle\int \frac{dx}{x^2 - a^2}$ **(90)** $\displaystyle\int \frac{dx}{a^2 + x^2}$

(91) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$ **(92)** $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$ **(93** $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}}$

Solution:

(88) $$\int \frac{dx}{a^2 - x^2} = \int \frac{1}{(a-x)\,(a+x)}\,dx$$

$$= \frac{1}{2a} \int \frac{2a}{(a-x)\,(a+x)}\,dx$$

$$= \frac{1}{2a} \int \frac{(a-x)+(a+x)}{(a-x)\,(a+x)}\,dx$$ or use Partial fraction method

$$= \frac{1}{2a} \int \left[\frac{1}{a+x} + \frac{1}{a-x}\right] dx$$

$$= \frac{1}{2a} \left[\log\,(a+x) - \log(a-x)\right]$$

$$\therefore \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-\mathrm{x}}\right) + c$$

(89) $$\int \frac{dx}{x^2 - a^2}\,dx = \int \frac{dx}{(x-a)\,(x+a)}$$

$$= \frac{1}{2a} \int \frac{2a}{(x-a)\,(x+a)}\,dx = \frac{1}{2a} \int \frac{(x+a)-(x-a)}{(x-a)\,(x+a)}\,dx$$

$$= \frac{1}{2a} \int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$$

$$= \frac{1}{2a} \left[\log\,(x-a) - \log(x+a)\right]$$

$$\therefore \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$$

(90) $$\text{Let I} = \int \frac{dx}{a^2 + x^2}$$

$$\text{put } x = a\tan\theta \;\Rightarrow\; \theta = \tan^{-1}(x/a)$$

$$dx = a\sec^2\theta\,d\theta$$

$$\therefore\ I = \int \frac{a \sec^2\theta d\theta}{a^2 + a^2 \tan^2\theta} = \int \frac{a \sec^2\theta d\theta}{a^2 \sec^2\theta} = \frac{1}{a} \int d\theta$$

$$I = \frac{1}{a}\ \theta + c$$

$$\therefore\ \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\ \tan^{-1} \frac{x}{a} + c$$

(91) Let $I = \int \frac{dx}{\sqrt{a^2 - x^2}}$

$$\text{put } x = a \sin\theta \Rightarrow \theta = \sin^{-1} (x / a)$$

$$dx = a \cos\theta\ d\theta$$

$$\therefore\ I = \int \frac{a \cos\theta\ d\theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} = \int \frac{a \cos\theta\ d\theta}{a \sqrt{1 - \sin^2\theta}}$$

$$= \int \frac{1}{\cos\theta} \cos\theta\ d\ \theta = \int d\theta$$

$$I = \theta + c$$

$$\therefore\ \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$$

(92) Let $I = \int \frac{1}{\sqrt{x^2 - a^2}}\ dx$

$$\text{put } u = x + \sqrt{x^2 - a^2}$$

$$du = \left(1 + \frac{(2x)}{2\sqrt{x^2 - a^2}}\right) dx = \left(\frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2}}\right) dx$$

$$\therefore\ dx = \frac{\sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}}\ du = \frac{\sqrt{x^2 - a^2}}{u}\ du$$

$$\therefore\ I = \int \frac{1}{\sqrt{x^2 - a^2}} \cdot \left(\frac{\sqrt{x^2 - a^2}}{u}\ du\right)$$

$$= \int \frac{1}{u}\ du$$

$$I = \log u + c$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left(x + \sqrt{x^2 - a^2}\right) + c$$

(Try the above problem by substituting $x = a \sec\theta$)

(93) Let $I = \int \frac{dx}{\sqrt{x^2 + a^2}}$

$$\text{put } u = x + \sqrt{x^2 + a^2}$$

$$du = \left(1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right) dx = \left(\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right) dx$$

$$\therefore\ dx = \frac{\sqrt{x^2 + a^2}}{x + \sqrt{x^2 + a^2}}\, du = \frac{\sqrt{x^2 + a^2}}{u}\, du$$

$$\therefore\ I = \int \frac{1}{\sqrt{x^2 + a^2}} \cdot \left(\frac{\sqrt{x^2 + a^2}}{u}\, du\right)$$

$$= \int \frac{1}{u}\, du$$

$$I = \log u + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right) + c$$

(Try the above problem by substituting $x = a \tan\theta$)

Remark: Remember the following useful substitution of the given integral as a functions of

Given	Substitution
$a^2 - x^2$	$x = a \sin\theta$
$a^2 + x^2$	$x = a \tan\theta$
$x^2 - a^2$	$x = a \sec\theta$

Example 9.94 – 9.105 :

Integrate :

(94) $\frac{1}{1 + 9x^2}$ (95) $\frac{1}{1 - 9x^2}$ (96) $\frac{1}{1 + \frac{x^2}{16}}$ (97) $\frac{1}{1 - 4x^2}$

(98) $\dfrac{1}{(x+2)^2-4}$ (99) $\dfrac{1}{(2x+1)^2-9}$ (100) $\dfrac{1}{\sqrt{25-x^2}}$ (101) $\dfrac{1}{\sqrt{1-\dfrac{x^2}{16}}}$

(102) $\dfrac{1}{\sqrt{1-16x^2}}$ (103) $\dfrac{1}{\sqrt{x^2-9}}$ (104) $\dfrac{1}{\sqrt{4x^2-25}}$ (105) $\dfrac{1}{\sqrt{9x^2+16}}$

Solution:

(94)
$$\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$$
$$= \left[\tan^{-1}\left(\frac{3x}{1}\right)\right] \times \frac{1}{3} + c$$
$$= \frac{1}{3}\tan^{-1} 3x + c$$

(95)
$$\int \frac{1}{1-9x^2}\,dx = \int \frac{1}{1-(3x)^2}\,dx$$
$$= \frac{1}{2.1}\log\left(\frac{1+3x}{1-3x}\right) \times \frac{1}{3}$$
$$= \frac{1}{6}\log\left(\frac{1+3x}{1-3x}\right) + c$$

(96)
$$\int \frac{1}{1+\dfrac{x^2}{16}}\,dx = \int \frac{1}{1+\left(\dfrac{x}{4}\right)^2}\,dx$$
$$= \left[\frac{1}{1}\tan^{-1}\left(\frac{x}{4}\right)\right]\frac{1}{(1/4)}$$
$$= 4\tan^{-1}\left(\frac{x}{4}\right) + c$$

(97)
$$\int \frac{1}{1-4x^2}\,dx = \int \frac{1}{1-(2x)^2}\,dx$$
$$= \left[\frac{1}{2.1}\log\left(\frac{1+2x}{1-2x}\right)\right] \times \frac{1}{2}$$

$$= \frac{1}{4} \log\left(\frac{1+2x}{1-2x}\right) + c$$

(98) $$\int \frac{dx}{(x+2)^2 - 4} = \int \frac{dx}{(x+2)^2 - 2^2}$$

$$= \frac{1}{2.(2)} \log\left(\frac{(x+2)-2}{(x+2)+2}\right)$$

$$= \frac{1}{4} \log\left(\frac{x}{x+4}\right) + c$$

(99) $$\int \frac{1}{(2x+1)^2 - 9}\, dx = \int \frac{1}{(2x+1)^2 - 3^2}\, dx$$

$$= \left[\frac{1}{2.(3)} \log\left(\frac{(2x+1)-3}{(2x+1)+3}\right)\right] \times \frac{1}{2}$$

$$= \frac{1}{12} \log\left(\frac{2x-2}{2x+4}\right)$$

$$= \frac{1}{12} \log\left(\frac{x+1}{x+2}\right) + c$$

(100) $$\int \frac{1}{\sqrt{25 - x^2}}\, dx = \int \frac{1}{\sqrt{5^2 - x^2}}\, dx$$

$$= \sin^{-1} \frac{x}{5} + c$$

(101) $$\int \frac{1}{\sqrt{1 - \frac{x^2}{16}}}\, dx = \int \frac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}}\, dx$$

$$= \left[\sin^{-1}\left(\frac{x}{4}\right)\right] \cdot \frac{1}{1/4}$$

$$= 4 \sin^{-1}\left(\frac{x}{4}\right) + c$$

(102) $$\int \frac{1}{\sqrt{1 - 16x^2}}\, dx = \int \frac{1}{\sqrt{1 - (4x)^2}}\, dx$$

$$= \left[\sin^{-1}(4x)\right] \frac{1}{4}$$

$$= \frac{1}{4} \sin^{-1}(4x) + c$$

(103) $$\int \frac{1}{\sqrt{x^2 - 9}}\, dx = \int \frac{1}{\sqrt{x^2 - 3^2}}\, dx$$

$$= \log\left(x + \sqrt{x^2 - 9}\right) + c$$

(104) $$\int \frac{1}{\sqrt{4x^2 - 25}}\, dx = \int \frac{1}{\sqrt{(2x)^2 - 5^2}}\, dx$$

$$= \log\left[2x + \sqrt{(2x)^2 - 5^2}\right] \times \frac{1}{2} + c$$

$$= \frac{1}{2} \log\left[2x + \sqrt{4x^2 - 25}\right] + c$$

(105) $$\int \frac{1}{\sqrt{9x^2 + 16}}\, dx = \int \frac{1}{\sqrt{(3x)^2 + 4^2}}\, dx$$

$$= \log\left[3x + \sqrt{(3x)^2 + 4^2}\right] \times \frac{1}{3} + c$$

$$= \frac{1}{3} \log\left[3x + \sqrt{9x^2 + 16}\right] + c$$

Type II: integral of the form $\int \frac{dx}{ax^2 + bx + c}$ **and** $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

In this case, we have to express $ax^2 + bx + c$ as sum or difference of two square terms to get the integrand in one of the standard forms of Type 1 mentioned earlier.

We first make the co-efficient of x^2 numerically one. Complete the square interms containing x^2 and x by adding and subtracting the square of half the coefficient of x.

i.e. $$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$$

OR We can directly use a formula for

$$\boxed{\mathbf{ax^2 + bx + c = \frac{1}{4a}\left[(2ax + b)^2 + (4ac - b^2)\right]}}$$

Example 9.106 – 9.113: Integrate the following:

(106) $\dfrac{1}{x^2 + 5x + 7}$ (107) $\dfrac{1}{x^2 - 7x + 5}$ (108) $\dfrac{1}{\sqrt{x^2 + 16x + 100}}$

(109) $\dfrac{1}{\sqrt{9 + 8x - x^2}}$ (110) $\dfrac{1}{\sqrt{6 - x - x^2}}$ (111) $\dfrac{1}{3x^2 + 13x - 10}$

(112) $\dfrac{1}{2x^2 + 7x + 13}$ (113) $\dfrac{1}{\sqrt{18 - 5x - 2x^2}}$

Solution:

(106) $$\int \frac{1}{x^2 + 5x + 7}\,dx = \int \frac{1}{\left(x + \frac{5}{2}\right)^2 + 7 - \left(\frac{5}{2}\right)^2}\,dx = \int \frac{1}{\left(x + \frac{5}{2}\right)^2 + \frac{3}{4}}\,dx$$

$$= \int \frac{1}{\left(x + \frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx = \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x + \frac{5}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$\int \frac{1}{x^2 + 5x + 7}\,dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 5}{\sqrt{3}}\right) + c$$

(107) $$\int \frac{1}{x^2 - 7x + 5}\,dx = \int \frac{1}{\left(x - \frac{7}{2}\right)^2 + 5 - \left(\frac{7}{2}\right)^2}\,dx = \int \frac{1}{\left(x - \frac{7}{2}\right)^2 - \left(\frac{\sqrt{29}}{2}\right)^2}\,dx$$

$$= \frac{1}{2.\frac{\sqrt{29}}{2}}\log\left(\frac{\left(x - \frac{7}{2}\right) - \frac{\sqrt{29}}{2}}{\left(x - \frac{7}{2}\right) + \frac{\sqrt{29}}{2}}\right) + c$$

$$\int \frac{1}{x^2 - 7x + 5}\,dx = \frac{1}{\sqrt{29}}\log\left(\frac{2x - 7 - \sqrt{29}}{2x - 7 + \sqrt{29}}\right) + c$$

(108) $$\int \frac{1}{\sqrt{x^2 + 16x + 100}}\,dx = \int \frac{1}{\sqrt{(x + 8)^2 + 100 - (8)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{(x + 8)^2 + 6^2}}\,dx$$

$$= \log\left[(x+8) + \sqrt{(x+8)^2 + 6^2}\right] + c$$

$$= \log\left((x+8) + \sqrt{x^2 + 16x + 100}\right) + c$$

(109) $$\int \frac{1}{\sqrt{9 + 8x - x^2}}\, dx = \int \frac{1}{\sqrt{9 - (x^2 - 8x)}}\, dx = \int \frac{1}{\sqrt{9 - \left\{(x-4)^2 - 4^2\right\}}}\, dx$$

$$= \int \frac{1}{\sqrt{9 + 16 - (x-4)^2}}\, dx = \int \frac{1}{\sqrt{5^2 - (x-4)^2}}\, dx$$

$$\int \frac{1}{\sqrt{9 + 8x - x^2}}\, dx = \sin^{-1} \frac{x-4}{5} + c$$

(110) $$\int \frac{1}{\sqrt{6 - x - x^2}}\, dx = \int \frac{1}{\sqrt{6 - (x^2 + x)}}\, dx = \int \frac{1}{\sqrt{6 - \left\{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(6 + \frac{1}{4}\right) - \left(x + \frac{1}{2}\right)^2}}\, dx = \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{x + \frac{1}{2}}{\frac{5}{2}}\right) + c = \sin^{-1}\left(\frac{2x+1}{5}\right) + c$$

$$\int \frac{1}{\sqrt{6 - x - x^2}}\, dx = \sin^{-1}\left(\frac{2x+1}{5}\right) + c$$

For the following problems 111 to 113 the direct formula

$ax^2 + bx + c = \frac{1}{4a}\left[(2ax + b)^2 + (4ac - b^2)\right]$ is used.

(111) $$\int \frac{1}{3x^2 + 13x - 10}\, dx = \int \frac{4 \times 3}{(2 \times 3x + 13)^2 - 4 \times 3 \times 10 - 13^2}\, dx$$

$$= \int \frac{12}{(6x+13)^2 - 289}\, dx = 12 \int \frac{1}{(6x+13)^2 - 17^2}\, dx$$

$$= 12\times\frac{1}{2\times 17}\left[\log\left(\frac{6x+13-17}{6x+13+17}\right)\right]\times\left(\frac{1}{6}\right) + c \qquad \left(\begin{matrix}\text{6 is the coefficient}\\ \text{of } x\end{matrix}\right)$$

$$= \frac{1}{17}\log\left(\frac{6x-4}{6x+30}\right) + c = \frac{1}{17}\log\left(\frac{3x-2}{3x+15}\right) + c$$

$$\int\frac{1}{3x^2+13x-10}\,dx = \frac{1}{17}\log\left(\frac{3x-2}{3x+15}\right) + c$$

(112) $$\int\frac{1}{2x^2+7x+13}\,dx = \int\frac{4\times 2}{(4x+7)^2+104-49}\,dx = 8\int\frac{1}{(4x+7)^2+\sqrt{55}^2}\,dx$$

$$= 8.\frac{1}{\sqrt{55}}\times\tan^{-1}\left(\frac{4x+7}{\sqrt{55}}\right)\times\left(\frac{1}{4}\right) \qquad \left(\begin{matrix}\text{4 is the coefficient}\\ \text{of } x\end{matrix}\right)$$

$$\int\frac{1}{2x^2+7x+13}\,dx = \frac{2}{\sqrt{55}}\tan^{-1}\left(\frac{4x+7}{\sqrt{55}}\right) + c$$

(113) $$\int\frac{1}{\sqrt{18-5x-2x^2}}\,dx = \int\frac{1}{\sqrt{-\{2x^2+5x-18\}}}dx \qquad \left(\begin{matrix}\text{negative sign}\\ \text{should not be taken}\\ \text{outside from the}\\ \text{square root}\end{matrix}\right)$$

$$= \int\frac{\sqrt{4\times 2}}{\sqrt{-\{(4x+5)^2-18\times 8-5^2\}}}\,dx$$

$$= \int\frac{2\sqrt{2}}{\sqrt{13^2-(4x+5)^2}}\,dx$$

$$= 2\sqrt{2}\left\{\sin^{-1}\left(\frac{4x+5}{13}\right)\right\}\times\left(\frac{1}{4}\right) + c$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x+5}{13}\right) + c$$

$$\therefore \quad \int\frac{1}{\sqrt{18-5x-2x^2}}\,dx = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x+5}{13}\right) + c$$

Type III :Integrals of the form $\displaystyle\int\frac{px+q}{ax^2+bx+c}\,dx$ **and** $\displaystyle\int\frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

To evaluate the above integrals, we have to express the numerator $px + q$ into two parts with suitable constants. One in terms of differential coefficient of denominator and the other without 'x' term.

Then the integrals will be separated into two standard form of known integrals and can easily be evaluated.

Let $(px + q) = A \frac{d}{dx}(ax^2 + bx + c) + B$

i.e. $(px + q) = A(2ax + b) + B$

(A & B can be found by equating coefficients of x and constant terms separately.

