MATHEMATICS

PART - 1

CLASS XI

SYLLABUS

(1) **MATRICES AND DETERMINANTS :** ***Matrix Algebra*** **–** Definitions, types, operations, algebraic properties. ***Determinants*** – Definitions, properties, evaluation, factor method, product of determinants, co-factor determinants. **(18 periods)**

(2) **VECTOR ALGEBRA :** Definitions, types, addition, subtraction, scalar multiplication, properties, position vector, resolution of a vector in two and three dimensions, direction cosines and direction ratios. **(15 periods)**

(3) **ALGEBRA :** ***Partial Fractions*** – Definitions, linear factors, none of which is repeated, some of which are repeated, quadratic factors (none of which is repeated). ***Permutations*** – Principles of counting, concept, permutation of objects not all distinct, permutation when objects can repeat, circular permutations**.** ***Combinations, Mathematical induction, Binomial theorem for positive integral index***–finding middle and particular terms. **(25 periods)**

(4) **SEQUENCE AND SERIES :** Definitions, special types of sequences and series, harmonic progression, arithmetic mean, geometric mean, harmonic mean. **Binomial theorem** for rational number other than positive integer, **Binomial series**, approximation, summation of Binomial series, **Exponential series, Logarithmic series** (simple problems) **(15 periods)**

(5) **ANALYTICAL GEOMETRY :** Locus, ***straight lines*** **–** normal form, parametric form, general form, perpendicular distance from a point, ***family of straight lines***, angle between two straight lines, **pair of straight lines**. ***Circle*** – general equation, parametric form, tangent equation, length of the tangent, condition for tangent. Equation of chord of contact of tangents from a point, **family of circles –** concetric circles, orthogonal circles. **(23 periods)**

(6) **TRIGONOMETRY :** Trigonometrical **ratios and identities**, signs of T-ratios, **compound angles** A ± B, multiple angles 2A, 3A, sub multiple (half) angle A/2, transformation of a product into a sum or difference, conditional identities, **trigonometrical equations**, **properties of triangles**, **solution of triangles** (SSS, SAA and SAS types only), **inverse trigonometrical functions**. **(25 periods)**

(7) **FUNCTIONS AND GRAPHS :** Constants, variables, intervals, neighbourhood of a point, Cartesian product, relation. **Function** – graph of a function, vertical line test. ***Types of functions*** – Onto, one-to-one, identity, inverse, composition of functions, sum, difference product, quotient of two functions, constant function, linear function, polynomial function, rational function, exponential function, reciprocal function, absolute value function, greatest integer function, least integer function, signum function, odd and even functions, trigonometrical functions, quadratic functions. ***Quadratic inequation*** – Domain and range. **(15 periods)**

(8) **DIFFERENTIAL CALCULUS :** ***Limit of a function*** – Concept, fundamental results, important limits, ***Continuity of a function*** – at a point, in an interval, discontinuous function. ***Concept of Differentiation*** – derivatives, slope, relation between continuity and differentiation. ***Differentiation techniques*** – first principle, standard formulae, product rule, quotient rule, chain rule, inverse functions, method of substitution, parametric functions, implicit function, third order derivatives. **(30 periods)**

(9) **INTEGRAL CALCULUS :** Concept, integral as anti-derivative, integration of linear functions, properties of integrals. ***Methods of integration*** – decomposition method, substitution method, integration by parts. ***Definite integrals*** – integration as summation, simple problems. **(32 periods)**

(10) **PROBABILITY :** Classical definitions, axioms, basic theorems, conditional probability, total probability of an event, Baye’s theorem (statement only), simple problems. **(12 periods)**

CONTENTS

1. MATRICES AND DETERMINANTS

1.1 Matrix Algebra

1.1.1 Introduction

The term 'matrix' was first introduced by Sylvester in 1850. He defined a matrix to be an arrangement of terms. In 1858 Cayley outlined a matrix algebra defining addition, multiplication, scalar multiplication and inverses. Knowledge of matrix is very useful and important as it has a wider application in almost every field of Mathematics. Economists are using matrices for social accounting, input – output tables and in the study of inter-industry economics. Matrices are also used in the study of communication theory, network analysis in electrical engineering.

For example let us consider the marks scored by a student in different subjects and in different terminal examinations. They are exhibited in a tabular form as given below.

	Tamil	English	Maths	Science	Social Science
Test 1	70	81	88	83	64
Test 2	68	76	93	81	70
Test 3	80	86	100	98	78

The above statement of marks can also be re-recorded as follows :

$$\begin{array}{l} \text{First row} \\ \text{Second row} \\ \text{Third row} \end{array} \begin{bmatrix} 70 & 81 & 88 & 83 & 64 \\ 68 & 76 & 93 & 81 & 70 \\ 80 & 86 & 100 & 98 & 78 \end{bmatrix}$$

$$\begin{array}{ccccc} \text{First Column} & \text{second Column} & \text{Third Column} & \text{Fourth Column} & \text{Fifth Column} \end{array}$$

This representation gives the following informations.

(i) The elements along the first, second, and third rows represent the test marks of the different subjects.

(ii) The elements along the first, second, third, fourth and fifth columns represent the subject marks in the different tests.

The purpose of matrices is to provide a kind of mathematical shorthand to help the study of problems represented by the entries. The matrices may represent transformations of co-ordinate spaces or systems of simultaneous linear equations.

1.1.2 Definitions:

A matrix is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket or parenthesis. The entries or elements may be any kind of numbers (real or complex), polynomials or other expressions. Matrices are denoted by the capital letters like A, B, C…

Here are some examples of Matrices.

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} & \begin{array}{l} \textbf{First Row} \\ \textbf{Second Row} \\ \textbf{Third Row} \end{array} \\ \begin{array}{cc} \textbf{First Column} & \textbf{Second Column} \end{array} & \end{array} \qquad B = \begin{array}{cc} \begin{bmatrix} 1 & -4 & 2 \\ 6 & 9 & 4 \\ 3 & -2 & 6 \end{bmatrix} & \begin{array}{l} \textbf{First row } (R_1) \\ \textbf{Second row } (R_2) \\ \textbf{Third row } (R_3) \end{array} \\ \begin{array}{ccc} \textbf{First Column} & \textbf{Second Column} & \textbf{Third Column} \\ C_1 & C_2 & C_3 \end{array} & \end{array}$$

Note : In a matrix, rows are counted from top to bottom and the columns are counted from left to right.

i.e. (i) The horizontal arrangements are known as rows.

(ii) The vertical arrangements are known as columns.

To identify an entry or an element of a matrix two suffixes are used. The first suffix denotes the row and the second suffix denotes the column in which the element occurs.

From the above example the elements of A are $a_{11} = 1$, $a_{12} = 4$, $a_{21} = 2$, $a_{22} = 5$, $a_{31} = 3$ and $a_{32} = 6$

Order or size of a matrix

The order or size of a matrix is the number of rows and the number of columns that are present in a matrix.

In the above examples order of A is 3×2, (to be read as 3-by-2) and order of B is 3×3, (to be read as 3-by-3).

In general a matrix A of order $m \times n$ can be represented as follows :

$$A = \begin{array}{cc} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \to i^{\text{th}} \text{ row} \\ \\ \\ \end{array} \\ \begin{array}{c} j^{\text{th}} \\ \text{column} \end{array} & \end{array}$$

This can be symbolically written as A = $[a_{ij}]_{m \times n}$.

The element a_{ij} belongs to i^{th} row and the j^{th} column. i being the row index and j being the column index. The above matrix A is an $m \times n$ or m-by-n matrix. The expression $m \times n$ is the order or size or dimension of the matrix.

Example 1.1: Construct a 3 × 2 matrix whose entries are given by $a_{ij} = i - 2j$

Solution: The general 3 × 2 matrix is of the form

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ where } i = 1, 2, 3 \text{ (rows)}, \quad j = 1, 2 \text{ (columns)}$$

It is given that $a_{ij} = i - 2j$

$a_{11} = 1 - 2 = -1$ $\quad a_{12} = 1 - 4 = -3$

$a_{21} = 2 - 2 = 0$ $\quad a_{22} = 2 - 4 = -2$ $\quad \therefore$ The required matrix is $A = \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{bmatrix}$

$a_{31} = 3 - 2 = 1$ $\quad a_{32} = 3 - 4 = -1$

1.1.3 Types of matrices

(1) Row matrix: A matrix having only one row is called a row matrix or a row vector.

Examples (i) $A = [a_{ij}]_{1 \times 3} = [1 \quad -7 \quad 4]$ is a row matrix of order 1 × 3.

(ii) $B = [b_{ij}]_{1 \times 2} = [5 \quad 8]$ is a row matrix of order 1 × 2

(iii) $C = [c_{ij}]_{1 \times 1} = [100]$ is a row matrix of order 1 × 1

(2) Column matrix:

A matrix having only one column is called a column matrix or a column vector.

Examples (i) $A = [a_{ij}]_{3 \times 1} = \begin{bmatrix} 1 \\ -7 \\ 4 \end{bmatrix}$ is a column matrix of order 3 × 1

(ii) $B = [b_{ij}]_{2 \times 1} = \begin{bmatrix} 25 \\ 30 \end{bmatrix}$ is a column matrix of order 2 × 1

(iii) $C = [c_{ij}]_{1 \times 1} = [68]$ is a column matrix of order 1 × 1

Note : Any matrix of order 1 × 1 can be treated as either a row matrix or a column matrix.

(3) Square matrix

A square matrix is a matrix in which the number of rows and the number of columns are equal. A matrix of order $n \times n$ is also known as a square matrix of order n.

In a square matrix A of order $n \times n$, the elements a_{11}, a_{22}, a_{33} … a_{nn} are called principal diagonal or leading diagonal or main diagonal elements.

$A = [a_{ij}]_{2 \times 2} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ is a square matrix of order 2

$B = [b_{ij}]_{3 \times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a square matrix of order 3.

Note: In general the number of elements in a square matrix of order n is n^2. We can easily verify this statement from the above two examples.

(4) Diagonal Matrix:

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be a diagonal matrix if $a_{ij} = 0$ when $i \neq j$

In a diagonal matrix all the entries except the entries along the main diagonal are zero.

For example $A = [a_{ij}]_{3 \times 3} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ is a diagonal matrix.

(5) Triangular matrix: A square matrix in which all the entries above the main diagonal are zero is called a lower triangular matrix. If all the entries below the main diagonal are zero, it is called an upper triangular matrix.

$A = \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ is an upper triangular matrix and $B = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & -5 & 7 \end{bmatrix}$ is a lower triangular matrix.

(6) Scalar matrix:

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be scalar matrix if

$$a_{ij} = \begin{cases} a & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

i.e. A scalar matrix is a diagonal matrix in which all the entries along the main diagonal are equal.