(i) $$\int \frac{px + q}{ax^2 + bx + c} = \int \frac{A(2ax + b) + B}{ax^2 + bx + c}\, dx$$

$$= A \int \left(\frac{2ax + b}{ax^2 + bx + c}\right) dx + B \int \frac{1}{ax^2 + bx + c}\, dx$$

$$\left(\because \int \frac{f'(x)}{f(x)}\, dx = \log f(x) \Rightarrow \int \left(\frac{2ax + b}{ax^2 + bx + c}\right) dx = [\log(ax^2 + bx + c)]\right)$$

$$\therefore \int \frac{px + q}{ax^2 + bx + c}\, dx = A\,[\log(ax^2 + bx + c)] + B \int \frac{1}{ax^2 + bx + c}\, dx$$

(ii) $$\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx = A \int \frac{(2ax + b)}{\sqrt{ax^2 + bx + c}}\, dx + B \int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$$

$$\left(\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} \Rightarrow \int \frac{(2ax + b)}{\sqrt{ax^2 + bx + c}}\, dx = 2\sqrt{ax^2 + bx + c}\right)$$

$$\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx = A\left(2\sqrt{ax^2 + bx + c}\right) + B \int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$$

Example 114:

Integrate the followings:

(114) $\frac{4x - 3}{x^2 + 3x + 8}$ (115) $\frac{3x + 2}{x^2 + x + 1}$ (116) $\frac{5x - 2}{x^2 - x - 2}$

(117) $\frac{3x + 1}{\sqrt{2x^2 + x + 3}}$ (118) $\frac{x + 1}{\sqrt{8 + x - x^2}}$ (119) $\frac{4x - 3}{\sqrt{x^2 + 2x - 1}}$

Solution:

(114) $\int \frac{4x-3}{x^2+3x+8}\ dx$

let $\quad 4x - 3 = A\frac{d}{dx}(x^2+3x+8) + B$

$4x - 3 = A(2x+3) + B \quad \dots \text{(i)}$

rearranging $\quad 4x - 3 = (2A)\,x + (3A + B)$

Equating like terms $\quad 2A = 4 \quad \Rightarrow \quad A = 2$

$3A + B = -3 \quad \Rightarrow \quad B = -3 - 3A = -9$

$\therefore$ (i) $\Rightarrow$

$$(4x-3) = 2(2x+3) + (-9)$$

$$\therefore \int \frac{4x-3}{x^2+3x+8}\ dx = \int \frac{2(2x+3)+(-9)}{x^2+3x+8}\ dx$$

$$= 2\int \frac{(2x+3)}{x^2+3x+8}\,dx - 9\int \frac{dx}{x^2+3x+8}$$

$$\int \frac{4x-3}{x^2+3x+8}\ dx = 2I_1 - 9I_2 \quad \dots (1)$$

Where $\quad I_1 = \int \frac{(2x+3)}{x^2+3x+8}\ dx$ and $I_2 = \int \frac{dx}{x^2+3x+8}\,dx$

$$I_1 = \int \frac{(2x+3)}{x^2+3x+8}\ dx$$

put $\quad x^2 + 3x - 18 = u \quad \therefore \quad (2x+3)dx = du$

$$\therefore \quad I_1 = \int \frac{du}{u} = \log(x^2+3x+8) \quad \dots (2)$$

$$I_2 = \int \frac{dx}{x^2+3x+8} = \int \frac{4(1)}{(2x+3)^2 + 4\times 8 - 3^2}\ dx$$

$$= \int \frac{4}{(2x+3)^2 + (\sqrt{23})^2}\,dx = 4\times\frac{1}{\sqrt{23}}\times\frac{1}{2}\ \tan^{-1}\frac{2x+3}{\sqrt{23}}$$

$$I_2 = \frac{2}{\sqrt{23}}\ \tan^{-1}\frac{2x+3}{\sqrt{23}} \quad \dots (3)$$

Substituting (2) and (3) in (1), we get

$$\therefore \quad \int \frac{4x-3}{x^2+3x+8}\,dx = 2\log(x^2+3x+8) - \frac{18}{\sqrt{23}}\tan^{-1}\frac{2x+3}{\sqrt{23}}$$

(115) $\int \frac{3x+2}{x^2+x+1}\,dx$

Let $\quad 3x+2 = A\frac{d}{dx}(x^2+x+1) + B$

$$(3x+12) = \quad A(2x+1) + B \qquad \ldots (1)$$

i.e. $\quad 3x+2 = (2A)x + (A+B)$

Equating like terms

$$2A = 3 \quad ; \quad A + B = 2$$

$$\therefore A = \frac{3}{2} \quad ; \quad \frac{3}{2} + B = 2 \qquad \Rightarrow B = 2 - \frac{3}{2} = \frac{1}{2}$$

Substituting $A = \frac{3}{2}$ and $B = \frac{1}{2}$ in (1) we get

$$\therefore (3x+2) = \frac{3}{2}(2x+1) + \left(\frac{1}{2}\right)$$

$$\therefore \int \frac{3x+2}{x^2+x+1}\,dx = \int \frac{\frac{3}{2}(2x+1) + \left(\frac{1}{2}\right)}{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$\therefore \int \frac{3x+2}{x^2+x+1}\,dx = \frac{3}{2}\left\{\log(x^2+x+1)\right\} + I \quad \ldots (2)$$

Where $\quad I = \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{4\times 1}{(2x+1)^2 + 4\times 1\times 1 - 1^2}\,dx$

$$= 2\int \frac{1}{(2x+1)^2 + (\sqrt{3})^2} = 2\times\frac{1}{\sqrt{3}}\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Substituting above I in (2), we get

$$\therefore \int \frac{3x+2}{x^2+x+1}\,dx = \frac{3}{2}\log(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

(116) $\int \frac{5x-2}{x^2-x-2}\, dx$

$$\text{Let } \quad 5x-2 = A\frac{d}{dx}(x^2-x-2) + B$$

$$5x-2 = A(2x-1) + B \qquad \ldots (1)$$

$$5x-2 = (2A)x - A + B$$

equating like terms $\quad 2A = 5 \;;\; -A + B = -2$

$$\therefore\ A = \frac{5}{2} \quad ; \; -\frac{5}{2} + B = -2 \;\Rightarrow\; B = -2 + \frac{5}{2} = \frac{1}{2}$$

Substituting $A = \frac{5}{2}$ and $B = \frac{1}{2}$ in (1), we get

$$(5x-2) = \frac{5}{2}(2x-1) + \frac{1}{2}$$

$$\therefore \int \frac{5x-2}{x^2-x-2}\, dx = \int \frac{\frac{5}{2}(2x-1) + \left(\frac{1}{2}\right)}{x^2-x-2}\, dx$$

$$= \frac{5}{2}\int \frac{2x-1}{x^2-x-2}\, dx + \frac{1}{2}\int \frac{1}{x^2-x-2}\, dx$$

$$\therefore \int \frac{5x-2}{x^2-x-2}\, dx = \frac{5}{2}\left\{\log(x^2-x-2)\right\} + I \ldots (2)$$

Where $\quad I = \frac{1}{2}\int \frac{1}{x^2-x-2}\, dx = \frac{1}{2}\int \frac{4\times 1}{(2x-1)^2 - 8 - 1}\, dx$

$$= \frac{1}{2}\int \frac{4}{(2x-1)^2 - 3^2} = \frac{4}{2} \times \frac{1}{2\times 3}\,\frac{1}{2}\log\left[\frac{2x-1-3}{2x-1+3}\right]$$

$$I = \frac{1}{3\times 2}\log\left[\frac{2x-4}{2x+2}\right] = \frac{1}{6}\log\left(\frac{x-2}{x+1}\right)$$

Substituting I in (2), we get

$$\int \frac{5x-2}{x^2-x-2}\, dx = \frac{5}{2}\log(x^2-x-2) + \frac{1}{6}\log\left(\frac{x-2}{x+1}\right) + c$$

Note : Resolve into partial fractions and then integrate.

(117) $\int \frac{3x+1}{\sqrt{2x^2+x+3}}\, dx$

$$\text{Let } 3x + 1 = A\frac{d}{dx}(2x^2 + x + 3) + B$$

$$3x + 1 = A(4x + 1) + B \qquad \ldots (1)$$

$$3x + 1 = 4Ax + A + B$$

equating like terms $4A = 3$; $A + B = 1$

$$\therefore A = \frac{3}{4} \qquad B = 1 - A = 1 - \frac{3}{4} = \frac{1}{4}$$

by (i) $\Rightarrow \therefore 3x + 1 = \frac{3}{4}(4x + 1) + \frac{1}{4}$

$$\therefore \int \frac{3x + 1}{\sqrt{2x^2 + x + 3}}\, dx = \int \frac{\frac{3}{4}(4x + 1) + \frac{1}{4}}{\sqrt{2x^2 + x + 3}}\, dx$$

$$= \frac{3}{4}\int \frac{4x + 1}{\sqrt{2x^2 + x + 3}}\, dx + \frac{1}{4}\int \frac{1}{\sqrt{2x^2 + x + 3}}\, dx$$

$$\therefore \int \frac{3x + 1}{\sqrt{2x^2 + x + 3}}\, dx = \frac{3}{4}\left\{2\sqrt{2x^2+x+3}\right\} + \mathbf{I} \ldots (2) \quad \left(\because \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)}\right)$$

Where
$$\mathbf{I} = \frac{1}{4}\int \frac{1}{\sqrt{2x^2 + x + 3}}\, dx$$

$$= \frac{1}{4}\int \frac{\sqrt{4.2}}{\sqrt{(4x + 1)^2 + 24 - 1}}\, dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(4x + 1)^2 + (\sqrt{23})^2}}\, dx$$

$$\mathbf{I} = \frac{1}{\sqrt{2}}\left[\log(4x + 1) + \sqrt{(4x + 1)^2 + 23}\right] \times \frac{1}{4}$$

substituting in (2) we get

$$\int \frac{3x + 1}{\sqrt{2x^2 + x + 3}}\, dx = \frac{3}{2}\sqrt{2x^2 + x + 3} + \frac{1}{4\sqrt{2}}\left\{\log(4x+1)+\sqrt{(4x+1)^2+23}\right\} + c$$

(118) $\int \frac{x + 1}{\sqrt{8 + x - x^2}}\, dx$

$$\text{Let } x + 1 = A\frac{d}{dx}(8 + x - x^2) + B$$

$$x + 1 = A(1 - 2x) + B \qquad \ldots (1)$$

$$= (-2A)x + A + B$$

equating like terms $\quad -2A = 1 \; ; \quad A + B = 1$

$$\therefore A = -\frac{1}{2} \qquad B = 1 - A = 1 - \frac{1}{2} = \frac{3}{2}$$

Substituting $\quad A = -\frac{1}{2}$ and $B = \frac{3}{2}$

by (1) $\quad x + 1 = -\frac{1}{2}(1 - 2x) + \frac{3}{2}$

$$\therefore \int \frac{x+1}{\sqrt{8+x-x^2}}\, dx = \int \frac{-\frac{1}{2}(1-2x) + \frac{3}{2}}{\sqrt{8+x-x^2}}\, dx$$

$$= -\frac{1}{2} \int \frac{(1-2x)}{\sqrt{8+x-x^2}}\, dx + \frac{3}{2} \int \frac{1}{\sqrt{8+x-x^2}}\, dx$$

$$\therefore \int \frac{x+1}{\sqrt{8+x-x^2}}\, dx = -\frac{1}{2} \left\{ 2\sqrt{8+x-x^2} \right\} + \quad \mathbf{I} \ldots (2)$$

Where $\quad \mathbf{I} = \frac{3}{2} \int \frac{1}{\sqrt{8+x-x^2}}\, dx$

$$= \frac{3}{2} \int \frac{1}{\sqrt{-\{x^2 - x - 8\}}}\, dx$$

$$= \frac{3}{2} \int \frac{\sqrt{4 \times 1}}{\sqrt{-\{(2x-1)^2 - 32 - 1\}}}\, dx$$

$$= \frac{3}{2} \int \frac{2}{\sqrt{\left(\sqrt{33}\right)^2 - (2x-1)^2}}\, dx$$

$$= 3\left[\left(\frac{1}{2}\right) \sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)\right]$$

$$\mathbf{I} = \frac{3}{2} \sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)$$

substituting in (2) we get

$$\int \frac{x+1}{\sqrt{8+x-x^2}}\, dx = -\sqrt{8+x-x^2} + \frac{3}{2} \sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right) + c$$

(119) $\int \frac{4x-3}{\sqrt{x^2+2x-1}}\ dx$

$$\text{Let } 4x-1 = A(2x+2)+B \qquad \ldots (1)$$

$$4x-3 = (2A)x + 2A + B$$

equating like terms

$$4 = 2A \ ; \qquad 2A + B$$

$$\therefore\ A = 2, \qquad B = -3-2A = -3-4 = -7$$

Substituting $A = 2$ and $B = -7$ in (1), we get

$$4x-3 = 2(2x+2)-7$$

$$\therefore \int \frac{4x-3}{\sqrt{x^2+2x-1}}\ dx = \int \frac{2(2x+2)-7}{\sqrt{x^2+2x-1}}\ dx$$

$$= 2\int \frac{2x+2}{\sqrt{x^2+2x-1}}\ dx + (-7)\int \frac{1}{\sqrt{x^2+2x-1}}\ dx$$

$$\therefore \int \frac{4x-3}{\sqrt{x^2+2x-1}}\ dx = 2\left\{2\sqrt{x^2+2x-1}\right\} + \mathbf{I} \qquad \ldots (2)$$

Where

$$\mathbf{I} = -7\int \frac{1}{\sqrt{x^2+2x-1}}\ dx = -7 \frac{dx}{\sqrt{(x+1)^2-1-1}}$$

$$= -7 \int \frac{dx}{\sqrt{(x+1)^2-\left(\sqrt{2}\right)^2}}$$

$$= -7 \log \left\{(x+1) + \sqrt{(x+1)-\left(\sqrt{2}^2\right)}\right\}$$

$$\mathbf{I} = -7 \log \left\{(x+1) + \sqrt{x^2+2x-1}\right\}$$

substituting in (2) we get

$$\int \frac{4x-3}{\sqrt{x^2+2x-1}}\, dx = 4\sqrt{x^2+2x-1} - 7 \log \left\{(x+1)+\sqrt{x^2+2x-1}\right\} + c$$

We have already seen that

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left[x + \sqrt{x^2 - a^2}\right] + c$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log\left[x + \sqrt{x^2 + a^2}\right] + c$$

The three more standard forms similar to the above are

Type IV:

(120) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(121) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[x + \sqrt{x^2 - a^2}\right] + c$

(122) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left[x + \sqrt{x^2 + a^2}\right] + c$

(120) Let $\mathbf{I} = \int \sqrt{a^2 - x^2}\, dx$

Applying integration by parts rule

let $u = \sqrt{a^2 - x^2}$ $\quad dv = dx$

$du = \frac{-2x}{2\sqrt{a^2 - x^2}}\, dx$ $\quad v = x$

$$\mathbf{I} = x\sqrt{a^2 - x^2} - \int x\left(-\frac{x}{\sqrt{a^2 - x^2}}\right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} + \frac{(-a^2)}{\sqrt{a^2 - x^2}}\right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$\mathbf{I} = x\sqrt{a^2 - x^2} - \mathbf{I} \quad + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\, dx$$

$$\mathbf{I} + \mathbf{I} = x\sqrt{a^2 - x^2} + a^2 . \sin^{-1}\frac{x}{a}$$

$$\therefore\ 2\mathbf{I} = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}$$

$$\mathbf{I} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\boxed{\therefore \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c}$$

(121) Let $\mathbf{I} = \int \sqrt{x^2 - a^2}\, dx$

Applying integration by parts rule

let $u = \sqrt{x^2 - a^2}$, $du = \dfrac{2x}{2\sqrt{x^2 - a^2}}\, dx$; $dv = dx$, $v = x$

$$\mathbf{I} = x\sqrt{x^2 - a^2} - \int x\left(\frac{x}{\sqrt{x^2 - a^2}}\right) dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}}\, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int\sqrt{x^2 - a^2}\, dx - a^2 \int \frac{1}{\sqrt{x^2 - a^2}}\, dx$$

$$\mathrm{I} = x\sqrt{x^2 - a^2} - \mathrm{I} - a^2 \log\left[x + \sqrt{x^2 - a^2}\right]$$

$$\therefore\ 2\mathrm{I} = x\sqrt{x^2 - a^2} - a^2 \log\left[x + \sqrt{x^2 - a^2}\right]$$

$$\therefore\ \mathrm{I} = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[x + \sqrt{x^2 - a^2}\right] + c$$

$$\boxed{\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[x + \sqrt{x^2 - a^2}\right] + c}$$

(122) Let $\mathbf{I} = \int \sqrt{x^2 + a^2}\, dx$

Applying integration by parts rule

$$\mathbf{I} = x\sqrt{x^2+a^2} - \int\left(\frac{x^2}{\sqrt{x^2+a^2}}\right)dx \qquad \text{let } u = \sqrt{x^2+a^2}, \quad du = \frac{2x}{2\sqrt{x^2+a^2}}\,dx, \quad dv = dx, \quad v = x$$

$$= x\sqrt{x^2+a^2} - \int\frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{x^2+a^2} - \int\frac{x^2+a^2}{\sqrt{x^2+a^2}}\,dx + \int\frac{a^2}{\sqrt{x^2+a^2}}\,dx$$

$$= x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\int\frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$\mathbf{I} = x\sqrt{x^2+a^2} - \mathbf{I} + a^2\log\left[x+\sqrt{x^2+a^2}\right] + c$$

$$\therefore\ 2\mathbf{I} = x\sqrt{x^2+a^2} + a^2\log\left[x+\sqrt{x^2+a^2}\right] + c$$

$$\therefore\ \mathbf{I} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right] + c$$

$$\boxed{\therefore \int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right] + c}$$

Example: 9.123 – 9.131:

Integrate the following :

(123) $\sqrt{4-9x^2}$ (124) $\sqrt{16x^2-25}$ (125) $\sqrt{9x^2+16}$ (126) $\sqrt{2x-x^2}$

(127) $\sqrt{x^2-4x+6}$ (128) $\sqrt{x^2+4x+1}$ (129) $\sqrt{4+8x-5x^2}$

(130) $\sqrt{(2-x)(1+x)}$ (131) $\sqrt{(x+1)(x-2)}$

Solution:

(123) $\sqrt{4-9x^2}\,dx = \int\sqrt{2^2-(3x)^2}dx = \frac{1}{3}\left[\frac{(3x)}{2}\sqrt{2^2-(3x)^2} + \frac{2^2}{2}\sin^{-1}\frac{3x}{2}\right]+c$

$$= \frac{1}{3}\left[\frac{3x}{2}\sqrt{4-9x^2} + 2\sin^{-1}\frac{3x}{2}\right] + c$$

(124) $\int\sqrt{16x^2 - 25}\ dx = \int\sqrt{(4x)^2 - 5^2}\ dx$

$$= \frac{1}{4}\left[\frac{(4x)}{2}\sqrt{(4x)^2 - 5^2} - \frac{25}{2}\log\left[4x + \sqrt{(4x)^2 - 5^2}\right]\right]$$

$$= \frac{1}{8}\ \left[4x\sqrt{16x^2 - 25} - 25\log\left(4x + \sqrt{16x^2 - 25}\right)\right] + c$$

(125) $\int\sqrt{9x^2 + 16}\ \ dx = \int\sqrt{(3x)^2 + 4^2}\ dx$

$$= \frac{1}{3}\left[\frac{(3x)}{2}\sqrt{(3x)^2 + 4^2} + \frac{4^2}{2}\log\left[3x + \sqrt{(3x)^2 + 4^2}\right]\right]$$

$$= \frac{1}{6}\ \left[3x\sqrt{9x^2 + 16} + 16\log\left(3x + \sqrt{9x^2 + 16}\right)\right] + c$$

(126) $\int\sqrt{2x - x^2}\ \ dx = \int\sqrt{1 - \{x^2 - 2x + 1\}}\ dx = \int\sqrt{1^2 - (x-1)^2}\ \ dx$

$$= \frac{(x-1)}{2}\ \sqrt{1 - (x-1)^2}\ + \frac{1^2}{2}\ \sin^{-1}\left(\frac{x-1}{1}\right) + c$$

$$= \frac{x-1}{2}\ \sqrt{2x - x^2}\ + \frac{1}{2}\ \sin^{-1}(x-1) + c$$

(127) $\int\sqrt{x^2 - 4x + 6}\ dx = \int\sqrt{x^2 - 4x + 4 + 2}\ \ dx = \int\sqrt{(x-2)^2 + \left(\sqrt{2^2}\right)}\ \ dx$

$$= \frac{(x-2)}{2}\ \sqrt{(x-2)^2 + \left(\sqrt{2}\right)^2} + \frac{\left(\sqrt{2}\right)^2}{2}\log\left[(x-2) + \sqrt{(x-2)^2 + \left(\sqrt{2}\right)^2}\right] + c$$

$$= \frac{(x-2)}{2}\sqrt{x^2 - 4x + 6}\ + \log\left[(x-2) + \sqrt{x^2 - 4x + 6}\right]\ + c$$

(128) $\int\sqrt{x^2 + 4x + 1}\ dx = \int\sqrt{(x+2)^2 - \left(\sqrt{3}\right)^2}\ dx$

$$= \frac{(x+2)}{2}\ \sqrt{(x+2)^2 - \left(\sqrt{3}\right)^2} - \frac{\left(\sqrt{3}\right)^2}{2}\ \log\left[(x+2) + \sqrt{(x+2)^2 - \left(\sqrt{3}\right)^2}\right] + c$$

$$= \frac{(x+2)}{2}\ \sqrt{x^2 + 4x + 1}\ - \frac{3}{2}\ \log\left[(x+2) + \sqrt{x^2 + 4x + 1}\right] + c$$

(129) $\int\sqrt{4 + 8x - 5x^2}\, dx = \int \sqrt{-\{5x^2 - 8x - 4\}}\ \ dx$

$$\left(\because ax^2 + bx + c = \frac{1}{4a}\ [(2ax + b)^2 + (4ac - b^2])\right)$$

$$= \int \frac{1}{\sqrt{4 \times 5}} \sqrt{-\{(10x - 8)^2 - 80 - 64\}}\ \ dx$$

$$= \frac{1}{\sqrt{20}} \int\sqrt{12^2 - (10x - 8)^2}\ \ dx$$

$$= \frac{1}{\sqrt{20}} \left[\left(\frac{1}{10}\right)\left(\frac{10x - 8}{2}\sqrt{12^2 - (10x - 8)^2} + \left(\frac{12^2}{2}\right)\sin^{-1}\frac{10x - 8}{12}\right)\right]$$

$$= \frac{1}{\sqrt{20}} \left[\frac{1}{10}(5x - 4)\sqrt{80 + 16x - 100x^2} + \frac{36}{5}\sin^{-1}\left(\frac{5x - 4}{6}\right)\right]$$

$$= \frac{1}{\sqrt{20}} \left[\left(\frac{5x - 4}{10}\right)\sqrt{20}\sqrt{(4 + 8x - 5x^2)} + \frac{36}{5}\sin^{-1}\frac{5x - 4}{6}\right]$$

$$= \frac{5x - 4}{10}\ \sqrt{4 + 8x - 5x^2}\ + \frac{36}{\sqrt{20} \times 5}\ \sin^{-1}\frac{5x - 4}{6}$$

$$\therefore \int\sqrt{4 + 8x - 5x^2}\, dx = \frac{5x - 4}{10}\ \sqrt{4 + 8x - 5x^2}\ + \frac{18}{5\sqrt{5}}\ \sin^{-1}\frac{5x - 4}{6}\ + c$$

(130) $\int\sqrt{(2 - x)(1 + x)}\, dx = \int \sqrt{2 + x - x^2}\ \ dx = \int \sqrt{-(x^2 - x - 2)}\ \ dx$

$$= \int \frac{\sqrt{-\{(2x - 1)^2 - 8 - 1\}}}{\sqrt{4.1}}\ dx\ = \frac{1}{2} \int \sqrt{3^2 - (2x - 1)^2}\ \ dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\frac{(2x - 1)}{2}\sqrt{3^2 - (2x - 1)^2} + \left(\frac{1}{2}\right)\frac{3^2}{2}\sin^{-1}\left(\frac{2x - 1}{3}\right)\right]$$

$$= \frac{1}{8}\left[(2x - 1)\sqrt{8 + 4x - 4x^2} + 9\sin^{-1}\left(\frac{2x - 1}{3}\right)\right]$$

$$= \frac{1}{8}\left[2(2x - 1)\sqrt{2 + x - x^2} + 9\sin^{-1}\left(\frac{2x - 1}{3}\right)\right]$$

(131) $\int\sqrt{(x+1)(x-2)}\,dx = \int\sqrt{x^2-x-2}\;dx = \int\frac{\sqrt{(2x-1)^2-8-1}}{\sqrt{4}}\;dx$

$$= \frac{1}{2}\int\sqrt{(2x-1)^2-3^2}\;dx$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\right)\left(\frac{2x-1}{2}\right)\sqrt{(2x-1)^2-3^2} - \left(\frac{1}{2}\right)\left(\frac{3^2}{2}\right)\log\left\{(2x-1)+\sqrt{(2x-1)^2-3^2}\right\}\right]$$

$$\int\sqrt{(x+1)(x-2)}dx = \frac{1}{2}\left[\frac{(2x-1)}{4}\sqrt{(2x-1)^2-9}-\frac{9}{4}\log\left\{(2x-1)+\sqrt{(2x-1)^2-9}\right\}\right]$$

EXERCISE 9.7

Integrate the followings

(1) $\frac{1}{x^2+25}$, $\frac{1}{(x+2)^2+16}$, $\frac{1}{(3x+5)^2+4}$, $\frac{1}{2x^2+7x+13}$, $\frac{1}{9x^2+6x+10}$

(2) $\frac{1}{16-x^2}$, $\frac{1}{9-(3-x)^2}$, $\frac{1}{7-(4x+1)^2}$, $\frac{1}{1+x-x^2}$, $\frac{1}{5-6x-9x^2}$

(3) $\frac{1}{x^2-25}$, $\frac{1}{(2x+1)^2-16}$, $\frac{1}{(3x+5)^2-7}$, $\frac{1}{x^2+3x-3}$, $\frac{1}{3x^2-13x-10}$

(4) $\frac{1}{\sqrt{x^2+1}}$, $\frac{1}{\sqrt{(2x+5)^2+4}}$, $\frac{1}{\sqrt{(3x-5)^2+6}}$, $\frac{1}{\sqrt{x^2+3x+10}}$, $\frac{1}{\sqrt{x^2+5x+26}}$

(5) $\frac{1}{\sqrt{x^2-91}}$, $\frac{1}{\sqrt{(x+1)^2-15}}$, $\frac{1}{\sqrt{(2x+3)^2-16}}$, $\frac{1}{\sqrt{x^2+4x-12}}$, $\frac{1}{\sqrt{x^2+8x-20}}$

(6) $\frac{1}{\sqrt{4-x^2}}$, $\frac{1}{\sqrt{25-(x-1)^2}}$, $\frac{1}{\sqrt{11-(2x+3)^2}}$, $\frac{1}{\sqrt{1+x-x^2}}$, $\frac{1}{\sqrt{8-x-x^2}}$

(7) $\frac{3-2x}{x^2+x+1}$, $\frac{x-3}{x^2+21x+3}$, $\frac{2x-1}{2x^2+x+3}$, $\frac{1-x}{1-x-x^2}$, $\frac{4x+1}{x^2+3x+1}$

(8) $\frac{x+2}{\sqrt{6+x-2x^2}}$, $\frac{2x-3}{\sqrt{10-7x-x^2}}$, $\frac{3x+2}{\sqrt{3x^2+4x+7}}$, $\sqrt{\frac{1+x}{1-x}}$, $\frac{6x+7}{\sqrt{(x-4)(x-5)}}$

(9) $\sqrt{1+x^2}$, $\sqrt{(x+1)^2+4)}$, $\sqrt{(2x+1)^2+9}$, $\sqrt{(x^2-3x+10)}$

(10) $\sqrt{4-x^2}$, $\sqrt{25-(x+2)^2}$, $\sqrt{169-(3x+1)^2}$, $\sqrt{1-3x-x^2}$, $\sqrt{(2-x)(3+x)}$

9.4 Definite integrals

A basic concept of integral calculus is limit, an idea applied by the Greeks in geometry.

To find the area of a circle, Archimedes inscribed an equilateral polygon in a circle. Upon increasing the numbers of sides, the area of the polygon approaches the area of the circle as a limit. The area of an irregular shaped plate also can be found by subdividing it into rectangles of equal width. If the number of rectangles is made larger and larger by reducing the width, the sum of the area of rectangles approaches the required area as a limit. The beauty and importance of the

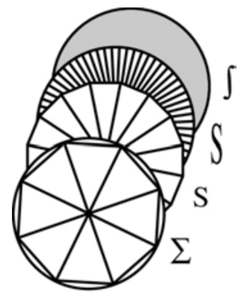


Fig. 9.2

integral calculus is that it provides a systematic way for the exact calculations of many areas, volumes and other quantities.

Integration as summation

To understand the concept of definite integral, let us take a simple case.

Consider the region R in the plane showing figure 9.3. The region R is bounded by the curve $y = f(x)$, the x-axis, and two vertical lines $x = a$ and $x = b$, where $b > a$

For simplicity, we assume $y = f(x)$ to be a continuous and increasing function on the closed interval $[a, b]$.

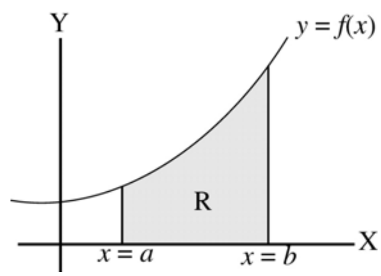


Fig. 9.3

We first define a polygon contained in R. Divided the closed interval $[a, b]$ into n sub intervals of equal length say Δx.

$$\therefore \ \Delta x = \frac{b-a}{n}$$

Denote the end points of these sub intervals by $x_0, x_1, x_2 \ldots\ldots\ldots x_r \ldots\ldots\ x_n$.

Where $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, …… $x_r = a + r\Delta x$, ……, $x_n = b$

The area of the polygon shown in figure 9.4 is the sum of the area of the rectangles (by taking the left hand ***x*** values of the such intervals).

$S_n = A_1 + A_2 + \ldots\ldots + A_n$

$= f(x_0)\,\Delta x + f(x_1)\Delta x \ldots + f(x_{n-1})\,\Delta x$

$= [f(a) + f(a + \Delta x) + \ldots f(a+r\ \Delta x) \ldots$

$+ f(a + (n-1)\Delta x)]\,\Delta x$

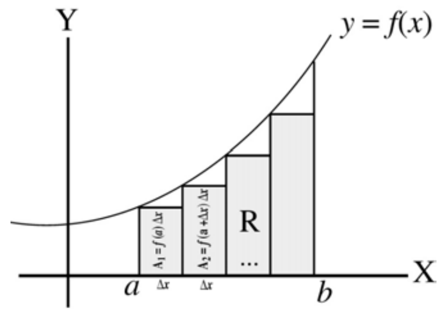


Fig. 9.4

$$= \sum_{r=1}^{n} f\{a + (r-1)\,\Delta x\}.(\Delta x) \quad = \Delta x \sum_{r=1}^{n} f\{a + (r-1)\,\Delta x\}$$

$$S_n = \frac{b-a}{n} \sum_{r=1}^{n} f\{a + (r-1)\,\Delta x\} \qquad \left(\because \Delta x = \frac{b-a}{n}\right)$$

Now increase the number of sub intervals multiply n by 2, then the number of rectangles is doubled, and width of each rectangle is halved as shown in figure. 9.5. By comparing the two figures, notice that the shaded region in fig.9.5 appears more approximate to the region R than in figure 9.4.

So sum of the areas of the rectangles S_n, approaches to the required region R as n increases.

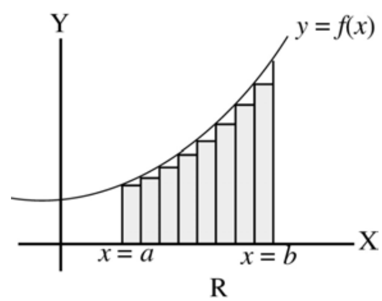


Fig. 9.5

Finally we get, $\underset{n \to \infty}{\mathrm{L}t}\ S_n = \underset{n \to \infty}{\mathrm{L}t}\ \frac{b-a}{n} \sum_{r=1}^{n} f\{a + (r-1)\,\Delta x\} \to \mathrm{R}$

Similarly, by taking the right hand values of x of the sub intervals, we can have,

$$\underset{n \to \infty}{\mathrm{L}t}\ S_n = \underset{n \to \infty}{\mathrm{L}t}\ \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\,\Delta x) \ \to\ \mathrm{R}$$

$$\text{i.e.}\ \boxed{\mathrm{R} = \underset{n \to \infty}{\mathrm{L}t}\ \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\,\Delta x)} \quad - \ \mathbf{I}$$

Definition: If a function $f(x)$ is defined on a closed interval $[a, b]$, then the definite integral of $f(x)$ from a to b is given by

$$\boxed{\underset{n \to \infty}{\text{Lt}} \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\,\Delta x)}, \text{ where } \Delta x = \frac{b-a}{n} \quad \text{(provided the limit exists)}$$

On the other hand the problem of finding the area of the region R is the problem of arguing from the derivative of a function back to the function itself.

Anti-derivative approach to find the area of the region R.

Let us consider the same region R (considering 9.6) bounded by the curve $y = f(x)$ the x-axis and the two vertical lines, $x = a$ and $x = b$, where $b > a$.

To evaluate the area of R, we need to consider the total area between the curve $y = f(x)$ and the x-axis from the left to the arbitrary point $P(x, y)$ on the curve.

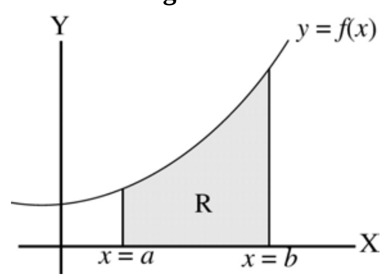


Fig. 9.6

Let us denote this area by $\mathbf{A}_x$.

Let Q $(x + \Delta x, y + \Delta y)$ be another point very close to $P(x, y)$.

Let $\Delta \mathbf{A}_x$ is the area enclosed by the strip under the arc PQ and x-axis.

If the strip is approximated by a rectangle of length y and width Δx, then the area of the strip is $y \,.\, \Delta x$.

Since P and Q are very close

$\Delta \mathbf{A}_x \approx y \,.\, \Delta x \qquad \therefore \quad \frac{\Delta \mathbf{A}_x}{\Delta x} \approx y$

If the width Δx is reduced, then the error is accordingly reduced.

If $\Delta x \to 0$ then $\Delta \mathbf{A}_x \to 0$

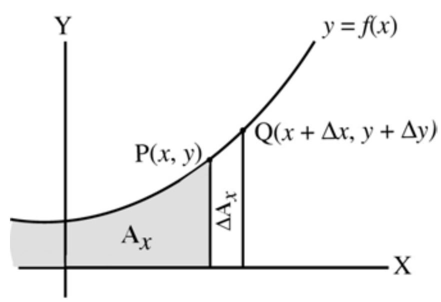


Fig. 9.7

$$\therefore \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta \mathbf{A}_x}{\Delta x} = y \quad \Rightarrow \quad \frac{d\mathbf{A}_x}{dx} = y$$

$\therefore$ By definition of anti derivative $\quad \dfrac{d\mathbf{A}_x}{dx} = y \;\Rightarrow\; \mathbf{A}_x = \int y\,dx$

is the total area $\mathbf{A}_x$, between the curve and x-axis upto the point P is given by the indefinite integral $\int y\,dx$

Let $\int y\,dx = F(x) + c$

If $x = a$, then the area upto $x = a$, $\mathbf{A}_a$ is

$\int y\,dx = F(a) + c$

If $x = b$, then the area upto $x = b$, $\mathbf{A}_b$ is

$\int y\,dx = F(b) + c$

$\therefore$ The required area of the region R is $\mathbf{A}_b - \mathbf{A}_a$

given by

$$\underset{\text{upto } x = b}{\int y\,dx} \; - \; \underset{\text{upto } x = a}{\int y\,dx}$$

$$= (F(b) + c) - (F(a) + c)$$

by notation $\displaystyle\int_a^b y\,dx = F(b) - F(a)$

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)} \qquad - \qquad \text{II}$$

gives the area of the region R bounded by the curve $y = f(x)$, x axis and between the lines $x = a$ and $x = b$.

***a* & *b* are called the lower and upper limits of the integral.**

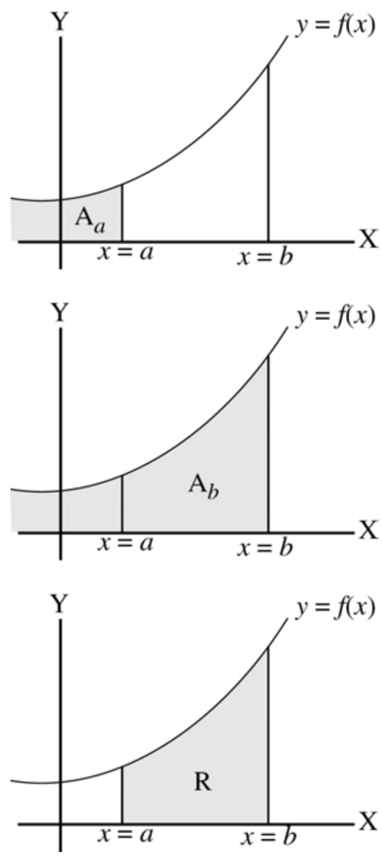


Fig. 9.8

From I & II, it is clear that

$$\boxed{R = \underset{n \to \infty}{Lt} \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\Delta x) = \int_a^b f(x)\,dx = F(b) - F(a)} \text{ , if the limit exists}$$

To evaluate the definite integrals under this method, the following four formulae will be very much useful.

(i) $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

(ii) $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) $$\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(iv) $$\sum_{r=1}^{n} a^r = a\left(\frac{a^n - 1}{a - 1}\right) \; ; (a \neq 1)$$

Illustration:

Consider the area A below the straightline $y = 3x$ above the x-axis and between the lines $x = 2$ and $x = 6$. as shown in the figure.

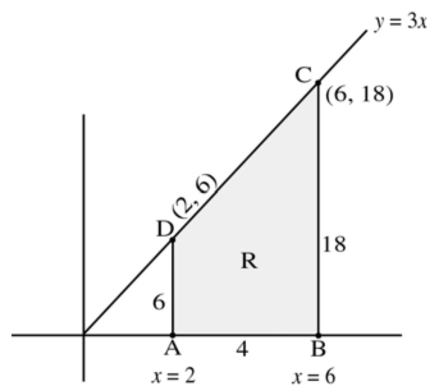


Fig. 9.9

(1) Using the formula for the area of the trapezium ABCD

$$R = \frac{h}{2}[a + b]$$

$$= \frac{4}{2}[6 + 18] = 2 \times 24$$

$$R = 48 \text{ sq. units} \quad \dots (i)$$

(2) Integration as summation

Let us divide the area ABCD into n strips with equal widths. Here $a = 2$, $b = 6$

∴ width of each strip

$$\Delta x = \frac{b-a}{n}$$

i.e. $$\Delta x = \frac{6-2}{n}$$

$$\Delta x = \frac{4}{n}$$

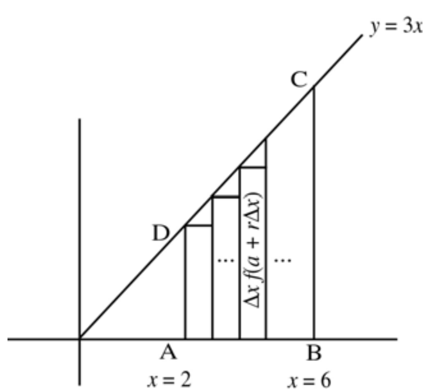


Fig. 9.10

By definite integral formula

$$R = \underset{n \to \infty}{\text{Lt}} \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\,\Delta x)$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{4}{n} \sum_{r=1}^{n} f\left(2 + r\left(\frac{4}{n}\right)\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{12}{n} \sum_{r=1}^{n} \left(2 + \frac{4r}{n}\right) \;;\quad \left[\because f(x) = 3x,\ f\left(2 + r\frac{4}{n}\right) = 3\left[2 + r\left(\frac{4}{n}\right)\right]\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{12}{n} \left[\sum_{r=1}^{n} 2 + \frac{4}{n} \sum_{r=1}^{n} r\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{12}{n} \left[2n + \frac{4}{n} \frac{(n)\,(n+1)}{2}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{12}{n} \left[2n + 2(n+1)\right]$$

$$= \underset{n \to \infty}{\text{Lt}} 12\left[2 + 2\,\frac{(n+1)}{n}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} 12\left[2 + 2\left(1 + \frac{1}{n}\right)\right]$$

$$= 12\,[2 + 2\,(1 + 0)] \qquad \text{as } n \to \infty,\ \frac{1}{n} \to 0$$

$$= 12 \times 4$$

R = 48 square units … (ii)

(3) By anti derivative method

$$R = \int_a^b f(x)\,dx = \int_2^6 3x\,dx = 3\int_2^6 x\,dx = 3\left[\frac{x^2}{2}\right]_2^6$$

$$= 3\left[\frac{6^2 - 2^2}{2}\right] = 3\left[\frac{36-4}{2}\right] = 3 \times \frac{32}{2}$$

R = 48 square units … (iii)

From (i), (ii) and (iii) it is clear that the area of the region is

$$\boxed{R = \underset{n \to \infty}{\text{Lt}} \frac{b-a}{n} \sum f(a + r\,\Delta x) = \int_a^b f(x)\,dx}, \quad \text{if the limit exists}$$

Examples 9.132 – 9.134:

Evaluate the following definite integrals as limit of sums

(132) $\int_1^2 (2x + 5)\, dx$ (133) $\int_1^3 x^2\, dx$ (134) $\int_2^5 (3x^2 + 4)\, dx$

(132) $\int_1^2 (2x + 5)\, dx$

$$\text{Let } f(x) = 2x + 5 \text{ and } [a, b] = [1, 2]$$

$$\Delta x = \frac{b - a}{n} = \frac{2 - 1}{n} = \frac{1}{n}$$

$$\therefore \Delta x = \frac{1}{n}$$

$$f(x) = 2x + 5$$

$$\therefore f(a + r\,\Delta x) = f\left(1 + r\ \frac{1}{n}\right) = 2\left(1 + \frac{r}{n}\right) + 5$$

Let us divide the closed interval [1, 2] into n equal sub intervals of each length Δx.