$A = [a_{ij}]_{2 \times 2} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $\quad$ $B = [b_{ij}]_{3 \times 3} = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$ are examples for scalar matrices.

(7) Identity matrix or unit matrix:

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be an identity matrix if $a_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$

i.e. An identity matrix or a unit matrix is a scalar matrix in which entries along the main diagonal are equal to 1. We represent the identity matrix of order n as I_n

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices.

(8) Zero matrix or null matrix or void matrix

A matrix $A = [a_{ij}]_{m \times n}$ is said to be a zero matrix or null matrix if all the entries are zero, and is denoted by O i.e. $a_{ij} = 0$ for all the values of i, j

$[0 \quad 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are examples of zero matrices.

(9) Equality of Matrices:

Two matrices A and B are said to be equal if

(i) both the matrices A and B are of the same order or size.

(ii) the corresponding entries in both the matrices A and B are equal.

i.e. the matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are equal if $m = p$, $n = q$ and $a_{ij} = b_{ij}$ for every i and j.

Example 1.2 :

If $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix}$ then find the values of x, y, z, w.

Solution:

Since the two matrices are equal, their corresponding entries are also equal.

$\therefore x = 4 \quad y = 3 \quad z = 1 \quad w = 5$

(10) Transpose of a matrix:

The matrix obtained from the given matrix A by interchanging its rows into columns and its columns into rows is called the transpose of A and it is denoted by A′ or A^T.

If $A = \begin{bmatrix} 4 & -3 \\ 2 & 0 \\ 1 & 5 \end{bmatrix}$ then $A^T = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 0 & 5 \end{bmatrix}$

Note that if A is of order $m \times n$ then A^T is order $n \times m$.

(11) Multiplication of a matrix by a scalar

Let A be any matrix. Let k be any non-zero scalar. The matrix kA is obtained by multiplying all the entries of matrix A by the non zero scalar k.

i.e. $A = [a_{ij}]_{m \times n} \Rightarrow kA = [ka_{ij}]_{m \times n}$

This is called scalar multiplication of a matrix.

Note: If a matrix A is of order $m \times n$ then the matrix kA is also of the same order $m \times n$

For example If $A = \begin{bmatrix} 1 & 7 & 2 \\ -6 & 3 & 9 \end{bmatrix}$ then $2A = 2\begin{bmatrix} 1 & 7 & 2 \\ -6 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 14 & 4 \\ -12 & 6 & 18 \end{bmatrix}$

(12) Negative of a matrix:

Let A be any matrix. The negative of a matrix A is – A and is obtained by changing the sign of all the entries of matrix A.

i.e. $A = [a_{ij}]_{m \times n} \Rightarrow -A = [-a_{ij}]_{m \times n}$

Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $-A = \begin{bmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

1.1.4 Operations on matrices

(1) Addition and subtraction

Two matrices A and B can be added provided both the matrices are of the same order and their sum A + B is obtained by adding the corresponding entries of both the matrices A and B

i.e. $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$

Similarly $A - B = A + (-B) = [a_{ij}]_{m \times n} + [-b_{ij}]_{m \times n}$

$= [a_{ij} - b_{ij}]_{m \times n}$

Note:

(1) The matrices A + B and A – B have same order equal to the order of A or B.

(2) Subtraction is treated as negative addition.

(3) The additive inverse of matrix A is – A.

i.e. A + (– A) = (– A) + A = O = zero matrix

For example, if $A = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -7 \\ 3 & 1 \\ -8 & 5 \end{bmatrix}$

then $A + B = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix} + \begin{bmatrix} 4 & -7 \\ 3 & 1 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} 7+4 & 2-7 \\ 8+3 & 6+1 \\ 9-8 & -6+5 \end{bmatrix} = \begin{bmatrix} 11 & -5 \\ 11 & 7 \\ 1 & -1 \end{bmatrix}$ and

$A - B = A + (-B) = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix} + \begin{bmatrix} -4 & 7 \\ -3 & -1 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 7-4 & 2+7 \\ 8-3 & 6-1 \\ 9+8 & -6-5 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 5 & 5 \\ 17 & -11 \end{bmatrix}$

(2) Matrix multiplication:

Two matrices A and B are said to be conformable for multiplication if the number of columns of the first matrix A is equal to the number of rows of the second matrix B. The product matrix 'AB' is acquired by multiplying every row of matrix A with the corresponding elements of every column of matrix B element-wise and add the results. This procedure is known as row-by-column multiplication rule.

Let A be a matrix of order $m \times n$ and B be a matrix of order $n \times p$ then the product matrix AB will be of order $m \times p$

i.e. order of A is $m \times n$, order of B is $n \times p$

Then the order of AB is $m \times p = \begin{pmatrix}\text{number of rows}\\ \text{of matrix A}\end{pmatrix} \times \begin{pmatrix}\text{number of columns}\\ \text{of matrix B}\end{pmatrix}$

The following example describes the method of obtaining the product matrix AB

Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 7 & 3 & 6 \end{bmatrix}_{2 \times 3}$ $B = \begin{bmatrix} 6 & 4 & 3 \\ 3 & 2 & 5 \\ 7 & 3 & 1 \end{bmatrix}_{3 \times 3}$

It is to be noted that the number of columns of the first matrix A is equal to the number of rows of the second matrix B.

$\therefore$ Matrices A and B are conformable, i.e. the product matrix AB can be found.

$$AB = \begin{bmatrix} 2 & 1 & 4 \\ 7 & 3 & 6 \end{bmatrix} \begin{bmatrix} 6 & 4 & 3 \\ 3 & 2 & 5 \\ 7 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 6 \\ 3 \\ 7 \end{matrix} & \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 4 \\ 2 \\ 3 \end{matrix} & \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 3 \\ 5 \\ 1 \end{matrix} \\ \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 6 \\ 3 \\ 7 \end{matrix} & \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 4 \\ 2 \\ 3 \end{matrix} & \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 3 \\ 5 \\ 1 \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} (2)(6) + (1)(3) + (4)(7) & (2)(4) + (1)(2) + (4)(3) & (2)(3) + (1)(5) + (4)(1) \\ (7)(6) + (3)(3) + (6)(7) & (7)(4) + (3)(2) + (6)(3) & (7)(3) + (3)(5) + (6)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 3 + 28 & 8 + 2 + 12 & 6 + 5 + 4 \\ 42 + 9 + 42 & 28 + 6 + 18 & 21 + 15 + 6 \end{bmatrix} \qquad \therefore\ AB = \begin{bmatrix} 43 & 22 & 15 \\ 93 & 52 & 42 \end{bmatrix}$$

It is to be noticed that order of AB is 2×3, which is the number of rows of first matrix A 'by' the number of columns of the second matrix B.

Note : (i) If AB = AC, it is not necessarily true that B = C. (i.e.) the equal matrices in the identity cannot be cancelled as in algebra.

(ii) AB = O does not necessarily imply A = O or B = O

For example, $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq O$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq O$

but $AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

(iii) If A is a square matrix then A.A is also a square matrix of the same order. AA is denoted by A^2. Similarly $A^2A = AAA = A^3$

If I is a unit matrix, then $I = I^2 = I^3 = \ldots = I^n$.

1.1.5 Algebraic properties of matrices:

(1) Matrix addition is commutative:

If A and B are any two matrices of the same order then A + B = B + A. This property is known as commutative property of matrix addition.

(2) Matrix addition is associative:

i.e. If A, B and C are any three matrices of the same order thenA+(B + C) = (A+B)+C. This property is known as associative property of matrix addition.

(3) Additive identity:

Let A be any matrix then A + O = O + A = A. This property is known as identity property of matrix addition.

The zero matrix O is known as the identity element with respect to matrix addition.

(4) Additive inverse:

Let A be any matrix then A + (– A) = (– A) + A = O. This property is known as inverse property with respect to matrix addition.

The negative of matrix A i.e. – A is the inverse of A with respect to matrix addition.

(5) In general, matrix multiplication is not commutative i.e. $AB \neq BA$

(6) Matrix multiplication is associative i.e. A(BC) = (AB)C

(7) Matrix multiplication is distributive over addition

i.e. (i) A(B + C) = AB + AC (ii) (A + B)C = AC + BC

(8) AI = IA = A where I is the unit matrix or identity matrix. This is known as identity property of matrix multiplication.

Example 1.3: If $A = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix}$ $C = \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix}$

Prove that (i) $AB \neq BA$ (ii) A(BC) = (AB)C

(iii) A(B + C) = AB + AC (iv) AI = IA = A

Solution:

(i) $$AB = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} (1)(1) + (8)(7) & (1)(3) + (8)(4) \\ (4)(1) + (3)(7) & (4)(3) + (3)(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 56 & 3 + 32 \\ 4 + 21 & 12 + 12 \end{bmatrix} = \begin{bmatrix} 57 & 35 \\ 25 & 24 \end{bmatrix} \quad \ldots (1)$$

$$BA = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} (1)(1) + (3)(4) & (1)(8) + (3)(3) \\ (7)(1) + (4)(4) & (7)(8) + (4)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 12 & 8 + 9 \\ 7 + 16 & 56 + 12 \end{bmatrix} = \begin{bmatrix} 13 & 17 \\ 23 & 68 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2) we have $AB \neq BA$

(ii) $$(AB)C = \begin{bmatrix} 57 & 35 \\ 25 & 24 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix} \quad \ldots \text{from (1)}$$

$$= \begin{bmatrix} (57)(-4) + (35)(3) & (57)(6) + (35)(-5) \\ (25)(-4) + (24)(3) & (25)(6) + (24)(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -228+105 & 342-175 \\ -100+72 & 150-120 \end{bmatrix}$$

$$\therefore\ (AB)C = \begin{bmatrix} -123 & 167 \\ -28 & 30 \end{bmatrix} \qquad \dots (3)$$

$$BC = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-4)+(3)(3) & (1)(6)+(3)(-5) \\ (7)(-4)+(4)(3) & (7)(6)+(4)(-5) \end{bmatrix} = \begin{bmatrix} -4+9 & 6-15 \\ -28+12 & 42-20 \end{bmatrix}$$

$$BC = \begin{bmatrix} 5 & -9 \\ -16 & 22 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & -9 \\ -16 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(5)+(8)(-16) & (1)(-9)+(8)(22) \\ (4)(5)+(3)(-16) & (4)(-9)+(3)(22) \end{bmatrix} = \begin{bmatrix} 5-128 & -9+176 \\ 20-48 & -36+66 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} -123 & 167 \\ -28 & 30 \end{bmatrix} \qquad \dots (4)$$

From (3) and (4) we have, (AB)C = A(BC)