By the formula

$$\int_a^b f(x)\, dx = \underset{\Delta x \to 0}{\text{Lt}}\ \Delta x \sum_{r=1}^{n} f(a + r\,\Delta x)$$

$$\int_1^2 (2x + 5) = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{n}\right) \sum_{r=1}^{n} \left(2\left(1 + \frac{r}{n}\right) + 5\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{1}{n} \sum_{r=1}^{n} \left(7 + \frac{2}{n} r\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{1}{n} \cdot \left[\sum_{r=1}^{n} 7 + \frac{2}{n} \sum_{r=1}^{n} r\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{1}{n} \left[7n + \frac{2}{n} \cdot \frac{n(n + 1)}{2}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \left[7 + \frac{n + 1}{n}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \left[7 + \left(1 + \frac{1}{n}\right)\right]$$

$$= (7 + 1) \qquad 1/n \to 0 \quad \text{as } n \to \infty$$

$$\therefore \int_1^2 (2x + 5) = 8 \text{ square units}$$

Verification : $\int_1^2 (2x + 5)\, dx = \left[2\left(\frac{x^2}{2}\right) + 5x\right]_1^2$

$$= (2^2 - 1^2) + 5(2 - 1) = (4 - 1) + (5 \times 1)$$

$$\int_1^2 (2x + 5)\, dx = 8 \text{ square units}$$

(133) $\int_1^3 x^2\, dx$

Let $f(x) = x^2$ and $[a, b] = [1, 3]$

Let us divide the closed interval [1, 3] into n equal sub intervals of each length Δx.

$$\Delta x = \frac{3 - 1}{n} = \frac{2}{n}$$

$$\therefore \Delta x = \frac{2}{n}$$

$$f(x) = x^2$$

$$\therefore f(a + r\,\Delta x) = f\left(1 + r\,\frac{2}{n}\right)$$

$$= \left(1 + r\,\frac{2}{n}\right)^2$$

$$f(a + r\,\Delta x) = \left(1 + \frac{4}{n}r + \frac{4}{n^2}r^2\right)$$

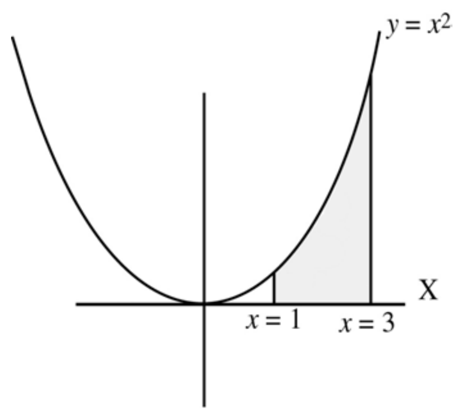


Fig. 9.11

By the formula

$$\int_a^b f(x)\, dx = \underset{\Delta x \to 0}{\text{Lt}} \Delta x \sum_{r=1}^{n} f(a + r\,\Delta x)$$

$$\int_1^3 x^2\,dx = \underset{n \to \infty}{\text{Lt}} \frac{2}{n} \sum_{r=1}^{n} \left(1 + \frac{4}{n} r + \frac{4}{n^2} r^2\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{2}{n}\left[\Sigma 1 + \frac{4}{n}\Sigma r + \frac{4}{n^2}\Sigma r^2\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{2}{n}\left[n + \frac{4}{n} \cdot \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{(n)(n+1)(2n+1)}{6}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} 2\left[1 + \frac{2(n+1)}{n} + \frac{2}{3}\left(\frac{n+1}{n}\right) \cdot \left(\frac{2n+1}{n}\right)\right]$$

$$= \underset{n \to \infty}{\text{Lt}} 2\left[1 + 2\left(1 + \frac{1}{n}\right) + \frac{2}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right]$$

$$= 2\left[1 + 2 + \frac{2}{3}(1)(2)\right] \text{ as } \underset{n \to \infty}{\text{Lt}} \to \frac{1}{n} \to 0$$

$$= 2\left[3 + \frac{4}{3}\right]$$

$$\therefore \int_1^3 x^2\,dx = \frac{26}{3} \text{ square units.}$$

(134) $\int_2^5 (3x^2 + 4)\,dx$

Let $f(x) = 3x^2 + 4$ and $[a, b] = [2, 5]$

Let us divide the closed interval [2, 5] into n equal sub intervals of each length Δx.

$$\Delta x = \frac{5-2}{n}$$

$$\therefore \Delta x = \frac{3}{n}$$

$$f(x) = 3x^2 + 4$$

$$\therefore f(a + r\,\Delta x) = f\left(2 + r \cdot \frac{3}{n}\right)$$

$$= 3\left(2 + \frac{3r}{n}\right)^2 + 4$$

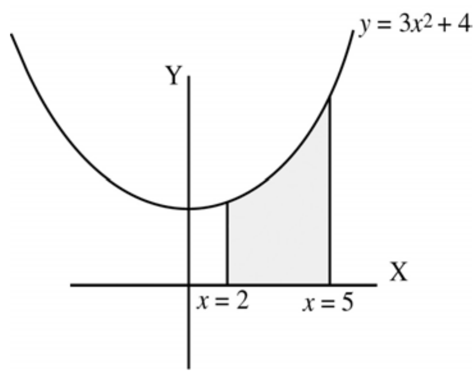


Fig. 9.12

By the formula

$$\int_a^b f(x)\,dx = \underset{\Delta x \to 0}{\text{Lt}} \; \Delta x \sum_{r=1}^{n} f(a + r\,\Delta x)$$

$$\int_2^5 (3x^2 + 4)\;dx = \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \sum_{r=1}^{n} \left(3\left(2 + \frac{3r}{n}\right)^2 + 4\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \sum_{r=1}^{n} \left(3\left(4 + \frac{12}{n}\,r + \frac{9}{n^2}\,r^2\right) + 4\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \sum_{r=1}^{n} \left(12 + \frac{36}{n}\,r + \frac{27}{n^2}\,r^2 + 4\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \sum_{r=1}^{n} \left[16 + \frac{36}{n}\,(r) + \frac{27}{n^2}\,(r^2)\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \left[\Sigma 16 + \frac{36}{n}\,\Sigma r + \frac{27}{n^2}\,\Sigma r^2\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \left[16n + \frac{36}{n}\,\frac{(n)\,(n+1)}{2} + \frac{27}{n^2}\,\frac{n(n+1)\,(2n+1)}{6}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \; 3\left[16 + 18\,\frac{(n+1)}{n} + \frac{9}{2}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \; 3\left[16 + 18\left(1 + \frac{1}{n}\right) + \frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right]$$

$$= 3\left[16 + (18 \times 1) + \frac{9}{2}\,(1)\,(2)\right] = 3\,[43]$$

$$\int_2^5 (3x^2 + 4)\;dx = 129 \text{ square units.}$$

10. PROBABILITY

"The theory of probability is nothing more than good sense confirmed by calculation"

– Pierre Laplace

10.1 Introduction:

The word probability and chance are quite familiar to everyone. Many a time we come across statements like "There is a bright **chance** for Indian cricket team to win the World Cup this time".

"It is **possible** that our school students may get state ranks in forthcoming public examination".

"**Probably** it may rain today".

The word chance, possible, probably, likely etc. convey some sense of uncertainty about the occurrence of some events. Our entire world is filled with uncertainty. We make decisions affected by uncertainty virtually every day.

In order to think about and measure uncertainty, we turn to a branch of mathematics called probability.

Before we study the theory of probability let us learn the definition of certain terms, which will be frequently used.

Experiment: An experiment is defined as a process for which its result is well defined.

Deterministic experiment: An experiment whose outcomes can be predicted with certain, under identical conditions.

Random experiment: An experiment whose all possible outcomes are known, but it is not possible to predict the outcome.

Example: (i) A fair coin is "tossed" (ii) A die is "rolled" are random experiments, since we cannot predict the outcome of the experiment in any trial.

A simple event (or elementary event): The most basic possible outcome of a random experiment and it cannot be decomposed further.

Sample space: The set of all possible outcomes of a random experiment is called a sample space.

Event: Every non-empty subset of the sample space is an event.

The sample space S is called **Sure event** or **Certain event**. The null set in S is called **Impossible event**.

Example: When a single, regular die is rolled once, the associated sample space is

S = {1, 2, 3, 4, 5, 6}

{1}, {2}, {3}, {4}, {5}, {6} are the simple events or elementary events.

{1}, {2, 3}, {1, 3, 5}, {2, 4, 5, 6} are some of the events.

Mutually exclusive events (or disjoint events)

Two or more events are said to be mutually exclusive if they have no simple events (or outcomes) in common. (i.e. They cannot occur simultaneously).

Example: When we roll a die the events {1, 2, 3} and {4, 5, 6} are mutually exclusive event

Exhaustive events:

A set of events is said to be exhaustive if no event outside this set occurs and atleast one of these events must happen as a result of an experiment.

Example:

When a die is rolled, the set of events {1, 2, 3}, {2, 3, 5}, {5, 6} and {4, 5} are exhaustive events.

Equally likely events:

A set of events is said to be equally likely if none of them is expected to occur in preference to the other.

Example:When a coin is tossed, the events {head} and {tail} are equally likely.

Example:

Trial	Random Experiment	Total Number of Outcomes	Sample space
(1)	Tossing of a fair coin	$2^1 = 2$	{H, T}
(2)	Tossing of two coins	$2^2 = 4$	{HH, HT, TH, TT}
(3)	Tossing of three coins	$2^3 = 8$	{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}
(4)	Rolling of fair die	$6^1 = 6$	{1, 2, 3, 4, 5, 6}
(5)	Rolling of two dice	$6^2 = 36$	{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}
(6)	Drawing a card from a pack of 52 playing cards	$52^1 = 52$	Heart ♥ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Diamond ♦ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Spade ♠ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour Club ♣ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour

Notations:

Let A and B be two events.

(i) $A \cup B$ stands for the occurrence of A or B or both.

(ii) $A \cap B$ stands for the simultaneous occurrence of A and B.

(iii) $\bar{A}$ or A' or A^c stands for non-occurrence of A

(iv) $(A \cap \overline{B})$ stands for the occurrence of only A.

Example: Suppose a fair die is rolled, the sample space is S = {1, 2, 3, 4, 5, 6}

Let A = {1, 2}, B = {2, 3}, C = {3, 4}, D = {5, 6}, E = {2, 4, 6} be some events.

(1) The events A, B, C and D are equally likely events, because they have equal chances to occur (but not E).

(2) The events A, C, D are mutually exclusive because
$A \cap C = C \cap D = A \cap D = \phi$.

(3) The events B and C are not mutually exclusive since $B \cap C = \{3\} \neq \phi$.

(4) The events A, C and D are exhaustive events, since $A \cup C \cup D = S$

(5) The events A, B and C are not exhaustive events since the event {5, 6} occurs outside the totality of the events A, B and C.
(i.e. $A \cup B \cup C \neq S$).

10.2 Classical definition of probability:

If there are ***n*** exhaustive, mutually exclusive and equally likely outcomes of an experiment and ***m*** of them are favourable to an event A, then the mathematical probability of A is defined as the ratio $\frac{m}{n}$ i.e. $P(A) = \frac{m}{n}$

In other words,

let S be the sample space and A be an event associated with a random experiment.

Let $n(S)$ and n (A) be the number of elements of S and A respectively. Then the probability of event A is defined as

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to A}}{\text{Exhaustive Number of cases in S}}$$

Axioms of probability

Given a finite sample space S and an event A in S, we define P(A), the probability of A, satisfies the following three conditions.

(1) O ≤ P(A) ≤ 1
(2) P(S) = 1
(3) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$

Note:

If A_1, A_2 …… A_n are mutually exclusively events in a sample space S, then

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$

Example 10.1:

If an experiment has exactly the three possible mutually exclusive outcomes A, B and C, check in each case whether the assignment of probability is permissible.

(i) $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{3}$

(ii) $P(A) = \frac{1}{4}$, $P(B) = \frac{3}{4}$, $P(C) = \frac{1}{4}$

(iii) P(A) = 0.5 , P(B) = 0.6 , P(C) = – 0.1

(iv) P(A) = 0.23 , P(B) = 0.67 , P(C) = 0.1

(v) P(A) = 0.51 , P(B) = 0.29 , P(C) = 0.1

Solution:

(i) The values of P(A), P(B) and P(C) are all lying in the interval from [0, 1]

Also their sum $P(A) + P(B) + P(C) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

∴ The assignment of probability is permissible.

(ii) Given that O ≤ P(A), P(B), P(C) ≤ 1

But the sum $P(A) + P(B) + P(C) = \frac{1}{4} + \frac{3}{4} + \frac{1}{4} = \frac{5}{4} > 1$

∴ The assignment is not permissible.

(iii) Since P(C) = – 0.1, is negative, the assignment is not permissible.

(iv) The assignment is permissible because 0 ≤ P(A), P(B), P(C) ≤ 1 and their sum P(A) + P(B) + P(C) = 0.23 + 0.67 + 0.1 = 1

(v) Eventhough 0 ≤ P(A), P(B), P(C) ≤ 1, their sum P(A) + P(B) + P(C) = 0.51 + 0.29 + 0.1 = 0.9 ≠ 1. Therefore, the assignment is not permissible

Note:

In the above examples each experiment has exactly three possible outcomes. Therefore they must be exhaustive events (i.e. totality must be sample space) and the sum of probabilities is equal to 1.

Examples 10.2: Two coins are tossed simultaneously, what is the probability of getting

(i) exactly one head (ii) atleast one head (iii) atmost one head.

Solution:

The sample space is S = {HH, HT, TH, TT}, $n(S) = 4$

Let A be the event of getting one head, B be the event of getting atleast one head and C be the event of getting atmost one head.

$$\therefore A = \{HT, TH\}, \quad n(A) = 2$$
$$B = \{HT, TH, HH\}, \quad n(B) = 3$$
$$C = \{HT, TH, TT\}, \quad n(C) = 3$$

(i) $P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$ (ii) $P(B) = \frac{n(B)}{n(S)} = \frac{3}{4}$ (iii) $P(C) = \frac{n(C)}{n(S)} = \frac{3}{4}$

Example 10.3: When a pair of balanced dice is rolled, what are the probabilities of getting the sum (i) 7 (ii) 7 or 11 (iii) 11 or 12

Solution:

The sample space $S = \{(1,1), (1,2) \ldots (6,6)\}$

Number of possible outcomes $= 6^2 = 36 = n(S)$

Let A be the event of getting sum 7, B be the event of getting the sum 11 and C be the event of getting sum 12

$$\therefore A = \{(1,6), (2,5), (3,4), (4,3), (5,2),(6,1)\}, \ n(A) = 6.$$
$$B = \{(5,6), (6,5)\}, \ n(B) = 2$$
$$C = \{(6, 6)\}, \ n(C) = 1$$

(i) P(getting sum 7) = $P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

(ii) $P(7 \text{ or } 11) = P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B \quad (\because A \text{ and } B \text{ are mutually exclusive i.e. } A \cap B = \phi)$

$= \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$

$P(7 \text{ or } 11) = \frac{2}{9}$

(iii) $P(11 \text{ or } 12) = P(B \text{ or } C) = P(B \cup C)$

$= P(B) + P(C) \quad (\because B \text{ and } C \text{ are mutually exclusive})$

$= \frac{2}{36} + \frac{1}{36} = \frac{3}{36} = \frac{1}{12}$

$P(11 \text{ or } 12) = \frac{1}{12}$

Example 10.4: Three letters are written to three different persons and addresses on three envelopes are also written. Without looking at the addresses, what is the probability that (i) all the letters go into right envelopes, (ii) none of the letters goes into right envelopes?

Solution:

Let A, B and C denote the envelopes and 1, 2 and 3 denote the corresponding letters.

The different combination of letters put into the envelopes are shown below:

Outcomes

		c_1	c_2	c_3	c_4	c_5	c_6
Envelopes	A	1	1	2	2	3	3
	B	2	3	1	3	1	2
	C	3	2	3	1	2	1

Let X be the event of putting the letters go into right envelopes.

Y be the event of putting none of the letters go into right envelope.

$S = \{c_1, c_2, c_3, c_4, c_5, c_6\}$, $n(S) = 6$

$X = \{c_1\}$, $n(X) = 1$ $Y = \{c_4, c_5\}$, $n(Y) = 2$

$$\therefore P(X) = \frac{1}{6}$$

$$P(Y) = \frac{2}{6} = \frac{1}{3}$$

Example 10.5: A cricket club has 15 members, of whom only 5 can bowl. What is the probability that in a team of 11 members atleast 3 bowlers are selected?

Let A, B and C be the three possible events of selection. The number of combinations are shown below.

Event	Combination of 11 players		Number of ways the combination formed		Total number of ways the selection can be done
	5 Bowlers	10 Others	5 Bowlers	10 Others	
A	3	8	$5c_3$	$10c_8$	$5c_3 \times 10c_8$
B	4	7	$5c_4$	$10c_7$	$5c_4 \times 10c_7$
C	5	6	$5c_5$	$10c_6$	$5c_5 \times 10c_6$

Solution:

Total number of exhaustive cases = Combination of 11 players from 15 members

$n(S) = 15c_{11}$

$$\text{P (atleast 3 bowlers)} = P[A \text{ or } B \text{ or } C]$$
$$= P[A \cup B \cup C]$$
$$= P(A) + P(B) + P(C) \quad \left(\because \begin{array}{l} \text{A, B and C are} \\ \text{mutually exclusive events} \end{array}\right)$$
$$= \frac{5C_3 \times 10C_8}{15C_{11}} + \frac{5C_4 \times 10C_7}{15C_{11}} + \frac{5C_5 \times 10C_6}{15C_{11}}$$
$$= \frac{5C_2 \times 10C_2}{15C_4} + \frac{5C_1 \times 10C_3}{15C_4} + \frac{5C_0 \times 10C_4}{15C_4} \quad (\because nCr = nCn{-}r)$$
$$= \frac{450}{1365} + \frac{600}{1365} + \frac{210}{1365} = \frac{1260}{1365}$$
$$\text{P (atleast 3 bowlers)} = \frac{12}{13}$$

EXERCISE 10.1

(1) An experiment has the four possible mutually exclusive outcomes A, B, C and D. Check whether the following assignments of probability are permissible.

(i) P(A) = 0.37, P(B) = 0.17, P(C) = 0.14, P(D) = 0.32

(ii) P(A) = 0.30, P(B) = 0.28, P(C) = 0.26, P(D) = 0.18

(iii) P(A) = 0.32, P(B) = 0.28, P(C) = – 0.06, P(D) = 0.46

(iv) P(A) = 1/2, P(B) = 1/4, P(C) = 1/8, P(D) = 1/16

(v) P(A) = 1/3, P(B) = 1/6, P(C) = 2/9, P(D) = 5/18

(2) In a single throw of two dice, find the probability of obtaining (i) sum of less than 5 (ii) a sum of greater than 10, (iii) a sum of 9 or 11.

(3) Three coins are tossed once. Find the probability of getting (i) exactly two heads (ii) atleast two heads (iii) atmost two heads.

(4) A single card is drawn from a pack of 52 cards. What is the probability that

(i) the card is a jack or king (ii) the card will be 5 or smaller

(iii) the card is either queen or 7.

(5) A bag contains 5 white and 7 black balls. 3 balls are drawn at random. Find the probability that (i) all are white (ii) one white and 2 black.

(6) In a box containing 10 bulbs, 2 are defective. What is the probability that among 5 bulbs chosen at random, none is defective.

(7) 4 mangoes and 3 apples are in a box. If two fruits are chosen at random, the probability that (i) one is a mango and the other is an apple (ii) both are of the same variety.

(8) Out of 10 outstanding students in a school there are 6 girls and 4 boys. A team of 4 students is selected at random for a quiz programme. Find the probability that there are atleast 2 girls.

(9) What is the chance that (i) non-leap year (ii) leap year should have fifty three Sundays?

(10) An integer is chosen at random from the first fifty positive integers. What is the probability that the integer chosen is a prime or multiple of 4.

10.3 Some basic theorems on probability

In the development of probability theory, all the results are derived directly or indirectly using only the axioms of probability. Here we study some of the important theorems on probability.

Theorem 10.1: The probability of the impossible event is zero i.e. $\boxed{\mathbf{P(\phi) = 0}}$

Proof:

Impossible event contains no sample point.

$$\therefore S \cup \phi = S$$

$$P(S \cup \phi) = P(S)$$

$$P(S) + P(\phi) = P(S) \quad (\because S \text{ and } \phi \text{ are mutually exclusive})$$

$$\therefore P(\phi) = 0$$

Theorem 10.2:

If $\bar{A}$ is the complementary event of A, $\boxed{\mathbf{P(\bar{A}) = 1 - P(A)}}$

Proof:

Let S be a sample space, we have

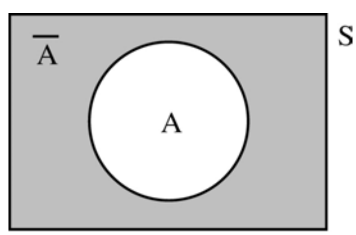


Fig. 10. 1

$$A \cup \bar{A} = S$$

$$P(A \cup \bar{A}) = P(S)$$

$$P(A) + P(\bar{A}) = 1$$

($\because$ A and $\bar{A}$ are mutually exclusive and P(S) = 1)

$$\therefore P(A) = 1 - P(\bar{A})$$

Theorem 10.3: If A and B are any two events and $\overline{B}$ is the complimentary event of B

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

Proof: A is the union of two mutually exclusive events $(A \cap \overline{B})$ and $(A \cap B)$ (see fig 10.2)

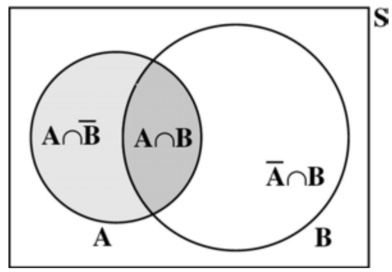


Fig. 10. 2

i.e. $A = (A \cap \overline{B}) \cup (A \cap B)$

$\therefore$ P(A) = $P[(A \cap \overline{B}) \cup (A \cap B)]$

($\because (A \cap \overline{B})$ and $(A \cap B)$ are mutually exclusive)

$$P(A) = P(A \cap \overline{B}) + P(A \cap B)$$

rearranging, we get $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Similarly $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

Theorem 10.4: (Additive theorem on probability) If A and B are any two events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: We have

$A \cup B = (A \cap \overline{B}) \cup B$ (See fig. 10.3)

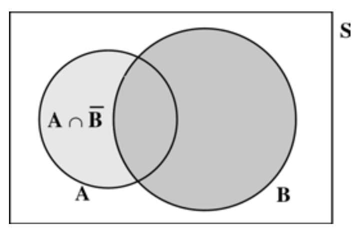


Fig. 10. 3

$P(A \cup B) = P[(A \cap \overline{B}) \cup B]$

($\because A\cap \overline{B}$ and B are mutually exclusive event)

$= P(A \cap \overline{B}) + P(B)$

$= [P(A) - P(A \cap B)] + P(B)$ (by theorem 3)

$\therefore P(A\cup B) = P(A) + P(B) - P(A \cap B)$

Note: The above theorem can be extended to any 3 events.

$$P(A\cup B\cup C) = \{P(A)+P(B)+P(C)\} - \{P(A \cap B)+P(B \cap C)+P(C \cap A)\} + P(A\cap B\cap C)$$

Example 10.6:

Given that P(A) = 0.35, P(B) = 0.73 and $P(A \cap B) = 0.14$, find

(i) $P(A \cup B)$ (ii) $P(\bar{A} \cap B)$ (iii) $P(A \cap \bar{B})$ (iv) $P(\bar{A} \cup \bar{B})$ (v) $P(\overline{A \cup B})$

Solution:

(i)
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.35 + 0.73 - 0.14 = 0.94$$
$$P(A \cup B) = 0.94$$

(ii)
$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$
$$= 0.73 - 0.14 = 0.59$$
$$P(\bar{A} \cap B) = 0.59$$

(iii)
$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$
$$= 0.35 - 0.14 = 0.21$$
$$P(A \cap \bar{B}) = 0.21$$

(iv)
$$P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - 0.14$$
$$P(\bar{A} \cup \bar{B}) = 0.86$$

(v)
$$P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.94 = 0.06 \qquad \text{(by (1))}$$
$$P(\overline{A \cup B}) = 0.06$$

Example 10.7: A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability of drawing (i) a king or a queen (ii) a king or a spade (iii) a king or a black card

Solution:

Total number of cases = 52

i.e. $n(S) = 52$

Let A be the event of drawing a king ; B be the event of drawing a queen

C be the event of drawing a spade; D be the event of drawing a black card

$\therefore$ $n(A) = 4$, $n(B) = 4$, $n(C) = 13$, $n(D) = 26$

also we have $n(A \cap C) = 1$, $n(A \cap D) = 2$

(i) P [king or queen] = [A or B] = $P(A \cup B)$

$$= P(A) + P(B) \qquad \left(\because \text{A and B are mutually exclusive i.e. } A \cap B = \phi\right)$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$$

$$= \frac{4}{52} + \frac{4}{52} = \frac{2}{13}$$

(ii) P[king or spade] $= P(A \text{ or } C) = P(A \cup C)$

$$= P(A)+P(C)-P(A\cap C)$$

$\left(\because \text{A and C are not mutually exclusive}\right)$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$$

$$= \frac{4}{13}$$

(iii)P[king or black card]= $P(A \text{ or } D) = P(A \cup D)$

$$= P(A) + P(D) - P(A \cap D) \qquad \left(\because \text{A and D are not mutually exclusive}\right)$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

$$= \frac{7}{13}$$

Example 10.8: The probability that a girl will get an admission in IIT is 0.16, the probability that she will get an admission in Government Medical College is 0.24, and the probability that she will get both is 0.11. Find the probability that (i) She will get atleast one of the two seats (ii) She will get only one of the two seats

Solution:

Let I be the event of getting admission in IIT and M be the event of getting admission in Government Medical College.

$\therefore$ P(I) = 0.16, P(M) = 0.24 and $P(I \cap M) = 0.11$

(i) P(atleast one of the two seats)

$$= P(I \text{ or } M) = P(I \cup M)$$

$$= P(I) + P(M) - P(I \cap M)$$

$$= 0.16 + 0.24 - 0.11$$

$$= 0.29$$

(ii) P(only one of two seats) = P[only I or only M].

$= P[(I \cap \overline{M}) \cup (\overline{I} \cap M)]$

$= P(I \cap \overline{M}) + P(\overline{I} \cap M)$

$= \{P(I) - P(I \cap M)\} + \{P(M) - P(I \cap M)\}$

$= \{0.16 - 0.11\} + \{0.24 - 0.11\}$

$= 0.05 + 0.13$

$= 0.18$

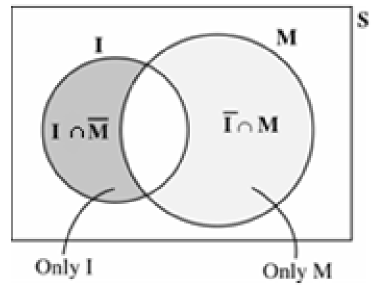


Fig. 10. 4

EXERCISE 10.2

(1) A and B are two events associated with random experiment for which P(A) = 0.36, P(A or B) = 0.90 and P(A and B) = 0.25. Find (i) P(B), (ii) $P(\overline{A} \cap \overline{B})$

(2) If A and B are mutually exclusive events P(A) = 0.28, P(B) = 0.44, find

(i) $P(\overline{A})$ (ii) $P(A \cup B)$ (iii) $(A \cap \overline{B})$ (iv) $P(\overline{A} \cap \overline{B})$

(3) Given P(A) = 0.5, P(B) = 0.6 and $P(A \cap B) = 0.24$.

Find (i) $P(A \cup B)$ (ii) $P(\overline{A} \cap B)$ (iii) $P(A \cap \overline{B})$

(iv) $P(\overline{A} \cup \overline{B})$ (v) $P(\overline{A} \cap \overline{B})$

(4) A die is thrown twice. Let A be the event. "First die shows 4' and B be the event, 'second die shows 4'. Find $P(A \cup B)$.

(5) The probability of an event A occurring is 0.5 and B occurring is 0.3. If A and B are mutually exclusive events, then find the probability of neither A nor B occurring

(6) A card is drawn at random from a deck of 52 cards. What is the probability that the drawn card is (i) a queen or club card (ii) a queen or a black card

(7) The probability that a new ship will get an award for its design is 0.25, the probability that it will get an award for the efficient use of materials is 0.35, and that it will get both awards is 0.15. What is the probability, that

(i) it will get atleast one of the two awards (ii) it will get only one of the awards

10.4 Conditional probability:

Consider the following example to understand the concept of conditional probability.

Suppose a fair die is rolled once. The sample space is S = {1, 2, 3, 4, 5, 6}.

Now we ask two questions:

Q 1: What is the probability that getting an even number which is less than 4?

Q2 : If the die shows an even number, then what is the probability that it is less than 4?

Case 1:

The event of getting an even number which is less than 4 is {2}

$$\therefore P_1 = \frac{n(\{2\})}{n(\{1, 2, 3, 4, 5, 6\})} = \frac{1}{6}$$

Case 2:

Here first we restrict our sample space S to a subset containing only even number i.e. to {2, 4, 6}. Then our interest is to find the probability of the event getting a number less than 4 i.e. to {2}.

$$\therefore P_2 = \frac{n(\{2\})}{n(\{2, 4, 6\})} = \frac{1}{3}$$

In the above two cases the favourable events are the same, but the number of exhaustive outcomes are different. In case 2, we observe that we have first imposed a condition on sample space, then asked to find the probability. This type of probability is called conditional probability.

Definition: (**Conditional probability**) : The conditional probability of an event B, assuming that the event A has already happened; is denoted by P (B/A) and defined as

$$\boxed{\mathbf{P(B/A) = \frac{P(A \cap B)}{P(A)}}} \quad \text{provided } P(A) \neq 0$$

Similarly

$$\boxed{\mathbf{P(A/B) = \frac{P(A \cap B)}{P(B)}}} \quad \text{provided } P(B) \neq 0$$

Example 10.9: If $P(A) = 0.4 \quad P(B) = 0.5 \quad P(A \cap B) = 0.25$

Find (i) $P(A/B)$ (ii) $P(B/A)$ (iii) $P(\overline{A}/B)$

(iv) $P(B/\overline{A})$ (v) $P(A/\overline{B})$ (vi) $P(\overline{B}/A)$

Solution:

(i) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.25}{0.50} = 0.5$

(ii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.25}{0.40} = 0.625$

(iii) $P(\bar{A}/B) = \dfrac{P(\bar{A} \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{0.5 - 0.25}{0.5} = 0.5$

(iv) $P(B/\bar{A}) = \dfrac{P(B \cap \bar{A})}{P(\bar{A})} = \dfrac{P(B) - P(A \cap B)}{1 - P(A)} = \dfrac{0.5 - 0.25}{1 - 0.4} = 0.4167$

(v) $P(A/\bar{B}) = \dfrac{P(A \cap \bar{B})}{P(\bar{B})} = \dfrac{P(A) - P(A \cap B)}{1 - P(B)} = \dfrac{0.4 - 0.25}{1 - 0.5} = 0.3$

(vi) $P(\bar{B}/A) = \dfrac{P(A \cap \bar{B})}{P(A)} = \dfrac{P(A) - P(A \cap B)}{P(A)} = \dfrac{0.4 - 0.25}{0.4} = 0.375$

Theorem 10.6 : (Multiplication theorem on probability)

The probability of the simultaneous happening of two events A and B is given by

$$\mathbf{P(A \cap B) = P(A) \,.\, P(B/A)}$$
$$\textbf{or} \quad \mathbf{P(A \cap B) = P(B) \,.\, P(A/B)}$$

Note: Rewriting the definition of conditional probability, we get the above 'multiplication theorem on probability'.

Independent Events:

Events are said to be independent if the occurrence or non occurrence of any one of the event does not affect the **probability** of occurrence or non-occurrence of the other events.

Definition: Two events A and B are **independent** if $\boxed{\mathbf{P(A \cap B) = P(A) \,.\, P(B)}}$

This definition is exactly equivalent to

$$P(A/B) = P(A), \quad P(B/A) = P(B)$$

Note: The events $A_1, A_2 \ldots\ldots A_n$ are mutually independent if

$$P(A_1 \cap A_2 \cap A_3 \ldots\ldots A_n) = P(A_1).\, P(A_2) \ldots (A_n)$$

Corollary 1: If A and B are independent then A and $\bar{B}$ are also independent.

Proof:

Since A and B are independent

$$P(A \cap B) = P(A) . (PB) \quad \ldots (1)$$

To prove A and $\overline{B}$ are independent, we have to prove

$$P(A \cap \overline{B}) = P(A) . P(\overline{B})$$

We know

$$\begin{aligned} P(A \cap \overline{B}) &= P(A) - P(A \cap B) \\ &= P(A) - P(A) . P(B) \quad \text{(by (1))} \\ &= P(A)\,[1 - P(B)] \end{aligned}$$

$$P(A \cap \overline{B}) = P(A) . P(\overline{B})$$

$\therefore$ A and $\overline{B}$ are independent.

Similarly, the following corollary can easily be proved.

Corollary 2: If A and B are independent, then $\overline{A}$ and $\overline{B}$ are also independent.

Note: If A_1, A_2… A_n are mutually independent then $\overline{A}_1$, $\overline{A}_2$, … $\overline{A}_n$ are mutually independent.

Example 10.10: Two cards are drawn from a pack of 52 cards in succession. Find the probability that both are kings when

(i) The first drawn card is replaced (ii) The card is not replaced

Solution:

Let A be the event of drawing a king in the first draw.

B be the event of drawing a king in the second draw.

Case i: Card is replaced:

$$n(A) = 4 \quad \text{(king)}$$
$$n(B) = 4 \quad \text{(king)}$$
$$\text{and } n(S) = 52 \quad \text{(Total)}$$

Clearly the event A will not affect the probability of the occurrence of event B and therefore A and B are independent.

$$\begin{aligned} P(A \cap B) &= P(A) . P(B) \\ &= \frac{4}{52} \times \frac{4}{52} \end{aligned}$$

$$P(A \cap B) = \frac{1}{169}$$

Case ii: (Card is not replaced)

In the first draw, there are 4 kings and 52 cards in total. Since the king, drawn at the first draw is not replaced, in the second draw there are only 3 kings and 51 cards in total. Therefore the first event A affects the probability of the occurrence of the second event B.

$\therefore$ A and B are not independent they are dependent events.

$$\therefore P(A \cap B) = P(A) . P(B/A)$$

$$P(A) = \frac{4}{52} \quad ; \quad P(B/A) = \frac{3}{51}$$

$$P(A \cap B) = P(A) . P(B/A) = \frac{4}{52} \cdot \frac{3}{51}$$

$$P(A \cap B) = \frac{1}{221}$$

Example 10.11: A coin is tossed twice. Event E and F are defined as follows : E = Head on first toss, F = head on second toss.

Find (i) $P(E \cap F)$ (ii) $P(E \cup F)$ (iii) $P(E/F)$

(iv) $P(\bar{E}/F)$ (v) Are the events E and F independent ?

Solution: The sample space is

$$S = \{(H,H), (H, T), (T, H), (T, T)\}$$

$$\text{and} \quad E = \{(H, H), (H T)\}$$

$$F = \{(H, H), (T, H)\}$$

$$\therefore E \cap F = \{(H, H)\}$$

(i) $$P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{1}{4}$$

(ii) $$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4}$$

$$P(E \cup F) = \frac{3}{4}$$

(iii) $$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(iv) $$P(\bar{E}/F) = \frac{P(\bar{E} \cap F)}{P(F)} = \frac{P(F) - P(E \cap F)}{P(F)}$$

$$= \frac{2/4 - 1/4}{2/4} = \frac{1}{2}$$

$$P(\bar{E}/F) = \frac{1}{2}$$

(v) $$P(E) = \frac{2}{4} = \frac{1}{2}, \quad P(F) = \frac{2}{4} = \frac{1}{2}$$

$$P(E \cap F) = \frac{1}{4}$$

$$\therefore \; P(E)\,P(F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Since $P(E \cap F) = P(E) \,.\, P(F)$, E and F are independent.

In the above example the events E and F are not mutually exclusive but they are independent.

Important Note:

Independence is a property of probability but **mutually exclusion is a set-theoretic property**. Therefore independent events can be identified by their probabilities and mutually exclusive events can be identified by their events.

Theorem 10.7: Suppose A and B are two events, such that $P(A) \neq 0$, $P(B) \neq 0$

(i) If A and B are mutually exclusive, they cannot be independent.

(ii) If A and B are independent they cannot be mutually exclusive.

(Proof not required)

Example 10.12: If A and B are two independent events such that $P(A) = 0.5$ and $P(A \cup B) = 0.8$. Find P(B).

Solution:

We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = p(A)+P(B)-P(A).P(B)$ ($\because$ A and B are independent)

i.e. $0.8 = 0.5 + P(B) - (0.5)\,P(B)$

$0.8 - 0.5 = (1 - 0.5)\,P(B)$

$$\therefore \; P(B) = \frac{0.3}{0.5} = 0.6$$

$P(B) = 0.6$

Example 10.13: A problem is given to 3 students X, Y and Z whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{5}$ respectively. What is the probability that the problem is solved?

Solution:

Let A, B and C be the events of solving the problem by X, Y and Z respectively.

$$\therefore\ P(A) = \frac{1}{2}\ ;\ P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(B) = \frac{1}{3}\ ;\ P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(C) = \frac{2}{5}\ ;\ P(\overline{C}) = 1 - P(C) = 1 - \frac{2}{5} = \frac{3}{5}$$

P[problem is solved] = P[the problem is solved by atleast one of them]

$$= P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C})$$

$$= 1 - P(\overline{A} \cap \overline{B} \cap \overline{C}) \text{ (By De Morgan's Law)}$$

$$= 1 - P(\overline{A}) \, . \, P(\overline{B}) \, . \, P(\overline{C})$$

($\because$ A, B, C are independent $\overline{A}$, $\overline{B}$, $\overline{C}$ are also independent)

$$= 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{5} = 1 - \frac{1}{5}$$

$$\text{P[problem is solved]} = \frac{4}{5}$$

Examples 10.14 : X speaks truth in 95 percent of cases, and Y in 90 percent of cases. In what percentage of cases are they likely to contradict each other in stating the same fact.

Solution: Let A be the event of X speaks the truth, B be the event of Y speaks the truth.

$\therefore$ $\overline{A}$ and $\overline{B}$ are the events of not speaking the truth by X and Y respectively.

Let C be the event that they will contradict each other.

Given that

$$P(A) = 0.95 \quad \therefore P(\overline{A}) = 1 - P(A) = 0.05$$

$$P(B) = 0.90 \quad \therefore P(\overline{B}) = 1 - P(B) = 0.10$$

C = (A speaks truth and B does not speak truth
or
B Speaks truth and A does not speak truth)

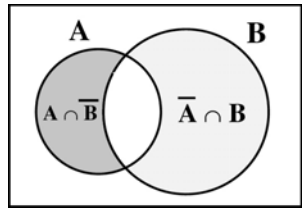


Fig. 10. 5

$C = [(A \cap \overline{B}) \cup (\overline{A} \cap B)]$

$\therefore P(C) = P[(A \cap \overline{B}) \cup (\overline{A} \cap B)]$

$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$

($\because \overline{A} \cap B$ and $A \cap \overline{B}$ are mutually exclusive)

$= P(A) . P(\overline{B}) + P(\overline{A}) . P(B)$ ($\because A, \overline{B}$ are independent event also $\overline{A}$, B are independent events)

$= (0.95) \times (0.10) + (0.05) (0.90)$
$= 0.095 + 0.045$
$= 0.1400$
P(C) =14%

EXERCISE 10.3

(1) Define independent and mutually exclusive events. Can two events be mutually exclusive and independent simultaneously.