(iii) $$B + C = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1-4 & 3+6 \\ 7+3 & 4-5 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 10 & -1 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -3 & 9 \\ 10 & -1 \end{bmatrix} = \begin{bmatrix} -3+80 & 9-8 \\ -12+30 & 36-3 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 77 & 1 \\ 18 & 33 \end{bmatrix} \qquad \dots (5)$$

$$AB = \begin{bmatrix} 57 & 35 \\ 25 & 24 \end{bmatrix} \dots \text{from (1)}$$

$$AC = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -4+24 & 6-40 \\ -16+9 & 24-15 \end{bmatrix} = \begin{bmatrix} 20 & -34 \\ -7 & 9 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 57 & 35 \\ 25 & 24 \end{bmatrix} + \begin{bmatrix} 20 & -34 \\ -7 & 9 \end{bmatrix} = \begin{bmatrix} 57+20 & 35-34 \\ 25-7 & 24+9 \end{bmatrix}$$

$$= \begin{bmatrix} 77 & 1 \\ 18 & 33 \end{bmatrix} \qquad \dots (6)$$

From equations (5) and (6) we have A(B + C) = AB + AC

(iv) Since order of A is 2×2, take $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$AI = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+8(0) & 1(0)+8(1) \\ 4(1)+3(0) & 4(0)+3(1) \end{bmatrix} = \begin{bmatrix} 1+0 & 0+8 \\ 4+0 & 0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} = A \qquad \ldots (7)$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)+0(4) & 1(8)+0(3) \\ 0(1)+1(4) & 0(8)+1(3) \end{bmatrix} = \begin{bmatrix} 1+0 & 8+0 \\ 0+4 & 0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} = A \qquad \ldots (8)$$

$\therefore$ From(7) and (8) AI = IA = A

Example 1.4: If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ find $A^2 - 7A - 2I$

Solution: $A^2 = AA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4+12 & 6+15 \\ 8+20 & 12+25 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix} \qquad \ldots (1)$$

$$-7A = -7\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -14 & -21 \\ -28 & -35 \end{bmatrix} \qquad \ldots (2)$$

$$-2I = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad \ldots (3)$$

(1) + (2) + (3) gives $A^2 - 7A - 2I = A^2 + (-7A) + (-2I)$

$$= \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix} + \begin{bmatrix} -14 & -21 \\ -28 & -35 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

i.e. $$A^2 - 7A - 2I = \begin{bmatrix} 16-14-2 & 21-21+0 \\ 28-28+0 & 37-35-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Example 1.5: If $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix}$,

show that $(A + B)^2 \neq A^2 + 2AB + B^2$

Solution: $A + B = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1+5 & 4+0 \\ 0+3 & 3+9 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & 12 \end{bmatrix}$

$$(A + B)^2 = (A + B)\,(A + B) = \begin{bmatrix} 6 & 4 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 3 & 12 \end{bmatrix} = \begin{bmatrix} 36+12 & 24+48 \\ 18+36 & 12+144 \end{bmatrix}$$

$$(A + B)^2 = \begin{bmatrix} 48 & 72 \\ 54 & 156 \end{bmatrix} \qquad \ldots (1)$$

$$A^2 = A.A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1+0 & 4+12 \\ 0+0 & 0+9 \end{bmatrix} = \begin{bmatrix} 1 & 16 \\ 0 & 9 \end{bmatrix}$$

$$B^2 = B.B = \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 25+0 & 0+0 \\ 15+27 & 0+81 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 42 & 81 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 5+12 & 0+36 \\ 0+9 & 0+27 \end{bmatrix} = \begin{bmatrix} 17 & 36 \\ 9 & 27 \end{bmatrix}$$

$$2AB = 2\begin{bmatrix} 17 & 36 \\ 9 & 27 \end{bmatrix} = \begin{bmatrix} 34 & 72 \\ 18 & 54 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 16 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 34 & 72 \\ 18 & 54 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 42 & 81 \end{bmatrix} = \begin{bmatrix} 1+34+25 & 16+72+0 \\ 0+18+42 & 9+54+81 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 60 & 88 \\ 60 & 144 \end{bmatrix} \qquad \ldots (2)$$

From (1) and (2) we have

$(A + B)^2 \neq A^2 + 2AB + B^2$

Example 1.6: Find the value of x if $[2x \quad 3]\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = O$

Solution: $[2x - 9 \quad 4x + 0] \begin{bmatrix} x \\ 3 \end{bmatrix} = O$ (Multiplying on first two matrices)

$$\Rightarrow \quad [(2x - 9)x + 4x(3)] = O \quad \Rightarrow \quad [2x^2 - 9x + 12x] = O$$

$$\Rightarrow \quad [2x^2 + 3x] = O$$

i.e. $2x^2 + 3x = 0 \Rightarrow x(2x + 3) = 0$

Hence we have $x = 0$, $\quad x = \dfrac{-3}{2}$

Example 1.7: Solve: $X + 2Y = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix}$; $X - Y = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$

Solution: Given $X + 2Y = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix}$ … (1)

$$X - Y = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \quad \ldots (2)$$

$(1) - (2) \Rightarrow (X + 2Y) - (X - Y) = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$

$$3Y = \begin{bmatrix} 3 & 6 \\ -6 & 12 \end{bmatrix} \Rightarrow Y = \frac{1}{3}\begin{bmatrix} 3 & 6 \\ -6 & 12 \end{bmatrix}$$

$$\Rightarrow \quad Y = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$$

Substituting matrix Y in equation (1) we have

$$X + 2\begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix}$$

$$\Rightarrow \quad X + \begin{bmatrix} 2 & 4 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & 2 \end{bmatrix}$$

$$\therefore \; X = \begin{bmatrix} 2 & 2 \\ -4 & 2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$$

EXERCISE 1.1

(1) Construct a 3×3 matrix whose elements are (i) $a_{ij} = i + j$ (ii) $a_{ij} = i \times j$

(2) Find the values of x, y, z if $\begin{bmatrix} x & 3x - y \\ 2x + z & 3y - w \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$

(3) If $\begin{bmatrix} 2x & 3x - y \\ 2x + z & 3y - w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$ find x, y, z, w

(4) If $A = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix}$ find each of the following

(i) $-2A + (B + C)$ (ii) $A - (3B - C)$ (iii) $A + (B + C)$ (iv) $(A + B) + C$
(v) $A + B$ (vi) $B + A$ (vii) AB (viii) BA

(5) Given $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

verify the following results:

(i) $AB \neq BA$ (ii) $(AB)\,C = A(BC)$ (iii) $A(B + C) = AB + AC$

(6) Solve : $2X + Y + \begin{bmatrix} -2 & 1 & 3 \\ 5 & -7 & 3 \\ 4 & 5 & 4 \end{bmatrix} = O$; $X - Y = \begin{bmatrix} 4 & 7 & 0 \\ -1 & 2 & -6 \\ -2 & 8 & -5 \end{bmatrix}$

(7) If $A=\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14\,I = O$ where I is the unit matrix of order 2.

(8) If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ find k so that $A^2 = kA - 2I$

(9) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I = O$

(10) Solve for x if $\begin{bmatrix} x^2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2x & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 7 \end{bmatrix}$

(11) Solve for x if $[x \quad 2 \quad -1]\begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = [0]$

(12) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$ verify the following:

(i) $(A + B)^2 = A^2 + AB + BA + B^2$ (ii) $(A - B)^2 \neq A^2 - 2AB + B^2$

(iii) $(A + B)^2 \neq A^2 + 2AB + B^2$ (iv) $(A - B)^2 = A^2 - AB - BA + B^2$

(v) $A^2 - B^2 \neq (A + B)\,(A - B)$

(13) Find matrix C if $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$ and $5C + 2B = A$

(14) If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, find x and y.

1.2 Determinants

1.2.1 Introduction:

The term determinant was first introduced by Gauss in 1801 while discussing quadratic forms. He used the term because the determinant determines the properties of the quadratic forms. We know that the area of a triangle with vertices (x_1, y_1) (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] \qquad \ldots (1)$$

Similarly the condition for a second degree equation in x and y to represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ … (2)

To minimize the difficulty in remembering these type of expressions, Mathematicians developed the idea of representing the expression in determinant form.

The above expression (1) can be represented in the form $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Similarly the second expression (2) can be expressed as $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Again if we eliminate x, y, z from the three equations

$a_1x + b_1y + c_1 z = 0$; $a_2x + b_2y + c_2z = 0$; $a_3x + b_3y + c_3z = 0$,

we obtain $a_1(b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2) = 0$

This can be written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$. Thus a determinant is a particular type of expression written in a special concise form. Note that the quantities are arranged in the form of a square between two vertical lines. This arrangement is called a determinant.

Difference between a matrix and a determinant

(i) A matrix cannot be reduced to a number. That means a matrix is a structure alone and is not having any value. But a determinant can be reduced to a number.

(ii) The number of rows may not be equal to the number of columns in a matrix. In a determinant the number of rows is always equal to the number of columns.

(iii) On interchanging the rows and columns, a different matrix is formed. In a determinant interchanging the rows and columns does not alter the value of the determinant.

1.2.2 Definitions:

To every square matrix A of order n with entries as real or complex numbers, we can associate a number called determinant of matrix A and it is denoted by | A | or det (A) or Δ.

Thus determinant formed by the elements of A is said to be the determinant of matrix A.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then its $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

To evaluate the determinant of order 3 or above we define minors and cofactors.

Minors:

Let $|A| = |[a_{ij}]|$ be a determinant of order n. The minor of an arbitrary element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is denoted by M_{ij}.

Cofactors:

The cofactor is a signed minor. The cofactor of a_{ij} is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

The minors and cofactors of a_{11}, a_{12}, a_{13} of a third order determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ are as follows:

(i) Minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$.