(2) If A and B are independent, prove that $\overline{A}$ and $\overline{B}$ are independent.

(3) If P(A) = 0.4, P(B) = 0.7 and P(B / A) = 0.5 find P(A / B) and $P(A \cup B)$.

(4) If for two events A and B, P(A) = 2/5, P(B) =3/4 and $A \cup B$ = (sample space), find the conditional probability P(A / B).

(5) If A and B are two independent events such that $P(A \cup B) = 0.6$, P(A) = 0.2 find P(B)

(6) If A and B are two events such that $P(A \cup B) = 5/6$, $P(A \cap B) = 1/3$, $P(\overline{B}) = 1/2$ show that A and B are independent.

(7) if the events A and B are independent and P(A) = 0.25, P(B) = 0.48,

find (i) $P(A \cap B)$ (ii) P(B / A) (iii) $P(\overline{A} \cap \overline{B})$

(8) Given P(A) = 0.50, P(B) = 0.40 and $P(A \cap B) = 0.20$.

Verify that (i) P(A / B) = P(A), (ii) $P(A / \overline{B}) = P(A)$

(iii) P(B / A) = P(B) (iv) $P(B / \overline{A}) = P(B)$

(9) $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cap B) = 0.25$

Find (i) $P(A \cup B)$ (ii) $P(A/B)$ (iii) $P(B/\bar{A})$ (iv) $P(\bar{A}/B)$ (v) $P(\bar{A}/\bar{B})$

(10) Given $P(A) = 0.45$ and $P(A \cup B) = 0.75$.

Find P(B) if (i) A and B are mutually exclusive (ii) A and B are independent events (iii) $P(A / B) = 0.5$ (iv) $P(B / A) = 0.5$

(11) Two cards are drawn one by one at random from a deck of 52 playing cards. What is the probability of getting two jacks if (i) the first card is replaced before the second is drawn (ii) the first cards is not replaced before the second card is drawn.

(12) If a card is drawn from a deck of 52 playing cards, what is the probability of drawing (i) a red king (ii) a red ace or a black queen.

(13) One bag contains 5 white and 3 black balls. Another bag contains 4 white and 6 black balls. If one ball is drawn from each bag, find the probability that (i) both are white (ii) both are black (iii) one white and one black.

(14) A husband and wife appear in an interview for two vacancies in the same post. The probability of husbands' selection is 1/6 and that of wife's selection is 1/5. What is the probability that

(i) both of them will be selected (ii) only one of them will be selected

(iii) none of them will be selected

(15) A problem in Mathematics is given to three students whose chances of solving it are 1/2, 1/3 and 1/4 (i) What is the probability that the problem is solved (ii) what is the probability that exactly one of them will solve it.

(16) A year is selected at random. What is the probability that (i) it contains 53 Sundays (ii) it is a leap year contains 53 Sundays

(17) For a student the probability of getting admission in IIT is 60% and probability of getting admission in Anna University is 75%. Find the probability that (i) getting admission in only one of these (ii) getting admission in atleast one of these.

(18) A can hit a target 4 times in 5 shots, B 3 times in 4 shots, C 2 times in 3 shots, they fire a volley. What is the chance that the target is damaged by exactly 2 hits?

(19) Two thirds of students in a class are boys and rest girls. It is known that the probability of a girl getting a first class is 0.75 and that of a boys is 0.70. Find the probability that a student chosen at random will get first class marks.

(20) A speaks truth in 80% cases and B in 75% cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

10.5 Total probability of an event

If A_1, A_2 … A_n are mutually exclusive and exhaustive events and B is any event in S then

$$\mathbf{P(B) = P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2) \; \dots + P(A_n) \, P(B/A_n)}$$

P(B) is called the total probability of event B

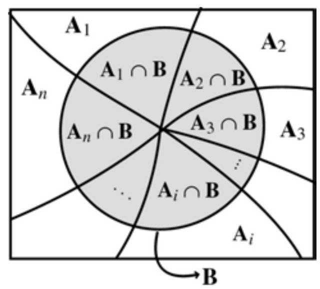


Fig. 10.7

Example 10.15: An urn contains 10 white and 5 black balls. While another urn contains 3 white and 7 black balls. One urn is chosen at random and two balls are drawn from it. Find the probability that both balls are white.

Solution:

Let A_1 be the event of selecting urn-I and A_2 be the event of selecting urn-II. Let B be the event of selecting 2 white balls.

We have to find the total probability of event B i.e. P(B). Clearly A_1 and A_2 are mutually exclusive and exhaustive events.

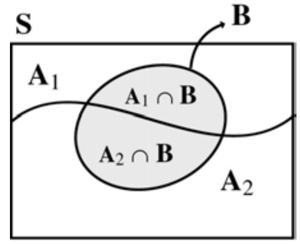


Fig. 10.8

$$P(B) = P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2) \quad \dots (1)$$

$$P(A_1) = \frac{1}{2} \; ; P(B/A_1) = \frac{10C_2}{15C_2}$$

$$P(A_2) = \frac{1}{2} \; ; \; P(B/A_2) = \frac{3C_2}{10C_2}$$

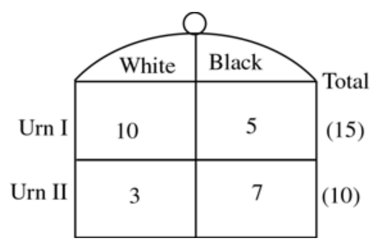


Fig. 10.9

Substituting in (1), $P(B) = P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2)$

$$= \left(\frac{1}{2}\right)\left(\frac{10C_2}{15C_2}\right) + \left(\frac{1}{2}\right)\left(\frac{3C_2}{10C_2}\right) = \frac{1}{2}\left[\frac{3}{7} + \frac{1}{15}\right]$$

$$P(B) = \frac{26}{105}$$

Example 10.16: A factory has two machines I and II. Machine I produces 30% of items of the output and Machine II produces 70% of the items. Further 3% of items produced by Machine I are defective and 4% produced by Machine II are defective. If an item is drawn at random, find the probability that it is a defective item.

Solution:

Let A_1 be the event that the items are produced by Machine I, A_2 be the event that items are produced by Machine II. Let B be the event of drawing a defective item.

$$\therefore P(A_1) = \frac{30}{100} \; ; \; P(B/A_1) = \frac{3}{100}$$

$$P(A_2) = \frac{70}{100} \; ; \; P(B/A_2) = \frac{4}{100}$$

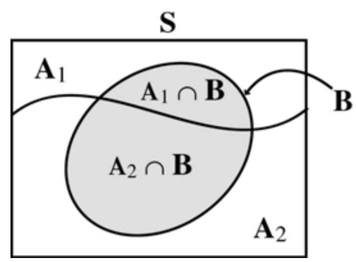


Fig. 10.10

We are asked to find the total probability of event B.

Since A_1, A_2 are mutually exclusive and exhaustive.

We have $P(B) = P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2)$

$$= \left(\frac{30}{100}\right)\left(\frac{3}{100}\right) + \left(\frac{70}{100}\right) \cdot \left(\frac{4}{100}\right)$$

$$= \frac{90 + 280}{10000}$$

$$P(B) = 0.0370$$

Theorem 10.8: (Bayes' Theorem):

Suppose $A_1, A_2, \ldots A_n$ are n mutually exclusive and exhaustive events such that $P(A_i) > 0$ for $i = 1, 2 \ldots n$. Let B be any event with $P(B) > 0$ then

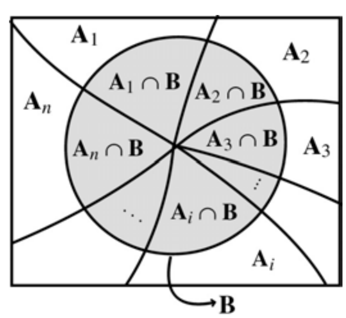


Fig. 10.11

$$\mathbf{P(A_i/B) = \frac{P(A_i)\ P(B/A_i)}{P(A_1)\ P(B/A_1) + P(A_2)\ P(B/A_2) + \ldots + P(A_n)\ P(B/A_n)}}$$

(Proof not required)

The above formula gives the relationship between $P(A_i/B)$ and $P(B/ A_i)$

Example 10.17: A factory has two machines I and II. Machine I and II produce 30% and 70% of items respectively. Further 3% of items produced by Machine I are defective and 4% of items produced by Machine II are defective. An item is drawn at random. If the drawn item is defective, find the probability that it was produced by Machine II. (See the previous example, compare the questions).

Solution:

Let A_1 and A_2 be the events that the items produced by Machine I & II respectively.

Let B be the event of drawing a defective item.

$$\therefore P(A_1) = \frac{30}{100} \quad ; P(B/ A_1) = \frac{3}{100}$$

$$P(A_2) = \frac{70}{100} \quad ; P(B/ A_2) = \frac{4}{100}$$

Now we are asked to find the conditional probability $P(A_2/ B)$

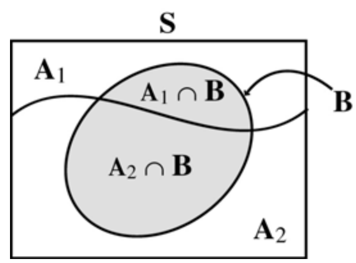


Fig. 10.12

Since A_1, A_2 are mutually exclusive and exhaustive events by Bayes' theorem

$$P(A_2/ B) = \frac{P(A_2) . P(B/ A_2)}{P(A_1) . P(B/ A_1) + P(A_2) . P(B/ A_2)}$$

$$= \frac{\left(\frac{70}{100}\right) \times \left(\frac{4}{100}\right)}{\left(\frac{30}{100}\right)\left(\frac{3}{100}\right) + \left(\frac{70}{100}\right)\left(\frac{4}{100}\right)} = \frac{0.0280}{0.0370} = \frac{28}{37}$$

$$P(A_2/ B) = \frac{28}{37}$$

Example 10.18: The chances of X, Y and Z becoming managers of a certain company are 4 : 2 : 3. The probabilities that bonus scheme will be introduced if X, Y and Z become managers are 0.3, 0.5 and 0.4 respectively. If the bonus scheme has been introduced, what is the probability that Z is appointed as the manager.

Solution:

Let A_1, A_2 and A_3 be the events of X, Y and Z becoming managers of the company respectively. Let B be the event that the bonus scheme will be introduced.

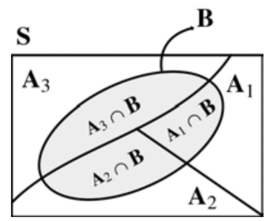


Fig. 10.13

$$\therefore P(A_1) = \frac{4}{9} \quad ; P(B/ A_1) = 0.3$$

$$P(A_2) = \frac{2}{9} \quad ; P(B/ A_2) = 0.5$$

$$P(A_3) = \frac{3}{9} \quad ; P(B/ A_3) = 0.4$$

We have to find the conditional probability $P(A_3/ B)$

A_1, A_2 and A_3 are mutually exclusive and exhaustive events. Applying Bayes' formula

$$P(A_3/ B) = \frac{P(A_3) \cdot P(B/ A_3)}{P(A_1) \cdot P(B/ A_1) + P(A_2) \cdot P(B/ A_2) + P(A_3) \cdot P(B/ A_3)}$$

$$= \frac{\left(\frac{3}{9}\right)(0.4)}{\left(\frac{4}{9}\right)(0.3) + \left(\frac{2}{9}\right)(0.5) + \left(\frac{3}{9}\right)(0.4)} = \frac{12}{34}$$

$$P(A_3/ B) = \frac{6}{17}$$

Example 10.19: A consulting firm rents car from three agencies such that 20% from agency X, 30% from agency Y and 50% from agency Z. If 90% of the cars from X, 80% of cars from Y and 95% of the cars from Z are in good conditions (1) what is the probability that the firm will get a car in good condition? Also (ii) If a car is in good condition, what is probability that it has came from agency Y?

Solution:

Let A_1, A_2, A_3 be the events that the cars are rented from the agencies X, Y and Z respectively.

Let G be the event of getting a car in good condition.

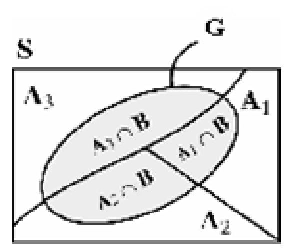


Fig. 10.14

$\therefore$ $P(A_1) = 0.20$; $P(G/A_1) = 0.90$

$P(A_2) = 0.30$; $P(G/A_2) = 0.80$

$P(A_3) = 0.50$; $P(G/A_3) = 0.95$

(i) We have to find the total probability of event G i.e. P(G)

Since A_1, A_2, A_3 are mutually exclusive and exhaustive events and G is an event in S.

We have

$$P(G) = P(A_1) . P(G/A_1) + P(A_2) . P(G/A_2) + P(A_3) . P(G/A_3)$$

$$= (0.2)(0.90) + (0.3)(0.80) + (0.5)(0.95)$$

$$= 0.180 + 0.240 + 0.475$$

$$P(G) = 0.895$$

(ii) We have to find the conditional probability A_2 given G i.e. $P(A_2/G)$

By Bayes' formula

$$P(A_2/G) = \frac{P(A_2) . P(G/A_2)}{P(A_1) . P(G/A_1) + P(A_2) . P(G/A_2) + P(A_3) . P(G/A_3)}$$

$$= \frac{(0.3)(0.80)}{(0.895)} \qquad \text{(by (1) Dr } = P(G) = 0.895)$$

$$= \frac{0.240}{0.895}$$

$$P(A_2/G) = 0.268 \text{ (Approximately)}$$

EXERCISE 10.4

(1) Bag A contains 5 white, 6 black balls and bag B contains 4 white, 5 black balls. One bag is selected at random and one ball is drawn from it. Find the probability that it is white.

(2) A factory has two Machines-I and II. Machine-I produces 25% of items and Machine-II produces 75% of the items of the total output. Further 3% of the items produced by Machine-I are defective whereas 4% produced by Machine-II are defective. If an item is drawn at random what is the probability that it is defective?

(3) There are two identical boxes containing respectively 5 white and 3 red balls, 4 white and 6 red balls. A box is chosen at random and a ball is drawn from it (i) find the probability that the ball is white (ii) if the ball is white, what is the probability that it is from the first box?

(4) In a factory, Machine-I produces 45% of the output and Machine-II produces 55% of the output. On the average 10% items produced by I and 5% of the items produced by II are defective. An item is drawn at random from a day's output. (i) Find the probability that it is a defective item (ii) If it is defective, what is the probability that it was produced by Machine-II.

(5) Three urns are given each containing red and white chips as given below.

Urn I : 6 red 4 white ; Urn II : 3 red 5 white ; Urn III : 4 red 6 white

An urn is chosen at random and a chip is drawn from the urn.

(i) Find the probability that it is white

(ii) If the chip is white find the probability that it is from urn II

OBJECTIVE TYPE QUESTIONS

(1) Identify the correct statement

(1) The set of real numbers is a closed set

(2) The set of all non-negative real numbers is represented by $(0, \infty)$

(3) The set $[3, 7]$ indicates the set of all natural numbers between 3 and 7

(4) $(2, 3)$ is a subset of $[2, 3]$.

(2) Identify the correct statements of the following

(i) a relation is a function

(ii) a function is a relation

(iii) 'a function which is not a relation' is not possible

(iv) 'a relation which is not a function' is possible

(1) (ii), (iii) and (iv) (2) (ii) and (iii) (3) (iii) and (iv) (4) all

(3) Which one of the following is a function which is 'onto'?

(1) $f : R \to R$; $f(x) = x^2$ (2) $f : R \to [1,\infty)$; $f(x) = x^2 + 1$

(3) $f : R \to \{1, -1\}$; $f(x) = \frac{|x|}{x}$ (4) $f : R \to R$; $f(x) = -x^2$

(4) Which of the following is a function which is not one-to-one?

(1) $f : R \to R$; $f(x) = x + 1$ (2) $f : R \to R$; $f(x) = x^2 + 1$

(3) $f : R \to \{1, -1\}$; $f(x) = x - 1$ (4) $f : R \to R$; $f(x) = -x$

(5) The inverse of $f : R \to R^+$; $f(x) = x^2$ is

(1) not onto (2) not one-to-one

(3) not onto and not one-to-one (4) not at all a function

(6) Identify the correct statements

(i) a constant function is a polynomial function.

(ii) a polynomial function is a quadratic function.

(iii) for linear function, inverse always exists.

(iv) A constant function is one-to-one only if the domain is a singleton set.

(1) (i) and (iii) (2) (i), (iii) and (iv) (3) (ii) and (iii) (4) (i) and (iii)

(7) Identify the correct statements

(i) the domain of circular functions are always R.

(ii) The range of tangent function is R.

(iii) The range of cosine function is same as the range of sine function.

(iv) The domain of cotangent function is $R - \{k\pi\}$

(1) all (2) (i) and (iii) (3) (ii), (iii) and (iv) (4) (iii) and (iv)

(8) The true statements of the following are

(i) The composition of function fog and the product of functions fg are same.

(ii) For the composition of functions fog, the co-domain of g must be the domain of f.

(iii) If fog, gof exist then $fog = gof$.