Cofactor of a_{11} is $A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

(ii) Minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

Cofactor of a_{12} is $A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$

(iii) Minor of a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$

Cofactor of a_{13} is $A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$

Note: A determinant can be expanded using any row or column as given below:

Let $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Delta = a_{11}\, A_{11} + a_{12}\, A_{12} + a_{13}\, A_{13}$ or $a_{11}\, M_{11} - a_{12}\, M_{12} + a_{13}\, M_{13}$ (expanding by R_1)

$\Delta = a_{11}\, A_{11} + a_{21} A_{21} + a_{31}\, A_{31}$ or $a_{11}\, M_{11} - a_{21}\, M_{21} + a_{31}\, M_{31}$ (expanding by C_1)

$\Delta = a_{21}\, A_{21} + a_{22}\, A_{22} + a_{23}\, A_{23}$ or $-\, a_{21}\, M_{21} + a_{22}\, M_{22} - a_{23}\, M_{23}$ (expanding by R_2)

Example 1.8:

Find the minor and cofactor of each element of the determinant $\begin{vmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{vmatrix}$

Solution: Minor of 3 is $M_{11} = \begin{vmatrix} -1 & 2 \\ -2 & 6 \end{vmatrix} = -6 + 4 = -2$

Minor of 4 is $M_{12} = \begin{vmatrix} 0 & 2 \\ 5 & 6 \end{vmatrix} = 0 - 10 = -10$

Minor of 1 is $M_{13} = \begin{vmatrix} 0 & -1 \\ 5 & -2 \end{vmatrix} = 0 + 5 = 5$

Minor of 0 is $M_{21} = \begin{vmatrix} 4 & 1 \\ -2 & 6 \end{vmatrix} = 24 + 2 = 26$

Minor of -1 is $M_{22} = \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} = 18 - 5 = 13$

Minor of 2 is $M_{23} = \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = -6 - 20 = -26$

Minor of 5 is $M_{31} = \begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix} = 8 + 1 = 9$

Minor of – 2 is $M_{32} = \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$

Minor of 6 is $M_{33} = \begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix} = -3 - 0 = -3$

Cofactor of 3 is $A_{11} = (-1)^{1+1} M_{11} = M_{11} = -2$

Cofactor of 4 is $A_{12} = (-1)^{1+2} M_{12} = -M_{12} = 10$

Cofactor of 1 is $A_{13} = (-1)^{1+3} M_{13} = M_{13} = 5$

Cofactor of 0 is $A_{21} = (-1)^{2+1} M_{21} = -M_{21} = -26$

Cofactor of – 1 is $A_{22} = (-1)^{2+2} M_{22} = M_{22} = 13$

Cofactor of 2 is $A_{23} = (-1)^{2+3} M_{23} = -M_{23} = 26$

Cofactor of 5 is $A_{31} = (-1)^{3+1} M_{31} = M_{31} = 9$

Cofactor of – 2 is $A_{32} = (-1)^{3+2} M_{32} = -M_{32} = -6$

Cofactor of 6 is $A_{33} = (-1)^{3+3} M_{33} = M_{33} = -3$

Singular and non-singular matrices:

A square matrix A is said to be singular if $|A| = 0$

A square matrix A is said to be non-singular matrix, if $|A| \neq 0$.

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a singular matrix.

$$\because \ |A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 12 - 9 = 0$$

$B = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 6 & 3 \\ 4 & 8 & 9 \end{bmatrix}$ is a non-singular matrix.

$$\because \ |B| = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 6 & 3 \\ 4 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 6 & 3 \\ 8 & 9 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} + 5\begin{vmatrix} 2 & 6 \\ 4 & 8 \end{vmatrix}$$

$= 1(54 - 24) - 7(18 - 12) + 5\,(16 - 24)$
$= 1(30) - 7(6) + 5(-8)$
$= -52 \neq 0$

$\therefore$ The matrix B is a non-singular matrix.

1.2.3 Properties of Determinants

There are many properties of determinants, which are very much useful in solving problems. The following properties are true for determinants of any order. But here we are going to prove the properties only for the determinant of order 3.

Property 1:

The value of a determinant is unaltered by interchanging its rows and columns.

Proof:

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Expanding Δ by the first row we get,

$$\Delta = a_1(b_2\,c_3 - b_3\,c_2) - b_1(a_2\,c_3 - a_3\,c_2) + c_1\,(a_2b_3 - a_3\,b_2)$$
$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \qquad \dots (1)$$

Let us interchange the rows and columns of Δ. Thus we get a new determinant.

$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Since determinant can be expanded by any row or any column we get

$$\Delta_1 = a_1(b_2c_3 - c_2b_3) - b_1\,(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$
$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \qquad \dots (2)$$

From equations (1) and (2) we have $\Delta = \Delta_1$ Hence the result.

Property 2:

If any two rows (columns) of a determinant are interchanged the determinant changes its sign but its numerical value is unaltered.

Proof:

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1 (a_2b_3 - a_3 b_2)$$

$$\Delta = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \qquad \ldots (1)$$

Let Δ_1 be the determinant obtained from Δ by interchanging the first and second rows. i.e. R_1 and R_2.

$$\Delta_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now we have to show that $\Delta_1 = -\Delta$.

Expanding Δ_1 by R_2, we have,

$$\Delta_1 = -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

$$= -[a_1b_2c_3 - a_1b_3c_2 + a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1] \quad \ldots (2)$$

From (1) and (2) we get $\Delta_1 = -\Delta$.

Similarly we can prove the result by interchanging any two columns.

Corollary:

The sign of a determinant changes or does not change according as there is an odd or even number of interchanges among its rows (columns).

Property 3:

If two rows (columns) of a determinant are identical then the value of the determinant is zero.

Proof:

Let Δ be the value of the determinant. Assume that the first two rows are identical. By interchanging R_1 and R_2 we obtain $-\Delta$ (by property2). Since R_1 and R_2 are identical even after the interchange we get the same Δ.

i.e. $\Delta = -\Delta \quad \Rightarrow \quad 2\Delta = 0 \quad$ i.e. $\quad \Delta = 0$

Property 4:

If every element in a row (or column) of a determinant is multiplied by a constant "*k*" then the value of the determinant is multiplied by *k*.

Proof:

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let Δ_1 be the determinant obtained by multiplying the elements of the first row by 'k' then $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Expanding along R_1 we get,

$$\begin{aligned}
\Delta_1 &= ka_1 (b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2) \\
&= k[a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1] \\
\Delta_1 &= k\Delta. \text{ Hence the result.}
\end{aligned}$$

Note:

(1) Let A be any square matrix of order n. Then kA is also a square matrix which is obtained by multiplying every entry of the matrix A with the scalar k. But the determinant k |A| means every entry in a row (or a column) is multiplied by the scalar k.

(2) Let A be any square matrix of order n then $|kA| = k^n|A|$.

Deduction from properties (3) and (4)

If two rows (columns) of a determinant are proportional i.e. one row (column) is a scalar multiple of other row (column) then its value is zero.

Property 5:

If every element in any row (column) can be expressed as the sum of two quantities then given determinant can be expressed as the sum of two determinants of the same order with the elements of the remaining rows (columns) of both being the same.

Proof: Let $\Delta = \begin{vmatrix} \alpha_1 + x_1 & \beta_1 + y_1 & \gamma_1 + z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding Δ along the first row, we get

$$\begin{aligned}
\Delta &= (\alpha_1 + x_1) \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - (\beta_1 + y_1) \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + (\gamma_1 + z_1) \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= \left\{\alpha_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \beta_1 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \gamma_1 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right\} \\
&\qquad + \left\{x_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - y_1 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + z_1 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right\}
\end{aligned}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hence the result.

Note: If we wish to add (or merge) two determinants of the same order we add corresponding entries of a particular row (column) provided the other entries in rows (columns) are the same.

Property 6:

A determinant is unaltered when to each element of any row (column) is added to those of several other rows (columns) multiplied respectively by constant factors.

i.e. A determinant is unaltered when to each element of any row (column) is added by the equimultiples of any parallel row (column).

Proof:

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let Δ_1 be a determinant obtained when to the elements of C_1 of Δ are added to those of second column and third column multiplied respectively by l and m.

$$\Delta_1 = \begin{vmatrix} a_1 + lb_1 + mc_1 & b_1 & c_1 \\ a_2 + lb_2 + mc_2 & b_2 & c_2 \\ a_3 + lb_3 + nc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \mathrm{l}b_1 & b_1 & c_1 \\ lb_2 & b_2 & c_2 \\ lb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mc_1 & b_1 & c_1 \\ mc_2 & b_2 & c_2 \\ mc_3 & b_3 & c_3 \end{vmatrix} \quad \text{(by property 5)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0 + 0 \quad \left[\because \begin{array}{l} C_1 \text{ is proportional to } C_2 \text{ in the second det.} \\ C_1 \text{ is proportional to } C_3 \text{ in the third det.} \end{array} \right]$$

Therefore $\Delta_1 = \Delta$. Hence the result.

Note:

(1) Multiplying or dividing all entries of any one row (column) by the same scalar is equivalent to multiplying or dividing the determinant by the same scalar.

(2) If all the entries above or below the principal diagonal are zero (upper triangular, lower triangular) then the value of the determinant is equal to the product of the entries of the principal diagonal.

For example, let us consider

$$|A| = \begin{vmatrix} 3 & 2 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 3(5-0) - 2(0-0) + 7(0-0) = 15$$

The value of the determinant A is 15.

The product of the entries of the principal diagonal is $3 \times 5 \times 1 = 15$.

Example 1.9: Solve $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

Solution: Since all the entries below the principal diagonal are zero, the value of the determinant is $(x-1)(x-2)(x-3)$

$\therefore \ (x-1)(x-2)(x-3) = 0 \Rightarrow x = 1, \ x = 2, \ x = 3$

Example 1.10: Solve for x if $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$

Solution : $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$

$\Rightarrow (x^2 - 35) + (1 - 2) = 0 \Rightarrow x^2 - 35 - 1 = 0 \Rightarrow x^2 - 36 = 0$

$\Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

Example 1.11: Solve for x if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$

Solution:

$(0)\begin{vmatrix} 2 & x \\ 3 & x \end{vmatrix} - 1\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} + (0)\begin{vmatrix} x & 2 \\ 1 & 3 \end{vmatrix} = 0 \quad \Rightarrow \quad 0 - 1[x^2 - x] + 0 = 0$

$-x^2 + x = 0$ i.e. $x(1-x) = 0 \Rightarrow x = 0, \ x = 1$

Example 1.12: Evaluate (i) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (ii) $\begin{vmatrix} x+2a & x+3a & x+4a \\ x+3a & x+4a & x+5a \\ x+4a & x+5a & x+6a \end{vmatrix}$

Solution:

(i) Let $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} \quad C_3 \rightarrow C_3 + C_2$

$= 0 \quad [\because C_1 \text{ is proportional to } C_3]$

(ii) Let $\Delta = \begin{vmatrix} x+2a & x+3a & x+4a \\ x+3a & x+4a & x+5a \\ x+4a & x+5a & x+6a \end{vmatrix} = \begin{vmatrix} x+2a & a & 2a \\ x+3a & a & 2a \\ x+4a & a & 2a \end{vmatrix}$ $\begin{array}{l} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{array}$

$= 0 \quad [\because C_2 \text{ is proportional to } C_3]$

Example 1.13: Prove that $\begin{vmatrix} 2x+y & x & y \\ 2y+z & y & z \\ 2z+x & z & x \end{vmatrix} = 0$

Solution: $\begin{vmatrix} 2x+y & x & y \\ 2y+z & y & z \\ 2z+x & z & x \end{vmatrix} = \begin{vmatrix} 2x & x & y \\ 2y & y & z \\ 2z & z & x \end{vmatrix} + \begin{vmatrix} y & x & y \\ z & y & z \\ x & z & x \end{vmatrix}$

$= 0 + 0 \quad \left[\because \begin{array}{l} C_1 \text{ is proportional to } C_2 \text{ in the first det.} \\ C_1 \text{ is identical to } C_3 \text{ in the second det.} \end{array}\right]$

$= 0$

Example 1.14: Prove that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:

$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$ $\begin{array}{l} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{array}$

$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$ Take $(a-b)$ and $(b-c)$ from R_1 and R_2 respectively.

$= (a-b)(b-c)\left[(1)(b+c) - (1)(a+b)\right] = (a-b)(b-c)(c-a)$

Example 1.15: Prove that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

Solution: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix}$ $\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$= xy \quad [\because \text{ upper diagonal matrix}]$

Example 1.16: Prove that $\begin{vmatrix} 1/a^2 & bc & b+c \\ 1/b^2 & ca & c+a \\ 1/c^2 & ab & a+b \end{vmatrix} = 0$

$$\begin{vmatrix} 1/a^2 & bc & b+c \\ 1/b^2 & ca & c+a \\ 1/c^2 & ab & a+b \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1/a & abc & a(b+c) \\ 1/b & abc & b(c+a) \\ 1/c & abc & c(a+b) \end{vmatrix} \quad \text{Multiply } R_1, R_2, R_3 \text{ by } a, b, c \text{ respectively}$$

$$= \frac{abc}{abc}\begin{vmatrix} 1/a & 1 & a(b+c) \\ 1/b & 1 & b(c+a) \\ 1/c & 1 & c(a+b) \end{vmatrix} \quad \text{Take } abc \text{ from } C_2$$

$$= \frac{1}{abc}\begin{vmatrix} bc & 1 & a(b+c) \\ ca & 1 & b(c+a) \\ ab & 1 & c(a+b) \end{vmatrix} \quad \text{Multiply } C_1 \text{ by } abc$$

$$= \frac{1}{abc}\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix} \quad C_3 \to C_3 + C_1$$

$$= \frac{(ab+bc+ca)}{abc}\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} \quad \text{Take } (ab+bc+ca) \text{ from } C_3$$

$$= \frac{(ab+bc+ca)}{abc}(0) \quad [\because C_2 \text{ is identical to } C_3]$$

$$= 0$$

Example 1.17: Prove that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

Solution: Let $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$

Multiply R_1, R_2 and R_3 by a, b and c respectively

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ab \\ ca^2b^2 & abc & ca+bc \end{vmatrix}$$

$$= \frac{(abc)^2}{abc} \begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ca+bc \end{vmatrix} \quad \text{Take } abc \text{ from } C_1 \text{ and } C_2$$

$$= abc \begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix} \quad C_3 \to C_3 + C_1$$

$$= abc\,(ab+bc+ca) \begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} \quad \text{Take } (ab+bc+ca) \text{ from } C_3$$

$$= abc\,(ab+bc+ca)\ (0) \qquad [\because C_2 \text{ is identical to } C_3]$$

$$= 0$$

Example 1.18 : Prove that $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)\,(b+c)\,(c+a)$

Solution:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = \begin{vmatrix} a+b & a+b & -(a+b) \\ -(b+c) & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix} \quad \begin{matrix} R_1 \to R_1 + R_2 \\ R_2 \to R_2 + R_3 \end{matrix}$$

$$= (a+b)\,(b+c) \begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix} \quad \text{Take } (a+b), (b+c) \text{ from } R_1 \text{ and } R_2 \text{ respectively}$$

$$= (a+b)\,(b+c) \begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix} \quad R_1 \to R_1 + R_2$$

$$= (a+b)\,(b+c) \times (-2) \begin{vmatrix} -1 & 1 \\ -b & a+b+c \end{vmatrix}$$

$$= (a+b)\,(b+c) \times (-2)\,\big[-(a+b+c) + b\big]$$

$$= (a+b)\,(b+c) \times (-2)\,[-a-c]$$

$$\Delta = 2(a+b)\,(b+c)\,(c+a)$$

Example 1.19: Prove that $\begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix} = \lambda^2\,(a^2+b^2+c^2+\lambda)$

Solution: Let $\Delta = \begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix}$

Multiply R_1, R_2 and R_3,by a, b and c respectively

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+\lambda) & a^2b & a^2c \\ ab^2 & b(b^2+\lambda) & b^2c \\ ac^2 & bc^2 & c(c^2+\lambda) \end{vmatrix}$$

Take a, b and c from C_1, C_2 and C_3 respectively

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2+\lambda & a^2 & a^2 \\ b^2 & b^2+\lambda & b^2 \\ c^2 & c^2 & c^2+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2+\lambda & a^2+b^2+c^2+\lambda & a^2+b^2+c^2+\lambda \\ b^2 & b^2+\lambda & b^2 \\ c^2 & c^2 & c^2+\lambda \end{vmatrix} \quad R_1 \to R_1 + R_2 + R_3$$

$$= (a^2+b^2+c^2+\lambda)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+\lambda & b^2 \\ c^2 & c^2 & c^2+\lambda \end{vmatrix}$$

$$= (a^2+b^2+c^2+\lambda)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & \lambda & 0 \\ c^2 & 0 & \lambda \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (a^2+b^2+c^2+\lambda)\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix}$$

$$\therefore \begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix} = \lambda^2(a^2+b^2+c^2+\lambda)$$

EXERCISE 1.2

(1) Find the value of the determinant $\begin{vmatrix} 2 & 6 & 4 \\ -5 & -15 & -10 \\ 1 & 3 & 2 \end{vmatrix}$ without usual expansion.

(2) Identify the singular and non-singular matrix

(i) $\begin{bmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{bmatrix}$

(3) Solve (i) $\begin{vmatrix} 2 & x & 4 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -3$ (ii) $\begin{vmatrix} 4 & 3 & 9 \\ 3 & -2 & 7 \\ 4 & 4 & x \end{vmatrix} = -1$

(4) Evaluate (i) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & ab & c(a+b) \\ 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \end{vmatrix}$

(5) Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(6) Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

where a, b, c are non zero real numbers and hence evaluate the value of $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$

(7) Prove that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(8) If x, y, z are all different and $\begin{vmatrix} x & x^2 & 1-x^3 \\ y & y^2 & 1-y^3 \\ z & z^2 & 1-z^3 \end{vmatrix} = 0$

then show that $xyz = 1$

(9) Prove that (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

(ii) $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)\ (x-z)^2$

(10) Prove that

(i) $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ (ii) $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

(iii) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

(iv) $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

1.2.4 Factor method

Application of Remainder theorem to determinants

Theorem:

If each element of a determinant (Δ) is polynomial in x and if Δ vanishes for $x = a$ then $(x - a)$ is a factor of Δ.

Proof:

Since the elements of Δ are polynomial in x, on expansion Δ will be a polynomial function in x. (say $p(x)$). For $x = a$, $\Delta = 0$

i.e. $p(x) = 0$ when $x = a$, i.e. $p(a) = 0$

$\therefore$ By Remainder theorem $(x - a)$ is a factor of $p(x)$.

i.e. $(x - a)$ is a factor of Δ.

Note:

(1) This theorem is very much useful when we have to obtain the value of the determinant in ‘factors’ form. Thus, for example if on putting $a = b$ in the determinant Δ any two of its rows or columns become identical then $\Delta = 0$ and hence by the above theorem $a - b$ will be a factor of Δ.

(2) If r rows (column) are identical in a determinant of order n ($n \geq r$) when we put $x = a$, then $(x - a)^{r-1}$ is a factor of Δ.

(3) $(x + a)$ is a factor of the polynomial $f(x)$ if and only if $x = -a$ is a root of the equation $f(x) = 0$.

Remark: In this section we deal certain problems with symmetric and cyclic properties.

Example 1.20: Prove that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution:

Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$. Put $a = b$, $\Delta = \begin{vmatrix} 1 & b & b^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$ [$\because$ R_1 is identical to R_2]

$\therefore$ $(a-b)$ is a factor of Δ.

Similarly we observe that Δ is symmetric in a, b, c, by putting $b = c$, $c = a$, we get $\Delta = 0$. Hence $(b-c)$ and $(c-a)$ are also factors of Δ. $\therefore$ The product $(a-b)(b-c)(c-a)$ is a factor of Δ. The degree of this product is 3. The product of leading diagonal elements is $1 . b . c^3$. The degree of this product is 4.

$\therefore$ By cyclic and symmetric properties, the remaining symmetric factor of first degree must be $k(a+b+c)$, where k is any non-zero constant.

Thus $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)\,k(a+b+c)$

To find the value of k, give suitable values for a, b, c so that both sides do not become zero. Take $a = 0$, $b = 1$, $c = 2$.

$\therefore$ $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 8 \end{vmatrix} = k(3)(-1)(-1)(2) \Rightarrow k = 1$

$\therefore$ $\Delta = (a-b)(b-c)(c-a)(a+b+c)$

Note: An important note regarding the remaining symmetric factor in the factorisation of cyclic and symmetric expression in a, b and c

If m is the difference between the degree of the product of the factors (found by the substitution) and the degree of the product of the leading diagonal elements and if

(1) m is zero then the other symmetric factor is a constant (k)

(2) m is one then the other symmetric factor of degree 1 is $k(a+b+c)$

(3) m is two then the other symmetric factor of degree 2 is $k(a^2+b^2+c^2)+l\,(ab+bc+ca)$

Example 1.21:

Prove by factor method $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Solution:

Let $\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ Put $a = b$ $\quad \Delta = \begin{vmatrix} 1 & b^2 & b^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = 0$ $\quad [\because R_1 \equiv R_2]$

$\therefore$ $(a-b)$ is a factor of Δ.

By symmetry on putting $b = c$ and $c = a$ we can easily show that Δ becomes zero and therefore $(b-c)$ and $(c-a)$ are also factors of Δ.

This means the product $(a-b)(b-c)(c-a)$ is a factor of Δ. The degree of this product is 3. The degree of the product of leading diagonal elements b^2c^3 is 5.

$\therefore$ The other factor is $k(a^2+b^2+c^2) + l(ab+bc+ca)$

$$\therefore \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \left[k(a^2+b^2+c^2) + l(ab+bc+ca)\right](a-b)(b-c)(c-a)$$

To determine k and l give suitable values for a, b and c so that both sides do not become zero. Take $a = 0$, $b = 1$ and $c = 2$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 8 \end{vmatrix} = [k(5) + l(2)]\ (-1)(-1)(2)$$

$\Rightarrow 4 = (5k + 2l)\,2 \quad \Rightarrow \quad 5k + 2l = 2$ … (1)

Again put $a = 0$, $b = -1$ and $c = 1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = [k(2) + l(-1)]\ (+1)(-2)(1)$$

$\Rightarrow 2 = (2k - l)(-2) \Rightarrow 2k - l = -1$ … (2)

On solving (1) and (2) we get $k = 0$ and $l = 1$

$\therefore\ \Delta = (ab + bc + ca)(a - b)(b - c)(c - a)$

$= (a - b)(b - c)(c - a)(ab + bc + ca)$

Example 1.22: Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a + b + c)^3$

Solution:

Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ Put $a = 0$ we get

$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ b^2 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = 0 \quad [\because C_2$ is porportional to $C_3]$

$\therefore\ (a - 0) = a$ is a factor of Δ.