(iv) If the function f and g are having same domain and co-domain then $fg = gf$

(1) all (2) (ii), (iii) and (iv) (3) (iii) and (iv) (4) (ii) and (iv)

(9) $\lim_{x \to -6} (-6)$ is

(1) 6 (2) -6 (3) 36 (4) -36

(10) $\lim_{x \to -1} (x)$ is

(1) -1 (2) 1 (3) 0 (4) 0.1

(11) The left limit as $x \to 1$ of $f(x) = -x + 3$ is

(1) 2 (2) 3 (3) 4 (4) -4

(12) Rf(0) for $f(x) = |x|$ is

(1) x (2) 0 (3) $-x$ (4) 1

(13) $\lim_{x \to 1} \frac{x^{1/3} - 1}{x - 1}$ is

(1) $\frac{2}{3}$ (2) $-\frac{2}{3}$ (3) $\frac{1}{3}$ (4) $-\frac{1}{3}$

(14) $\lim_{x \to 0} \frac{\sin 5x}{x}$ is

(1) 5 (2) $\frac{1}{5}$ (3) 0 (4) 1

(15) $\lim_{x \to 0} x \cot x$ is

(1) 0 (2) -1 (3) ∞ (4) 1

(16) $\lim_{x \to 0} \frac{2^x - 3^x}{x}$ is

(1) $\log\left(\frac{3}{2}\right)$ (2) $\log\left(\frac{2}{3}\right)$ (3) log 2 (4) log 3

(17) $\lim\limits_{x \to 1} \dfrac{e^x - e}{x - 1}$ is

(1) 1 (2) 0 (3) ∞ (4) e

(18) $\lim\limits_{x \to \infty} \left(1 - \dfrac{1}{x}\right)^x$ is

(1) e (2) $-e$ (3) $\dfrac{1}{e}$ (4) 0

(19) The function $f(x) = |x|$ is

(1) continuous at $x = 0$

(2) discontinuous at $x = 0$

(3) not continuous from the right at $x = 0$

(4) not continuous from the left at $x = 0$

(20) The function $f(x) = \begin{cases} \dfrac{\sin(x-2)}{x-2}, & x \neq 2 \\ 0, & x = 2 \end{cases}$ is discontinuous at

(1) $x = 0$ (2) $x = -1$ (3) $x = -2$ (4) $x = 2$

(21) The function $f(x) = \dfrac{x^2 + 1}{x^2 - 3x + 2}$ is continuous at all points of R except at

(1) $x = 1$ (2) $x = 2$ (3) $x = 1, 2$ (4) $x = -1, -2$

(22) Let $f(x) = \lfloor x \rfloor$ be the greatest integer function. Then

(1) $f(x)$ is continuous at all integral values

(2) $f(x)$ is discontinuous at all integral values

(3) $x = 0$ is the only discontinuous point

(4) $x = 1$ is the only continuous point

(23) The function $y = \tan x$ is continuous at

(1) $x = 0$ (2) $x = \dfrac{\pi}{2}$ (3) $x = \dfrac{3\pi}{2}$ (4) $x = -\dfrac{\pi}{2}$

(24) $f(x) = |x| + |x - 1|$ is

(1) continuous at $x = 0$ only (2) continuous at $x = 1$ only

(3) continuous at both $x = 0$ and $x = 1$ (4) discontinuous at $x = 0, 1$

(25) If $f(x) = \begin{cases} kx^2 \text{ for } x \leq 2 \\ 3 \text{ for } x > 2 \end{cases}$ is continuous at $x = 2$, the value of k is

(1) $\dfrac{3}{4}$ (2) $\dfrac{4}{3}$ (3) 1 (4) 0

(26) R$f'(0)$ for the function $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$ is

(1) 1 (2) 0 (3) – 1 (4) 2

(27) L$f'(\alpha)$ for the function $f(x) = |x - \alpha|$ is

(1) α (2) $-\alpha$ (3) – 1 (4) 1

(28) The function $f(x) = \begin{cases} 2, & x \leq 1 \\ x, & x > 1 \end{cases}$ is not differentiable at

(1) $x = 0$ (2) $x = -1$ (3) $x = 1$ (4) $x = -2$

(29) The derivative of $f(x) = x^2 |x|$ at $x = 0$ is

(1) 0 (2) – 1 (3) – 2 (4) 1

(30) $\int \sin^2 x \, dx =$

(1) $\frac{\sin^3 x}{3} + c$ (2) $-\frac{\cos^2 x}{2} + c$

(3) $\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$ (4) $\frac{1}{2}[1 + \sin 2x] + c$

(31) $\int \sin 7x \cos 5x \, dx =$

(1) $\frac{1}{35} \cos 7x \sin 5x + c$ (2) $-\frac{1}{2}\left[\frac{\cos 12x}{12} + \frac{\cos 2x}{2}\right] + c$

(3) $-\frac{1}{2}\left[\frac{\cos 6x}{6} + \cos x\right] + c$ (4) $\frac{1}{2}\left[\frac{\cos 12x}{12} + \frac{\cos 2x}{2}\right] + c$

(32) $\int \frac{e^x}{e^x + 1} \, dx =$

(1) $\frac{1}{2} x + c$ (2) $\frac{1}{2}\left(\frac{e^x}{1 + e^x}\right)^2 + c$ (3) $\log(e^x + 1) + c$ (4) $x + e^x + c$

(33) $\int \frac{1}{e^x} \, dx =$

(1) $\log e^x + c$ (2) $-\frac{1}{e^x} + c$ (3) $\frac{1}{e^x} + c$ (4) $x + c$

(34) $\int \log x \, dx =$

(1) $\frac{1}{x} + c$ (2) $\frac{(\log x)^2}{2} + c$ (3) $x \log x + x + c$ (4) $x \log x - x + c$

(35) $\int \frac{x}{1 + x^2}\, dx =$

(1) $\tan^{-1}x + c$ (2) $\frac{1}{2}\log(1 + x^2) + c$ (3) $\log(1 + x^2) + c$ (4) $\log x + c$

(36) $\int \tan x\, dx =$

(1) $\log \cos x + c$ (2) $\log \sec x + c$ (3) $\sec^2 x + c$ (4) $\frac{\tan^2 x}{2} + c$

(37) $\int \frac{1}{\sqrt{3 + 4x}}\, dx =$

(1) $\frac{1}{2}\sqrt{3 + 4x} + c$ (2) $\frac{1}{4}\log\sqrt{3 + 4x} + c$

(3) $2\sqrt{3 + 4x} + c$ (4) $-\frac{1}{2}\sqrt{3 + 4x} + c$

(38) $\int \left(\frac{x - 1}{x + 1}\right) dx =$

(1) $\frac{1}{2}\left(\frac{x - 1}{x + 1}\right)^2 + c$ (2) $x - 2\log(x + 1) + c$

(3) $\frac{(x - 1)^2}{2}\log(x + 1) + c$ (4) $x + 2\log(x + 1) + c$

(39) $\int \operatorname{cosec} x\, dx =$

(1) $\log \tan\frac{x}{2} + c$ (2) $-\log(\operatorname{cosec} x + \cot x) + c$

(3) $\log(\operatorname{cosec} x - \cot x) + c$ (4) all of them

(40) When three dice are rolled , number of elementary events are

(1) 2^3 (2) 3^6 (3) 6^3 (4) 3^2

(41) Three coins are tossed. The probability of getting atleast two heads is

(1) $\frac{3}{8}$ (2) $\frac{7}{8}$ (3) $\frac{1}{8}$ (4) $\frac{1}{2}$

(42) If $P(A) = 0.35$, $P(B) = 0.73$ and $P(A \cap B) = 0.14$. Then $P(\bar{A} \cup \bar{B}) =$

(1) 0.94 (2) 0.06 (3) 0.86 (4) 0.14

(43) If A and B are two events such that $P(A)$ = 0.16, $P(B)$ = 0.24 and $P(A \cap B)$ = 0.11, then the probability of obtaining only one of the two events is

(1) 0.29 (2) 0.71 (3) 0.82 (4) 0.18

(44) Two events A and B are independent, then $P(A/B)$ =

(1) $P(A)$ (2) $P(A \cap B)$ (3) $P(A) = P(B)$ (4) $\frac{P(A)}{P(B)}$

(45) A and B are two events such that $P(A) \neq 0$, $P(B) \neq 0$. If A and B are mutually exclusive, then

(1) $P(A \cap B) = P(A)\, P(B)$ (2) $P(A \cap B) \neq P(A) \,.\, P(B)$

(3) $P(A/B) = P(A)$ (4) $P(B/A) = P(A)$

(46) X speaks truth in 95 percent of cases and Y in 80 percent of cases. The percentage of cases they likely to contradict each other in stating same fact is

(1) 14% (2) 86% (3) 23% (4) 85.5%

(47) A problem is given to 3 students A, B and C whose chances of solving it are $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{1}{4}$. The probability to solve is

(1) $\frac{4}{5}$ (2) $\frac{3}{10}$ (3) $\frac{7}{10}$ (4) $\frac{1}{30}$

(48) Given $P(A) = 0.50$, $P(B) = 0.40$ and $P(A \cap B) = 0.20$ then $P(A/\overline{B})$ =

(1) 0.50 (2) 0.40

(3) 0.70 (4) 0.10

(49) An urn contains 10 white and 10 black balls. While another urn contains 5 white and 10 black balls. One urn is chosen at random and a ball is drawn from it. The probability that it is white, is

(1) $\frac{5}{11}$ (2) $\frac{5}{12}$ (3) $\frac{3}{7}$ (4) $\frac{4}{7}$

ANSWERS

EXERCISE 7.1

(1) (i) $x^2 + 1$ (ii) $(x + 1)^2$ (iii) $x + 2$

(iv) x^4 (v) 10 (vi) 16

(2) (i) $x^2 + x + 1$ (ii) $\frac{x+1}{x^2}$ for $x \neq 0$ (iii) $x^3 + x^2$

(iv) $1+x-x^2$ (v) $x^3 + x^2$

(3) $f^{-1}(x) = \frac{x-2}{3}$

(4) (i) $x \in [-3, 3]$ (ii) $x \in (-\infty, -3) \cup (6, \infty)$

(iii) $x \in (-\infty, -2) \cup (2, \infty)$ (iv) $x \in (-4, 3)$

(v) $x \in (-\infty, -3] \cup [4, \infty)$ (vi) no solution

(vii) $x \in (0, 1)$ (viii) $x \in (-\infty, 0) \cup (1/3, \infty)$

(ix) $x \in (-\infty, -1/3) \cup (2/3, \infty)$

EXERCISE 8.1

(1) 4 (2) 0 (3) $2x$ (4) m (5) $\frac{2\sqrt{2}}{3}$ (6) $\frac{q}{p}$

(7) $\frac{\sqrt[m]{a}}{ma}$ (8) $\frac{2}{3}$ (9) $\frac{1}{2}$ (10) $\frac{1}{9}$ (11) $2 \cos a$ (12) α

(13) e (14) yes ; $\lim_{x \to 3} f(x) = 27$ (15) $n = 4$ (16) 1

(18) -1 ; 1 ; $\lim_{x \to 0} \frac{\lfloor x \rfloor}{x}$ does not exist (19) $\log_e \left(\frac{a}{b}\right)$; $\log_e \left(\frac{5}{6}\right)$

EXERCISE 8.2

(1) continuous at $x = 2$ (2) continuous at $x = 0$

(3) discontinuous at $x = 1$ (4) discontinuous at $x = 0$

(5) $a = 3$; $b = -8$ (7) f is continuous at $x = 1$ and $x = 2$

EXERCISE 8.3

(2) No ; $Lf'(0) = -1$; $Rf'(0) = 1$

(3) f is continuous on R; not differentiable at $x = 0$ and $x = 1$).

(4) (i) f is not differentiable at $x = 1$

(ii) f is not differentiable at $x = 2$; but differentiable at $x = 4$

(5) $Lf'(0) = -1$; $Rf'(0) = 1$

EXERCISE 8.4

(1) $3x^2 - 12x + 7$ (2) $3x^2 - 8$; $f'(2) = 4$; $f'(10) = 292$ (3) $a = 1$; $b = 7$

(4) (i) $7x^6 + e^x$ (ii) $\dfrac{\log_7 e}{x}$

(iii) $3\cos x - 4\sin x - e^x$ (iv) $e^x + 3\sec^2 x + \dfrac{6}{x}$

(v) $\dfrac{\log_{10} e}{x} + 2\sec x \tan x$ (vi) $\dfrac{-3}{2x^2\sqrt{x}} + 7\sec^2 x$

(vii) $3\left(1 + x^2 - \dfrac{1}{x^2} - \dfrac{1}{x^4}\right)$ (viii) $\left(4x - 6 - \dfrac{12}{x^2}\right)$

EXERCISE 8.5

(1) $e^x(\cos x - \sin x)$ (2) $\dfrac{\sqrt[n]{x}}{2x}\left(1 + \dfrac{\log x}{n}\right)$

(3) $6\log_{10} e\left(\dfrac{\sin x}{x} + \cos x \log_e x\right)$

(4) $(7x^6 - 36x^5 + 35x^4 + 12x^3 + 24x^2 - 14x - 4)$

(5) $b(2\cos 2x - \cos x) + 2a\sin x$

(6) $-\operatorname{cosec} x(\cot^2 x + \operatorname{cosec}^2 x)$

(7) $\sin 2x$ (8) $-\sin 2x$

(9) $12x(3x^2 + 1)$

(10) $2(12x^2 + 12x - 1)$

(11) $6\tan^2 x + 20\cot^2 x + 26$

(12) $xe^x[x\cos x + x\sin x + 2\sin x]$

(13) $\dfrac{e^x}{\sqrt{x}}\left(1 + x\log x + \dfrac{\log x}{2}\right)$

EXERCISE 8.6

(1) $-\frac{10}{x^3}$ (2) $\frac{22}{(4x+5)^2}$ (3) $\frac{6x^7 - 28x^6 + 4^7}{(x-4)^2}$

(4) $\frac{e^x\left(\frac{1}{x} - \sin x - \cos x - \log x\right) - 2x(\cos x + \log x) + x - x^2 \sin x}{(x^2 + e^x)^2}$

(5) $\frac{4x(1 - 2\log x)}{(\log x + 2x^2)^2}$ (6) $\frac{\sin x - x \log x \cos x}{x \sin^2 x}$ (7) $\frac{-(2ax+b)}{(ax^2+bx+c)^2}$

(8) $\frac{-2\sec^2 x}{(\tan x - 1)^2}$ (9) $\frac{-(x^2+2)}{(x\sin x - \cos x)^2}$ (10) $e^{-x}\left(\frac{2}{x} - 2\log x\right)$

EXERCISE 8.7

(1) $\cot x$ (2) $\cos x \, e^{\sin x}$

(3) $\frac{-\operatorname{cosec}^2 x}{2\sqrt{1+\cot x}}$ (4) $\frac{\sec^2(\log x)}{x}$

(5) $\frac{e^{bx}(a\sin(ax+b) + b\cos(ax+b))}{\cos^2(ax+b)}$ (6) $\frac{1}{2}\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

(7) $(e^x + 4)\cot(e^x + 4x + 5)$ (8) $\frac{3}{2}\sqrt{x}\cos\left(x\sqrt{x}\right)$

(9) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$ (10) $\frac{\cos(\log x)\, e^{\sin(\log x)}}{x}$

EXERCISE 8.8

(1) $\frac{-1}{\sqrt{x}(1+x)}$ (2) $\frac{-2x\, e^{x^2}}{1+e^{2x^2}}$ (3) $\frac{1}{x(1+(\log x)^2)}$ (4) -2

EXERCISE 8.9

(1) $\frac{\sqrt{2}x^{\sqrt{2}}}{x}$ (2) $x^{x^2+1}(1 + 2\log x)$

(3) $x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x(\log x)\right)$

(4) $\sin x^{\sin x} \cos x\,(1 + \log \sin x)$

(5) $(\tan^{-1}x)^{\log x}\left(\dfrac{\log x}{(1 + x^2)\tan^{-1}x} + \dfrac{\log(\tan^{-1}x)}{x}\right)$

(6) $(\log x)^{\sin^{-1}x}\left[\dfrac{\log(\log x)}{\sqrt{1 - x^2}} + \dfrac{\sin^{-1}x}{x\log x}\right]$

(7) $\dfrac{(x^2 + 2)\left(x + \sqrt{2}\right)}{\sqrt{x + 4}\,(x - 7)}\left\{\dfrac{2x}{x^2 + 2} + \dfrac{1}{x + \sqrt{2}} - \dfrac{1}{2(x + 4)} - \dfrac{1}{x - 7}\right\}$

(8) $(x^2 + 2x + 1)^{\sqrt{x - 1}}\left[\dfrac{2\sqrt{x - 1}}{(x + 1)} + \dfrac{\log(x + 1)}{\sqrt{x - 1}}\right]$

(9) $\dfrac{\sin x \cos(e^x)}{e^x + \log x}\left[\cot x - e^x \tan(e^x) - \dfrac{(xe^x + 1)}{x(e^x + \log x)}\right]$

(10) $x^{\sin x}\left(\dfrac{\sin x}{x} + \log x \cos x\right) + (\sin x)^x\,(x \cot x + \log \sin x)$

EXERCISE 8.10

(1) $\frac{1}{2}$ (2) 1 (3) $\frac{1}{2}$ (4) 1 (5) $\dfrac{1}{2(1 + x^2)}$

(6) $\dfrac{2x}{1 + x^4}$ (7) $\dfrac{1}{2\sqrt{x}\,(1 + x)}$ (8) $\dfrac{1}{2\sqrt{1 - x^2}}$ (9) $-\frac{1}{2}$

EXERCISE 8.11

(1) $-\dfrac{b}{a}\cot\theta$ (2) $\dfrac{1}{t}$ (3) $\dfrac{b}{a}\sin\theta$ (4) $-\dfrac{1}{t^2}$

(5) $\tan\left(\dfrac{3\theta}{2}\right)$ (6) $\tan\theta$ (7) $\dfrac{t(2 - t^3)}{1 - 2t^3}$

EXERCISE 8.12

(1) $\dfrac{b^2x}{a^2y}$ (2) $\dfrac{\sin y}{1 - x\cos y}$ (3) $\dfrac{x^2(x - 3a^2y^3)}{y^2(3a^2x^3 - y)}$ (4) $\dfrac{2 + y(\sec^2 x + y\sin x)}{2y\cos x - \tan x}$

(5) $\dfrac{y\ \text{cosec}^2 x + (1 + y^2)\sec x \tan x - 2x}{\cot x - 2y \sec x}$

(6) $2\left(\dfrac{xy - (1 + x^2)^2 \tan x \sec^2 x}{(1 + x^2)\,[4y\,(1 + x^2) + (1 + x^2)\cos y + 1]}\right)$

(7) $-\dfrac{y}{x}$ (8) $\dfrac{y}{x}$ (9) $e^{x-y}\left(\dfrac{1 - e^y}{e^x - 1}\right)$ (10) $\dfrac{100 - y}{x - 100}$ (11) $\dfrac{y\,(x \log y - y)}{x\,(y \log x - x)}$

EXERCISE 8.13

(1) $2(3x + \tan x + \tan^3 x)$ (2) $-2(1 + 4\cot^2 x + 3\cot^4 x)$

(3) (i) (2) (ii) $2\cos x - x \sin x$ (iii) $\dfrac{2x}{(1 + x^2)^2}$

(4) (i) $m^3 e^{mx} + 6$ (ii) $x \sin x - 3 \cos x$

EXERCISE 9.1

Add an arbitrary constant '*c*' to all the answers from Exercise 9.1 to Exercise 9.9

(1) (i) $\dfrac{x^{17}}{17}$ (ii) $\dfrac{2}{7}\, x^{7/2}$ (iii) $\dfrac{2}{9}\, x^{9/2}$ (iv) $\dfrac{3}{7}\, x^{7/3}$ (v) $\dfrac{7}{17}\, x^{17/7}$

(2) (i) $-\dfrac{1}{4x^4}$ (ii) $\log x$ (iii) $-\dfrac{2}{3x^{3/2}}$ (iv) $-\dfrac{3}{2x^{2/3}}$ (v) $4x^{1/4}$

(3) (I) $-\cos x$ (ii) $\sec x$ (iii) $-\text{cosec}\, x$ (iv) $\tan x$ (v) e^x

EXERCISE 9.2

(1) (i) $\dfrac{x^5}{5}$ (ii) $\dfrac{(x + 3)^6}{6}$ (iii) $\dfrac{(3x + 4)^7}{21}$ (iv) $-\dfrac{(4 - 3x)^8}{24}$ (v) $\dfrac{(lx + m)^9}{9l}$

(2) (i) $-\dfrac{1}{5x^5}$ (ii) $-\dfrac{1}{3(x + 5)^3}$ (iii) $-\dfrac{1}{8(2x + 3)^4}$

(iv) $\dfrac{1}{30(4 - 5x)^6}$ (v) $-\dfrac{1}{7a(ax + b)^7}$

(3) (i) $\log(x + 2)$ (ii) $\dfrac{1}{3} \log (3x+2)$ (iii) $-\dfrac{1}{4} \log (3 - 4x)$

(iv) $\dfrac{1}{q} \log (p + qx)$ (v) $-\dfrac{1}{t} \log (s - tx)$

(4) (i) $-\cos(x + 3)$ (ii) $-\frac{1}{2}\cos(2x + 4)$ (iii) $\frac{1}{4}\cos(3 - 4x)$

(iv) $\frac{1}{4}\sin(4x + 5)$ (v) $-\frac{1}{2}\sin(5 - 2x)$

(5) (i) $-\tan(2 - x)$ (ii) $-\frac{1}{2}\cot(5 + 2x)$ (iii) $\frac{1}{4}\tan(3 + 4x)$

(iv) $\frac{1}{11}\cot(7 - 11x)$ (v) $-\frac{1}{q}\tan(p - qx)$

(6) (i) $\sec(3 + x)$ (ii) $\frac{1}{3}\sec(3x + 4)$ (iii) $-\sec(4 - x)$

(iv) $-\frac{1}{3}\sec(4 - 3x)$ (v) $\frac{1}{a}\sec(ax + b)$

(7) (i) $\operatorname{cosec}(2 - x)$ (ii) $-\frac{1}{4}\operatorname{cosec}(4x + 2)$ (iii) $\frac{1}{2}\operatorname{cosec}(3 - 2x)$

(iv) $-\frac{1}{l}\operatorname{cosec}(lx + m)$ (v) $\frac{1}{t}\operatorname{cosec}(s - tx)$