Similarly on putting $b = 0$, $c = 0$, we see that the value of Δ is zero.

$\therefore\ a, b, c$ are factors of Δ. Put $a + b + c = 0$, we have

$\Delta = \begin{vmatrix} (-a)^2 & a^2 & a^2 \\ b^2 & (-b)^2 & b^2 \\ c^2 & c^2 & (-c)^2 \end{vmatrix} = 0$

Since three columns are identical, $(a + b + c)^2$ is a factor of Δ.

$\therefore\ abc\,(a + b + c)^2$ is a factor of Δ and is of degree 5. The product of the leading diagonal elements $(b + c)^2\,(c + a)^2\,(a + b)^2$ is of degree 6.

$\therefore$ The other factor of Δ must be $k(a + b + c)$.

$\therefore\ \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = k\,abc\,(a + b + c)^3$

Take the values $a = 1$, $b = 1$ and $c = 1$

$\therefore\ \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k(1)(1)(1)(3)^3 \Rightarrow 54 = 27k \Rightarrow k = 2$

$\therefore\ \Delta = 2abc\,(a + b + c)^3$

Example 1.23: Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2\,(x+2a)$

Solution:

Let $\Delta = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$ Put $x = a$ $\therefore$ $\Delta = \begin{vmatrix} a & a & a \\ a & a & a \\ a & a & a \end{vmatrix} = 0$

Since all the three rows are identical $(x-a)^2$ is a factor of Δ.

Put $x = -2a$.

$$\Delta = \begin{vmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{vmatrix} = \begin{vmatrix} 0 & a & a \\ 0 & -2a & a \\ 0 & a & -2a \end{vmatrix} = 0 \quad [C_1 \to C_1 + C_2 + C_3]$$

$(x + 2a)$ is a factor of Δ.

$\therefore$ $(x-a)^2\,(x+2a)$ is a factor of Δ and is of degree 3. The degree of the product of leading diagonal element is also 3. Therefore the other factor must be k.

$$\therefore \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = k(x-a)^2\,(x+2a).$$

Equate x^3 term on both sides, $1 = k$ $\therefore$ $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2\,(x+2a)$

Example 1.24: Using factor method, prove $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2\,(x+9)$

Solution: Let $\Delta = \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix}$

Put $x = 1$, $\Delta = \begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 0$

Since all the three rows are identical, $(x-1)^2$ is a factor of Δ.

Put $x = -9$ in Δ, then $\Delta = \begin{vmatrix} -8 & 3 & 5 \\ 2 & -7 & 5 \\ 2 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 5 \\ 0 & -7 & 5 \\ 0 & 3 & -5 \end{vmatrix} = 0 \quad [\because C_1 \to C_1 + C_2 + C_3]$

$\therefore$ $(x + 9)$ is a factor of Δ.

The product $(x - 1)^2 (x + 9)$ is a factor of Δ and is of degree 3. The degree of the product of leading diagonal elements $(x + 1)(x + 2)(x + 4)$ is also 3.

$\therefore$ The remaining factor must be a constant "k"

$\therefore \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = k(x - 1)^2 (x + 9)$. Equating x^3 term on both sides we get $k = 1$

Thus $\Delta = (x - 1)^2 (x + 9)$

EXERCISE 1.3

(1) Using factor method show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

(2) Prove by factor method $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$

(3) Solve using factor method $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$

(4) Factorise $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

(5) Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a + b + c)(a - b)(b - c)(c - a)$

1.2.5 Product of determinants

Rule for multiplication of two determinants is the same as the rule for multiplication of two matrices.

While multiplying two matrices "row-by-column" rule alone can be followed. The process of interchanging the rows and columns will not affect the value of the determinant. Therefore we can also adopt the following procedures for multiplication of two determinants.

(1) Row-by-row multiplication rule
(2) Column-by-column multiplication rule
(3) Column-by-row multiplication rule

Note: The determinant of the product matrix is equal to the product of the individual determinant values of the square matrices of same order.

i.e. Let A and B be two square matrices of the same order.

We have | AB | = | A | | B |

This statement is verified in the following example.

Example 1.25: If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ are two square matrices then show that | AB | = | A | | B |

Solution:

Given that $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \ldots (1)$$

$$|A| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$|B| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$|A|\ \ |B| = 1 \times 1 = 1 \qquad \ldots (2)$$

From (1) and (2) | AB | = | A | | B |

Example 1.26: Show that $\begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

Solution: L.H.S. $= \begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}^2 = \begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix} \begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}$

$$= \begin{vmatrix} o + c^2 + b^2 & o + o + ab & o + ac + o \\ o + o + ab & c^2 + o + a^2 & bc + o + o \\ o + ac + o & bc + o + o & b^2 + a^2 + 0 \end{vmatrix}$$

$$= \begin{vmatrix} c^2+b^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & b^2+a^2 \end{vmatrix} = \text{R.H.S.}$$

Example 1.27: Prove that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1^2+a_2^2 & a_1b_1+a_2b_2 \\ a_1b_1+a_2b_2 & b_1^2+b_2^2 \end{vmatrix}$

Solution:

$$\text{L.H.S.} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \left(\begin{array}{l}\text{Interchange rows and} \\ \text{columns of the first determinant}\end{array}\right)$$

$$= \begin{vmatrix} a_1^2+a_2^2 & a_1b_1+a_2b_2 \\ a_1b_1+a_2b_2 & b_1^2+b_2^2 \end{vmatrix}$$

Example 1.28: Show that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

Solution:

$$\text{R.H.S.} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}; \begin{array}{l}\text{Interchanging } R_2 \text{ and } R_3 \\ \text{in the } 2^{\text{nd}} \text{ determinant}\end{array}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+cb & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ba \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \text{L.H.S.}$$

1.2.6 Relation between a determinant and its co-factor determinant

Consider $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let $A_1, B_1, C_1 \ldots \ldots$ be the co-factors of $a_1, b_1, c_1 \ldots \ldots$ in Δ

$\therefore$ The cofactor determinant is $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Let Δ be expanded by R_1 $\therefore \Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$\Rightarrow \quad \Delta = a_1$ (co-factor of a_1) $+ b_1$ (co-factor of b_1) $+ c_1$ (co-factor of c_1)

$\Rightarrow \quad \Delta = a_1A_1 + b_1 B_1 + c_1 C_1$

i.e. **The sum of the products of the elements of any row of a determinant with the corresponding row of co-factor determinant is equal to the value of the determinant.**

Similarly $\Delta = a_2A_2 + b_2B_2 + c_2C_2$ $\qquad \Delta = a_3A_3 + b_3B_3 + c_3C_3$

Now let us consider the sum of the product of first row elements with the corresponding second row elements of co-factor determinant i.e. let us consider the expression

$a_1A_2 + b_1B_2 + c_1C_2$

$$= -a_1 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1)$$

$$= 0$$

$\therefore$ The expression $a_1A_2 + b_1B_2 + c_1C_2 = 0$

Thus we have

$a_1A_3 + b_1B_3 + c_1C_3 = 0$; $a_2A_1 + b_2B_1 + c_2C_1 = 0$; $a_2A_3 + b_2B_3 + c_2C_3 = 0$

$a_3A_1 + b_3B_1 + c_3C_1 = 0$; $a_3A_2 + b_3B_2 + c_3C_2 = 0$

i.e. The sum of the products of the elements of any row of a determinant with any other row of co-factor determinant is equal to 0

Note: Instead of rows, if we take columns we get the same results.

$\therefore \Delta = a_1A_1 + a_2A_2 + a_3A_3$

$\Delta = b_1B_1 + b_2B_2 + b_3B_3$

$\Delta = c_1C_1 + c_2C_2 + c_3C_3$

Thus the above results can be put in a tabular column as shown below.

Row-wise

	R_1	R_2	R_3
r_1	Δ	0	0
r_2	0	Δ	0
r_3	0	0	Δ

Column-wise

	C_1	C_2	C_3
c_1	Δ	0	0
c_2	0	Δ	0
c_3	0	0	Δ

Where r_i's c_i's are i^{th} row and i^{th} column of the original determinant R_i's, C_i's are i^{th} row and i^{th} column respectively of the corresponding co-factor determinant.

Example 1.29: If A_1, B_1, C_1 are the co-factors of a_1, b_1, c_1 in $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then show that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$

Solution: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_1C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

i.e. $\Delta \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3 \quad \Rightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$

EXERCISE 1.4

(1) Show that $\begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2a^2 & -a^2 & -a^2 \\ -a^2 & -1 & a^2-2a \\ -a^2 & a^2-2a & -1 \end{vmatrix}$

(2) Show that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} a^2 & 1 & 2a \\ b^2 & 1 & 2b \\ c^2 & 1 & 2c \end{vmatrix} = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

2. VECTOR ALGEBRA

2.1 Introduction:

The development of the concept of vectors was influenced by the works of the German Mathematician H.G. Grassmann (1809 – 1877) and the Irish mathematician W.R. Hamilton (1805 – 1865). It is interesting to note that both were linguists, being specialists in Sanskrit literature. While Hamilton occupied high positions, Grassman was a secondary school teacher.

The best features of Quaternion Calculus and Cartesian Geometry were united, largely through the efforts of the American Mathematician J.B. Gibbs (1839 – 1903) and Q. Heariside (1850 – 1925) of England and new subject called Vector Algebra was created. The term vectors was due to Hamilton and it was derived from the Latin word 'to carry'. The theory of vectors was also based on Grassman's theory of extension.

It was soon realised that vectors would be the ideal tools for the fruitful study of many ideas in geometry and physics. Vector algebra is widely used in the study of certain type of problems in Geometry, Mechanics, Engineering and other branches of Applied Mathematics.

Physical quantities are divided into two categories – scalar quantities and vector quantities.

Definitions:

Scalar : A quantity having only magnitude is called a scalar. It is not related to any fixed direction in space.

Examples : mass, volume, density, work, temperature, distance, area, real numbers etc.

To represent a scalar quantity, we assign a real number to it, which gives its magnitude in terms of a certain basic unit of a quantity. Throughout this chapter, by scalars we shall mean real numbers. Normally, scalars are denoted by a, b, c…

Vector : A quantity having both magnitude and direction is called a vector.

Examples : displacement, velocity, acceleration, momentum, force, moment of a force, weight etc.