(8) (i) $\frac{e^{3x}}{3}$ (ii) $e^{x + 3}$ (iii) $\frac{1}{3}e^{3x + 2}$ (iv) $-\frac{1}{4}e^{5 - 4x}$ (v) $\frac{1}{a}e^{ax + b}$

(9) (i) $\frac{1}{p}\tan(px + a)$ (ii) $\frac{1}{m}\cot(l - mx)$ (iii) $-\frac{1}{7a}(ax + b)^{-7}$

(iv) $-\frac{1}{2}\log(3 - 2x)$ (v) $-e^{-x}$

(10) (i) $-\frac{1}{4}\sec(3 - 4x)$ (ii) $-\left(\frac{1}{q}\right)\frac{1}{e^{p + qx}}$ (iii) $-\frac{1}{2}\operatorname{cosec}(2x + 3)$

(iv) $\frac{2}{3l}(lx + m)^{3/2}$ (v) $-\frac{2}{15}(4 - 5x)^{3/2}$

EXERCISE 9.3

(1) $x^5 + \frac{3}{10}(2x + 3)^5 + \frac{1}{3}(4 - 3x)^6$

(2) $3\log x + \frac{m}{4}\log(4x + 1) + \frac{(5 - 2x)^6}{6}$

(3) $4x - 5\log(x + 2) + \frac{3}{2}\sin 2x$

(4) $\frac{3}{7}e^{7x} - \sec(4x + 3) - \frac{11}{4x^4}$

(5) $-\cot(px - q) + \frac{6}{5}(1 - x)^5 - e^{3 - 4x}$

(6) $\log(3 + 4x) + \frac{(10x + 3)^{10}}{100} + \frac{3}{2}\operatorname{cosec}(2x + 3)$

(7) $-\frac{6}{5}\cos 5x + \frac{1}{p(m - 1)(px + q)^{m - 1}}$

(8) $\frac{a}{b}\tan(bx + c) + \frac{q}{m\, e^{l - mx}}$

(9) $\frac{3}{2}\log\left(3 + \frac{2}{3}x\right) - \frac{2}{3}\sin\left(x - \frac{2}{3}\right) + \frac{9}{7}\left(\frac{x}{3} + 4\right)^7$

(10) $-49\cos\frac{x}{7} + 32\tan\left(4 - \frac{x}{4}\right) + 10\left(\frac{2x}{5} - 4\right)^{5/2}$

(11) $2\frac{x^{e + 1}}{e + 1} + 3e^x + xe^e$

(12) $\frac{(ae)^x}{1 + \log a} + \frac{a^{-x}}{\log a} + \frac{b^x}{\log b}$

EXERCISE 9.4

(1) $\frac{8}{3}x^3 + 26x^2 - 180x$

(2) $\frac{x^7}{7} + \frac{x^4}{2} + x$

(3) $\frac{x^2}{2} + 4x - 3\log x - \frac{2}{x}$

(4) $\frac{x^4}{4} - \frac{x^3}{3} + 2\log(x + 1)$

(5) $\frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 2\sqrt{x}$

(6) $e^x - \frac{e^{-3x}}{3} - 2e^{-x}$

(7) $\frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + \sin 4x$

(8) $\frac{1}{4}\left(\frac{3\sin 2x}{2} + \frac{\sin 6x}{6}\right) + \frac{\cos 6x}{6}$

(9) $\tan x - \sec x$

(10) $-\operatorname{cosec} x - \cot x$

(11) $\pm(\sin x + \cos x)$

(12) $\sqrt{2}\ \sin x$

(13) $\tan x - \cot x$

(14) $x - \sin x$

(15) $-\frac{1}{2}\left(\frac{\cos 12x}{12} + \frac{\cos 2x}{2}\right)$

(16) $\frac{1}{2}\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right)$

(17) $-\frac{1}{2}\left(\frac{\cos 6x}{6} + \frac{\cos 2x}{2}\right)$

(18) $\frac{1}{2}\left(\frac{\sin 8x}{8} - \frac{\sin 12x}{12}\right)$

(19) $-\frac{1}{2}\cot x$

(20) $-\left(\frac{e^{-2x}}{2} - \frac{2}{3}e^{-3x} + \frac{1}{4}e^{-4x}\right)$

(21) $2\tan x - 2\sec x - x$

(22) $-2\frac{3^{-x}}{\log 3} + \frac{2^{-x}}{3\log 2}$

(23) $\frac{(ae)^x}{1 + \log a}$

(24) $a\left[\frac{(a/c)^x}{\log a - \log c}\right] - \frac{1}{b}\left[\frac{(b/c)^x}{\log b - \log c}\right]$

(25) $\frac{x^2}{2} + 2x + \log x$

(26) $-\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{(m-n)}\right]$

(27) $\frac{1}{2}\left[\frac{\sin(p+q)x}{p+q} + \frac{\sin(p-q)x}{p-q}\right]$

(28) $-\frac{1}{2}\left[\frac{\cos 10x}{10} + \frac{\cos 20x}{40}\right]$

(29) $\frac{2}{9}\left[(x+1)^{3/2} + (x-2)^{3/2}\right]$

(30) $\frac{2}{3a(b-c)}\left[(ax+b)^{3/2} + (ax+c)^{3/2}\right]$

(31) $\frac{2}{5}(x+3)^{5/2} - \frac{4}{3}(x+3)^{3/2}$

(32) $\frac{2}{5}(x+7)^{5/2} - \frac{22}{3}(x+7)^{3/2}$

(33) $\frac{1}{5}(2x+3)^{5/2} - \frac{2}{3}(2x+3)^{3/2}$

(34) $2\log(x+3) - \log(x+2)$

(35) $\frac{5}{52}\log\left(\frac{x-2}{x+2}\right) + \frac{8}{39}\tan^{-1}\left(\frac{x}{3}\right)$

EXERCISE 9.5

(1) $\frac{(1+x^6)^8}{48}$ (2) $\log(lx^2 + mx + n)$ (3) $-\frac{2}{9(ax^2+bx+c)^9}$

(4) $\sqrt{x^2+3}$ (5) $\frac{2}{3}(x^2+3x-5)^{3/2}$ (6) $\log \sec x$

(7) $\log(\sec x + \tan x)$ (8) $-\frac{\cos^{15}x}{15}$ (9) $-\frac{\cos^5 x}{5} + \frac{2}{3}\cos^3 x - \cos x$

(10) $\frac{-1}{7}\sin^7 x + \frac{3}{5}\sin^5 x - \sin^3 x + \sin x$ (11) $\log(x + \log\sec x)$

(12) $\frac{1}{m}e^{m\tan^{-1}x}$ (13) $\frac{1}{4}(\sin^{-1}x^2)^2$ (14) $(x+\log x)^5$

(15) $-\cos(\log x)$ (16) $\log\log\sin x$ (17) $\frac{\sec^4 x}{4}$

(18) $-\sec x + \frac{\sec^3 x}{3}$ (19) $(x+a)\cos a - \sin a \log\sin(x+a)$

(20) $(x-a)\cos a + \sin a\log\cos(x-a)$

(21) $\frac{1}{b-a}\log(a\cos^2 x + b\sin^2 x)$

(22) $\log\cos\left(\frac{\pi}{4} - x\right)$ (23) $2\sqrt{\tan x}$ (24) $\frac{1}{3}(\log x)^3$

(25) $\frac{1}{4}e^{x^4}$

(26) $\frac{1}{e}\log(x^e + e^x + e^e)$ (27) $\frac{(l-x)^{18}}{18} - \frac{l(l-x)^{17}}{17}$

(28) $\frac{a}{m+1}\ (x-a)^{m+1} + \frac{1}{m+2}\ (x-a)^{m+2}$

(29) $-\frac{(2-x)^{18}}{18} + \frac{4}{17}\ (2-x)^{17} - \frac{1}{4}\ (2-x)^{16}$ (30) $-2\cos\sqrt{x}$

(31) $\frac{1}{2}\left[\frac{(2x+3)^{5/2}}{5} - \frac{(2x+3)^{3/2}}{3}\right]$

(32) $\frac{1}{2}\left[\frac{3}{5}(2x+1)^{5/2} + \frac{7}{3}(2x+1)^{3/2}\right]$

(33) $2\left[\frac{(x+1)^{7/2}}{7} - \frac{2}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2}\right]$

EXERCISE 9.6

(1) $-xe^{-x} - e^{-x}$ (2) $x\sin x + \cos x$

(3) $-x\cot x + \log\sin x$ (4) $x\sec x - \log(\sec x + \tan x)$

(5) $x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$ (6) $x\tan x + \log\cos x - \frac{x^2}{2}$

(7) $\frac{1}{2}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]$

(8) $\frac{1}{2}\left(\left(\frac{\sin 7x}{7} + \frac{\sin 3x}{3}\right) + \left(\frac{\cos 7x}{49} + \frac{\cos 3x}{9}\right)\right)$

(9) $2\left[\frac{1}{3}xe^{3x} - \frac{e^{3x}}{9}\right]$ (10) $\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)\ e^{2x}$

(11) $\frac{1}{3}\ x^2\sin 3x + \frac{2}{9}\ x\cos 3x - \frac{2}{27}\ \sin 3x$ (12) $(\sin^{-1}x - 1)\ e^{\sin^{-1}x}$

(13) $\frac{1}{2}\ (x^4 - 2x^2 + 2)e^{x^2}$ (14) $3\left[x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right]$

(15) $\frac{1}{2}\ \left[x^2\sin^{-1}(x^2) + \sqrt{1-x^4}\right]$ (16) $-\frac{1}{2}\ \operatorname{cosec} x\cot x + \frac{1}{2}\ \log\tan\frac{x}{2}$

(17) $\frac{e^{ax}}{a^2+b^2}\ (a\cos bx + b\sin bx)$ (18) $\frac{e^{2x}}{13}\ (2\sin 3x - 3\cos 3x)$

(19) $\frac{e^x}{5}$ (cos 2x + 2 sin 2x)

(20) $\frac{e^{3x}}{13}$ (3 sin2x – 2 cos 2x)

(21) $\frac{1}{4}$ [sec 2x tan2x + log (sec 2x + tan 2x)]

(22) $\frac{e^{4x}}{2}\left[\frac{1}{65}(4\sin 7x - 7\cos 7x) - \frac{1}{25}(4\sin 3x - 3\cos 3x)\right]$

(23) $\frac{e^{-3x}}{4}\left[\frac{3}{10}(-3\cos x + \sin x) + \frac{1}{6}(-\cos 3x + \sin 3x)\right]$

EXERCISE 9.7

(1) (i) $\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)$ (ii) $\frac{1}{4}\tan^{-1}\left(\frac{x+2}{4}\right)$ (iii) $\frac{1}{6}\tan^{-1}\left(\frac{3x+5}{2}\right)$

(iv) $\frac{2}{\sqrt{55}}\tan^{-1}\left(\frac{4x+7}{\sqrt{55}}\right)$ (v) $\frac{1}{9}\tan^{-1}\left(\frac{3x+1}{3}\right)$

(2) (i) $\frac{1}{8}\log\left(\frac{4+x}{4-x}\right)$ (ii) $\frac{1}{6}\log\left(\frac{x}{6-x}\right)$ (iii) $\frac{1}{8\sqrt{7}}\log\left(\frac{\sqrt{7}+1+4x}{\sqrt{7}-1-4x}\right)$

(iv) $\frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right)$ (v) $\frac{1}{6\sqrt{6}}\log\left(\frac{\sqrt{6}+1+3x}{\sqrt{6}-1-3x}\right)$

(3) (i) $\frac{1}{10}\log\left(\frac{x-5}{x+5}\right)$ (ii) $\frac{1}{16}\log\left(\frac{2x-3}{2x+5}\right)$ (iii) $\frac{1}{6\sqrt{7}}\log\left(\frac{3x+5-\sqrt{7}}{3x+5+\sqrt{7}}\right)$

(iv) $\frac{1}{\sqrt{21}}\log\left(\frac{2x+3-\sqrt{21}}{2x+3+\sqrt{21}}\right)$ (v) $\frac{1}{17}\log\left(\frac{3x-15}{3x+2}\right)$

(4) (i) $\log\left(x+\sqrt{x^2+1}\right)$ (ii) $\frac{1}{2}\log\left[(2x+5)+\sqrt{(2x+5)^2+4}\right]$

(iii) $\frac{1}{3}\log\left[(3x-5)+\sqrt{(3x-5)^2+6}\right]$

(iv) $\log\left[\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x+10}\right]$

(v) $\log\left[\left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 26}\right]$

(5) (i) $\log\left(x + \sqrt{x^2 - 91}\right)$ (ii) $\log\left[(x + 1) + \sqrt{(x + 1)^2 - 15}\right]$

(iii) $\frac{1}{2}\log\left[(2x + 3) + \sqrt{(2x + 3)^2 - 16}\right]$ (iv) $\log\left[(x + 2) + \sqrt{x^2 + 4x - 12}\right]$

(v) $\log\left[(x + 4) + \sqrt{x^2 + 8x - 20}\right]$

(6) (i) $\sin^{-1}\left(\frac{x}{2}\right)$ (ii) $\sin^{-1}\left(\frac{x - 1}{5}\right)$ (iii) $\frac{1}{2}\sin^{-1}\left(\frac{2x + 3}{\sqrt{11}}\right)$

(iv) $\sin^{-1}\left(\frac{2x - 1}{\sqrt{5}}\right)$ (v) $\sin^{-1}\left(\frac{2x + 1}{\sqrt{33}}\right)$

(7) (i) $-\log(x^2 + x + 1) + \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right)$

(ii) $\frac{1}{2}\log(x^2 + 21x + 3) - \frac{27}{2\sqrt{429}}\log\left(\frac{2x + 21 - \sqrt{429}}{2x + 21 + \sqrt{429}}\right)$

(iii) $\frac{1}{2}\log(2x^2 + x + 3) - \frac{3}{\sqrt{23}}\tan^{-1}\left(\frac{4x + 1}{\sqrt{23}}\right)$

(iv) $\frac{1}{2}\log(1 - x - x^2) + \frac{3}{2\sqrt{5}}\log\left(\frac{\sqrt{5} + 2x + 1}{\sqrt{5} - 2x - 1}\right)$

(v) $2\log(x^2 + 3x + 1) - \sqrt{5}\log\left(\frac{2x + 3 - \sqrt{5}}{2x + 3 + \sqrt{5}}\right)$

(8) (i) $-\frac{1}{2}\sqrt{6 + x - 2x^2} + \frac{9}{4\sqrt{2}}\sin^{-1}\left(\frac{4x - 1}{7}\right)$

(ii) $-2\sqrt{10 - 7x - x^2} - 10\sin^{-1}\left(\frac{2x + 7}{\sqrt{89}}\right)$

(iii) $\sqrt{3x^2 + 4x + 7}$ (iv) $\sin^{-1}x - \sqrt{1 - x^2} + c$

(v) $6\sqrt{x^2 - 9x + 20} + 34\log\left[(x - 9/2) + \sqrt{x^2 - 9x + 20}\right]$

(9) (i) $\frac{x}{2}\sqrt{1 + x^2} + \frac{1}{2}\log\left[x + \sqrt{1 + x^2}\right]$

(ii) $\frac{x+1}{2}\sqrt{(x+1)^2+4}+2\log\left[(x+1)+\sqrt{(x+1)^2+4}\right]$

(iii) $\frac{1}{4}\left[(2x+1)\sqrt{(2x+1)^2+9}+9\log\left\{(2x+1)+\sqrt{(2x+1)^2+9}\right\}\right]$

(iv) $\frac{2x-3}{4}\sqrt{x^2-3x+10}+\frac{31}{8}\log\left[(x-3/2)+\sqrt{x^2-3x+10}\right]$

(10) (i) $\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)$ (ii) $\left(\frac{x+2}{2}\right)\sqrt{25-(x+2)^2}+\frac{25}{2}\sin^{-1}\left(\frac{x+2}{5}\right)$

(iii) $\frac{1}{6}\left[(3x+1)\sqrt{169-(3x+1)^2}+169\sin^{-1}\left(\frac{3x+1}{13}\right)\right]$

(iv) $\frac{2x-3}{4}\sqrt{1-3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x+3}{\sqrt{13}}\right)$

(v) $\frac{2x+1}{4}\sqrt{6-x-x^2}+\frac{25}{8}\sin^{-1}\left(\frac{2x+1}{5}\right)$

EXERCISE 10.1

(1) (i) Yes (ii) No (iii) No, $\because$ P(C) is negative (iv) No, $\because \sum P \neq 1$ (v) Yes

(2) (i) $\frac{1}{6}$ (ii) $\frac{1}{12}$ (iii) $\frac{1}{6}$ (3) (i) $\frac{3}{8}$ (ii) $\frac{1}{2}$ (iii) $\frac{7}{8}$ (4) (i) $\frac{2}{13}$ (ii) $\frac{4}{13}$ (iii) $\frac{2}{13}$

(5) (i) $\frac{1}{22}$ (ii) $\frac{21}{44}$ (6) $\frac{2}{9}$ (7) (i) $\frac{4}{7}$ (ii) $\frac{3}{7}$ (8) $\frac{37}{42}$ (9) (i) $\frac{1}{7}$ (ii) $\frac{2}{7}$ (10) $\frac{27}{50}$

EXERCISE 10.2

(1) (i) 0.79 (ii) 0.10 (2) (i) 0.72 (ii) 0.72 (iii) 0.28 (iv) 0.28

(3) (i) 0.86 (ii) 0.36 (iii) 0.26 (iv) 0.76 (v) 0.14 (4) $\frac{11}{36}$ (5) 0.2

(6) (i) $\frac{4}{13}$ (ii) $\frac{7}{13}$ (7) (i) 0.45 (ii) 0.30

EXERCISE 10.3

(1) No, for non empty events and possible for any one being null event.

(3) (i) $\frac{9}{10}$ (ii) $\frac{2}{7}$ (4) $\frac{1}{5}$ (5) 0.5

(7) (i) 0.12 (ii) 0.48 (iii) 0.39 (9) (i) $\frac{13}{20}$ (ii) $\frac{5}{12}$ (iii) $\frac{1}{2}$ (iv) $\frac{7}{12}$ (v) $\frac{7}{8}$

(10) (i) $\frac{3}{10}$ (ii) $\frac{6}{11}$ (iii) 0.6 (iv) 0.525 (11) (i) $\frac{1}{169}$ (ii) $\frac{1}{221}$

(12) (i) $\frac{1}{26}$ (ii) $\frac{1}{13}$ (13) (i) $\frac{1}{4}$ (ii) $\frac{9}{40}$ (iii) $\frac{21}{40}$

(14) (i) $\frac{1}{30}$ (ii) $\frac{3}{10}$ (iii) $\frac{2}{3}$ (15) (i) $\frac{3}{4}$ (ii) $\frac{11}{24}$

(16) (i) $\frac{5}{28}$ (ii) $\frac{1}{14}$ (17) (i) 0.45 (ii) 0.9 (18) $\frac{13}{30}$ (19) $\frac{43}{60}$ (20) $\frac{7}{20}$

EXERCISE 10.4

(1) $\frac{89}{198}$ (2) $\frac{3}{80}$ (3) (i) $\frac{41}{80}$ (ii) $\frac{25}{41}$

(4) (i) $\frac{29}{400}$ (ii) $\frac{11}{29}$ (5) (i) $\frac{13}{24}$ (ii) $\frac{5}{13}$

Objective Type Questions – Answers (Key)

(1) 4	(2) 1	(3) 2	(4) 2	(5) 4	(6) 2
(7) 3	(8) 4	(9) 2	(10) 1	(11) 1	(12) 2
(13) 3	(14) 1	(15) 4	(16) 2	(17) 4	(18) 3
(19) 1	(20) 4	(21) 3	(22) 2	(23) 1	(24) 3
(25) 1	(26) 1	(27) 3	(28) 3	(29) 1	(30) 3
(31) 2	(32) 3	(33) 2	(34) 4	(35) 2	(36) 2
(37) 1	(38) 2	(39) 4	(40) 3	(41) 4	(42) 3
(43) 4	(44) 1	(45) 2	(46) 3	(47) 3	(48) 1
(49) 2					