Representation of vectors:

Vectors are represented by directed line segments such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end denotes the direction of the vector.

A vector denoted by $\vec{a} = \overrightarrow{AB}$ is determined by two points A, B such that the magnitude of the vector is the length of the

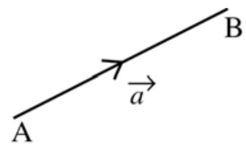


Fig. 2. 1

line segment AB and its direction is that from A to B. The point A is called initial point of the vector $\overrightarrow{AB}$ and B is called the terminal point. Vectors are generally denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$ … (read as vector *a*, vector *b*, vector *c*, …)

Magnitude of a vector

The modulus or magnitude of a vector $\vec{a} = \overrightarrow{AB}$ is a positive number which is a measure of its length and is denoted by $\left|\vec{a}\right| = \left|\overrightarrow{AB}\right| = AB$ The modulus of $\vec{a}$ is also written as ***'a'***

$$\text{Thus } \left|\vec{a}\right| = a \ ; \quad \left|\vec{b}\right| = b \ ; \quad \left|\vec{c}\right| = c$$

$$\left|\overrightarrow{AB}\right| = AB \ ; \quad \left|\overrightarrow{CD}\right| = CD \ ; \quad \left|\overrightarrow{PQ}\right| = PQ$$

Caution: The two end points A and B are not interchangeable.

Note: Every vector $\overrightarrow{AB}$ has three characteristics:

Length : The length of $\overrightarrow{AB}$ will be denoted by $\left|\overrightarrow{AB}\right|$ or AB.

Support : The line of unlimited length of which AB is a segment is called the support of the vector $\overrightarrow{AB}$,

Sense : The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ is from B to A. Thus the sense of a directed line segment is from its initial point to the terminal point.

Equality of vectors:

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a} = \vec{b}$, if they have the

(i) same magnitude (ii) same direction

In fig (2.2) AB || CD and AB = CD

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are in the same direction

Hence $\overrightarrow{AB} = \overrightarrow{CD}$ or $\vec{a} = \vec{b}$

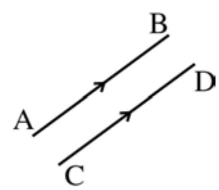


Fig. 2. 2

2.2 Types of Vectors

Zero or Null Vector:

A vector whose initial and terminal points are coincident is called a zero or null or a void vector. The zero vector is denoted by $\vec{O}$

Vectors other than the null vector are called proper vectors.

Unit vector:

A vector whose modulus is unity, is called a unit vector.

The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ (read as '*a* cap'). Thus $|\hat{a}| = 1$

The unit vectors parallel to $\vec{a}$ are $\pm\hat{a}$

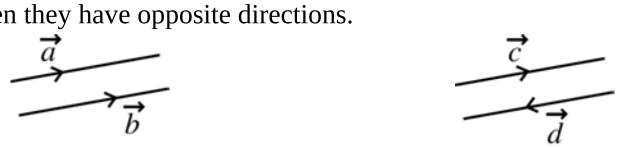
Result: $\vec{a} = |\vec{a}|\,\hat{a}$ [i.e. any vector = (its modulus) × (unit vector in that direction)]

$$\Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|}\ ;\ \left(\vec{a} \neq \vec{O}\right)$$

In general $\boxed{\text{unit vector in any direction} = \dfrac{\text{vector in that direction}}{\text{modulus of the vector}}}$

Like and unlike vectors:

Vectors are said to be **like** when they have the same sense of direction and **unlike** when they have opposite directions.



like vectors unlike vectors

Fig. 2. 3

Co-initial vectors:

Vectors having the same initial point are called co-initial vectors.

Co-terminus vectors:

Vectors having the same terminal point are called co-terminus vectors.

Collinear or Parallel vectors:

Vectors are said to be collinear or parallel if they have the same line of action or have the lines of action parallel to one another.

Coplanar vectors:

Vectors are said to be coplanar if they are parallel to the same plane or they lie in the same plane.

Negative vector:

The vector which has the same magnitude as that of $\vec{a}$ but opposite direction is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. Thus if $\overrightarrow{AB} = \vec{a}$ then $\overrightarrow{BA} = -\vec{a}$.

Reciprocal of a vector:

Let $\vec{a}$ be a non-zero vector. The vector which has the same direction as that of $\vec{a}$ but has magnitude reciprocal to that of $\vec{a}$ is called the reciprocal of $\vec{a}$ and is written as $(\vec{a})^{-1}$ where $\left|(\vec{a})^{-1}\right| = \dfrac{1}{a}$

Free and localised vector:

When we are at liberty to choose the origin of the vector at any point, then it is said to be a free vector. But when it is restricted to a certain specified point, then the vector is said to be localised vector.

2.3 Operations on vectors:

2.3.1 Addition of vectors:

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$ Join OB.

Then $\overrightarrow{OB}$ represents the addition (sum) of the vectors $\vec{a}$ and $\vec{b}$.

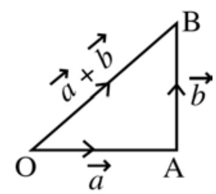


Fig. 2. 4

This is written as $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

Thus $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

This is known as the triangle law of addition of vectors which states that, if two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order.

Applying the triangle law of addition of vectors in ΔABC, we have

$$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$

$$\Rightarrow \quad \overrightarrow{BC} + \overrightarrow{CA} = -\overrightarrow{AB}$$

$$\Rightarrow \quad \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{0}$$

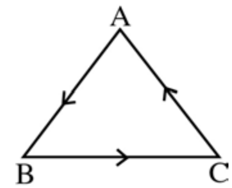


Fig. 2. 5

Thus the sum of the vectors representing the sides of a triangle taken in order is the null vector.

Parallelogram law of addition of vectors:

If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their sum $\vec{c}$ is represented by the diagonal of the parallelogram which is co-initial with the given vectors.

Symbolically we have $\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$

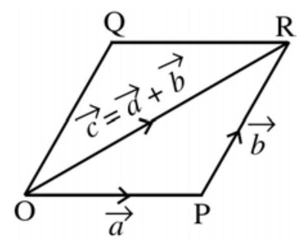


Fig. 2. 6

Thus if the vectors are represented by two adjacent sides of a parallelogram, the diagonal of the parallelogram will represent the sum of the vectors.

By repeated use of the triangle law we can find the sum of any number of vectors.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{BC} = \vec{c}$, $\overrightarrow{CD} = \vec{d}$, $\overrightarrow{DE} = \vec{e}$ be any five vectors as shown in the fig (2.7). We observe from the figure that each new vector is drawn from the terminal point of its previous one.

$$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{OE}$$

Thus the line joining the initial point of the first vector to the terminal point of the last vector is the sum of all the vectors. This is called the polygon law of addition of vectors.

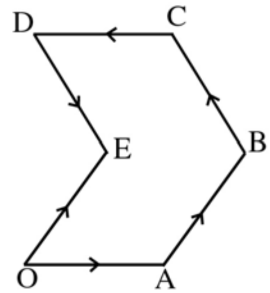


Fig. 2. 7

Note : It should be noted that the magnitude of $\vec{a} + \vec{b}$ is not equal to the sum of the magnitudes of $\vec{a}$ and $\vec{b}$.

2.3.2 Subtraction of vectors:

If $\vec{a}$ and $\vec{b}$ are two vectors, then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and $-\vec{b}$ and is denoted by $\vec{a} - \vec{b}$.

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$$

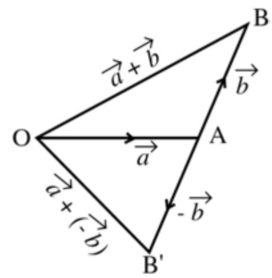


Fig. 2. 8

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$

Then $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

To subtract $\vec{b}$ from $\vec{a}$, produce BA to B′ such that AB = AB′. $\therefore$ $\overrightarrow{AB'} = -\overrightarrow{AB} = -\vec{b}$

Now by the triangle law of addition

$$\overrightarrow{OB'} = \overrightarrow{OA} + \overrightarrow{AB'} = \vec{a} + \left(-\vec{b}\right) = \vec{a} - \vec{b}$$

Properties of addition of vectors:

Theorem 2.1:

Vector addition is commutative i.e., if $\vec{a}$ and $\vec{b}$ are any two vectors then $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

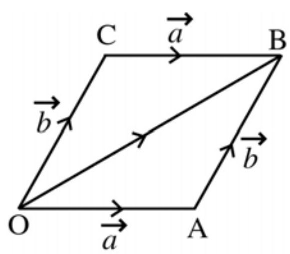


Fig. 2. 9

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$

In ΔOAB, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ (by triangle law of add.)

$\Rightarrow$ $\vec{a} + \vec{b} = \overrightarrow{OB}$ … (1)

Complete the parallelogram OABC

$\overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$; $\overrightarrow{OC} = \overrightarrow{AB} = \vec{b}$

In ΔOCB, we have $\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$ i.e. $\Rightarrow \vec{b} + \vec{a} = \overrightarrow{OB}$ … (2)

From (1) and (2) we have $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

$\therefore$ Vector addition is commutative.

Theorem 2.2:

Vector addition is associative

i.e. For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

$$\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$$

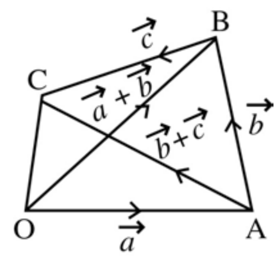


Fig. 2. 10

Proof :

Let $\overrightarrow{OA}=\vec{a}$; $\overrightarrow{AB}=\vec{b}$; $\overrightarrow{BC}=\vec{c}$

Join O and B ; O and C ; A and C

In ΔOAB, $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$

$\Rightarrow$ $\vec{a}+\vec{b}=\overrightarrow{OB}$ … (1)

In ΔOBC, $\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}$

$\Rightarrow$ $\left(\vec{a}+\vec{b}\right)+\vec{c}=\overrightarrow{OC}$ … (2) [using (1)]

In ΔABC, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow$ $\vec{b}+\vec{c}=\overrightarrow{AC}$ … (3)

In ΔOAC $\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$

$\Rightarrow$ $\vec{a}+\left(\vec{b}+\vec{c}\right)=\overrightarrow{OC}$ … (4) [using (3)]

From (2) and (4), we have $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$

$\therefore$ vector addition is associative.

Theorem 2.3:

For every vector $\vec{a}$, $\vec{a}+\vec{O}=\vec{O}+\vec{a}=\vec{a}$ where $\vec{O}$ is the null vector. [existence of additive identity]

Proof:

Let $\overrightarrow{OA}=\vec{a}$

Then $\vec{a}+\vec{O}=\overrightarrow{OA}+\overrightarrow{AA}=\overrightarrow{OA}=\vec{a}$

$\therefore\ \vec{a}+\vec{O}=\vec{a}$

Also $\vec{O}+\vec{a}=\overrightarrow{OO}+\overrightarrow{OA}=\overrightarrow{OA}=\vec{a}$

$$\therefore \ \vec{O} + \vec{a} = \vec{a}$$

$$\therefore \ \vec{a} + \vec{O} = \vec{O} + \vec{a} = \vec{a}$$

Theorem 2.4:

For every vector $\vec{a}$, there corresponds a vector $-\vec{a}$ such that

$\vec{a} + \left(-\vec{a}\right) = \vec{O} = \left(-\vec{a}\right) + \vec{a}$ [existence of additive inverse]

Proof: Let $\overrightarrow{OA} = \vec{a}$. Then $\overrightarrow{AO} = -\vec{a}$

$$\therefore \ \vec{a} + \left(-\vec{a}\right) = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO} = \vec{O}$$

$$\left(-\vec{a}\right) + \vec{a} = \overrightarrow{AO} + \overrightarrow{OA} = \overrightarrow{AA} = \vec{O}$$

Hence $\vec{a} + \left(-\vec{a}\right) = \left(-\vec{a}\right) + \vec{a} = \vec{O}$

2.3.3 Multiplication of a vector by a scalar

Let m be a scalar and $\vec{a}$ be any vector, then $m\vec{a}$ is defined as a vector having the same support as that of $\vec{a}$ such that its magnitude is $|\, m \,|$ times the magnitude of $\vec{a}$ and its direction is same as or opposite to the direction of $\vec{a}$ according as m is positive or negative.

Result : Two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel if and only if $\vec{a} = m\vec{b}$ for some non-zero scalar m.

For any vector $\vec{a}$ we define the following:

$$(1)\ \vec{a} = \vec{a} \quad ; \quad (-1)\ \vec{a} = -\vec{a} \quad ; \quad 0\vec{a} = \vec{O}$$

Note: If $\vec{a}$ is a vector then $5\vec{a}$ is a vector whose magnitude is 5 times the magnitude of $\vec{a}$ and whose direction is same as that of $\vec{a}$. But $-5\vec{a}$ is a vector whose magnitude is 5 times the magnitude of $\vec{a}$ and whose direction is opposite to $\vec{a}$.

Properties of Multiplication of vectors by a scalar

The following are properties of multiplication of vectors by scalars.

For vectors $\vec{a}$, $\vec{b}$ and scalars m, n we have

(i) $m\left(-\vec{a}\right) = (-m)\vec{a} = -\left(m\vec{a}\right)$ (ii) $(-m)\left(-\vec{a}\right) = m\vec{a}$

(iii) $m\left(n\vec{a}\right) = (mn)\vec{a} = n\left(m\vec{a}\right)$ (iv) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

Theorem 2.5 (Without Proof) :

If $\vec{a}$ and $\vec{b}$ are any two vectors and m is a scalar

then $m\left(\vec{a} + \vec{b}\right) = m\vec{a} + m\vec{b}$.

Result : $m\left(\vec{a} - \vec{b}\right) = m\vec{a} - m\vec{b}$

2.4 Position vector

If a point O is fixed as the origin in space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector (P.V.) of P with respect to O.

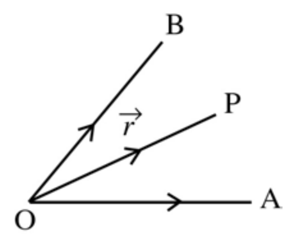


Fig. 2. 11

From the diagram $\overrightarrow{OP} = \vec{r}$

Similarly $\overrightarrow{OA}$ is called the position vector (P.V.) of A with respect to O and $\overrightarrow{OB}$ is the P.V. of B with respect to O.

Theorem 2.6: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ where $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the P.Vs of A and B respectively.

Proof: Let O be the origin. Let $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B respectively

Then $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$

In ΔOAB, we have by triangle law of addition

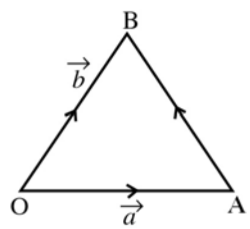


Fig. 2. 12

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$$

i.e. $\overrightarrow{AB}$ = (P.V of B) – (P.V of A)

Note : In $\overrightarrow{AB}$, the point B is the head of the vector and A is the tail of the vector.

$\therefore \overrightarrow{AB}$ = (P.V. of the head) – (P.V. of the tail). Similarly $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

The above rule will be very much useful in doing problems.

Theorem 2.7: [Section Formula – Internal Division]

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing AB internally in the ratio $m : n$. Then the position vector of P is given by

$$\overrightarrow{OP} = \frac{n\vec{a} + m\vec{b}}{m+n}$$

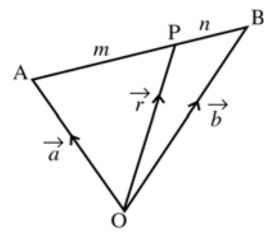


Fig. 2. 13

Proof:

Let O be the origin.

Then $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$

Let the position vector of P with respect to O be $\vec{r}$ i.e. $\overrightarrow{OP} = \vec{r}$

Let P divide AB internally in the ratio $m : n$

Then $\frac{AP}{PB} = \frac{m}{n} \Rightarrow n\,AP = m\,PB \quad \Rightarrow n\,\overrightarrow{AP} = m\,\overrightarrow{PB}$

$$\Rightarrow n\left(\overrightarrow{OP} - \overrightarrow{OA}\right) = m\left(\overrightarrow{OB} - \overrightarrow{OP}\right) \quad \Rightarrow n\left(\vec{r} - \vec{a}\right) = m\left(\vec{b} - \vec{r}\right)$$

$$\Rightarrow \quad n\vec{r} - n\vec{a} = m\vec{b} - m\vec{r} \quad\quad \Rightarrow m\vec{r} + n\vec{r} = m\vec{b} + n\vec{a}$$

$$\Rightarrow \quad (m+n)\vec{r} = m\vec{b} + n\vec{a}$$

$$\boxed{\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}}$$

Result (1): If P is the mid point of AB, then it divides AB in the ratio 1 : 1.

$\therefore$ The P.V. of P is $\quad \frac{1\,.\,\vec{b} + 1.\,\vec{a}}{1+1} = \frac{\vec{a} + \vec{b}}{2}$

$\therefore$ P.V. of the mid point P of AB is $\quad \overrightarrow{OP} = \vec{r} = \dfrac{\vec{a} + \vec{b}}{2}$

Result (2): Condition that three points may be collinear

Proof: Assume that the points A, P and B (whose P.Vs are $\vec{a}$, $\vec{r}$ and $\vec{b}$ respectively) are collinear

We have $\quad \vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

$$(m+n)\vec{r} = m\vec{b} + n\vec{a}$$

$$\Rightarrow \quad (m+n)\vec{r} - m\vec{b} - n\vec{a} = 0$$

In this vector equation, sum of the scalar coefficients in the L.H.S. $= (m+n) - m - n = 0$

Thus we have the result, if A, B, C are collinear points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively then there exists scalars x, y, z such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{O}$ and $x + y + z = 0$

Conversely if the scalars x, y, z are such that $x + y + z = 0$ and $x\vec{a} + y\vec{b} + z\vec{c} = \vec{O}$ then the points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear.

Result 3: [Section formula – External division]

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing AB externally in the ratio $m : n$. Then the position vector of P is given by

$$\overrightarrow{OP} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

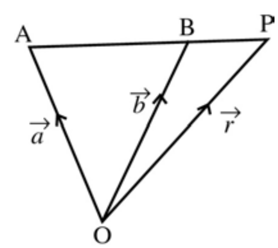


Fig. 2. 14

Proof:

Let O be the origin. A and B are the two points whose position vectors are $\vec{a}$ and $\vec{b}$

Then $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$

Let P divide AB externally in the ratio $m : n$. Let the position vector of P with respect to O be $\vec{r}$ i.e. $\overrightarrow{OP} = \vec{r}$

We have $\frac{\text{AP}}{\text{PB}} = \frac{m}{n}$ $\Rightarrow$ $n\,\text{AP} = m\,\text{PB}$

$\Rightarrow$ $n\,\overrightarrow{\text{AP}} = -m\,\overrightarrow{\text{PB}}$ $\left[\overrightarrow{\text{AP}}\ \&\ \overrightarrow{\text{PB}} \text{ are in the opposite direction}\right]$

$\Rightarrow$ $n\left(\overrightarrow{\text{OP}} - \overrightarrow{\text{OA}}\right) = -m\left(\overrightarrow{\text{OB}} - \overrightarrow{\text{OP}}\right)$ $\Rightarrow$ $n\left(\vec{r} - \vec{a}\right) = m\left(\vec{r} - \vec{b}\right)$

$\Rightarrow$ $n\vec{r} - n\vec{a} = m\vec{r} - m\vec{b}$ $\Rightarrow$ $m\vec{b} - n\vec{a} = m\vec{r} - n\vec{r}$

$\Rightarrow$ $m\vec{b} - n\vec{a} = (m-n)\,\vec{r}$

$$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Theorem 2.8: The medians of a triangle are concurrent.

Proof:

Let ABC be a triangle and let D, E, F be the mid points of its sides BC, CA and AB respectively. We have to prove that the medians AD, BE, CF are concurrent.

Let O be the origin and $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of A, B, C respectively.

The position vectors of D, E, F are

$$\frac{\vec{b}+\vec{c}}{2},\ \frac{\vec{c}+\vec{a}}{2},\ \frac{\vec{a}+\vec{b}}{2}$$

Let G_1 be the point on AD dividing it internally in the ratio 2 : 1

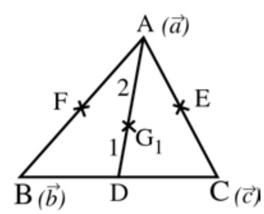


Fig. 2. 15

$\therefore$ P.V. of $G_1 = \frac{2\overrightarrow{\text{OD}} + 1\overrightarrow{\text{OA}}}{2+1}$

$$\overrightarrow{\text{OG}_1} = \frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right) + 1\,\vec{a}}{3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3} \qquad (1)$$

Let G_2 be the point on BE dividing it internally in the ratio 2 : 1

$\therefore$ $\overrightarrow{\text{OG}_2} = \frac{2\overrightarrow{\text{OE}} + 1\overrightarrow{\text{OB}}}{2+1}$

$$\overrightarrow{OG_2} = \frac{2\left(\frac{\vec{c} + \vec{a}}{2}\right) + 1.\,\vec{b}}{3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} \qquad (2)$$

Similarly if G_3 divides CF in the ratio 2 : 1 then

$$\overrightarrow{OG_3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} \qquad (3)$$

From (1), (2), (3) we find that the position vectors of the three points G_1, G_2, G_3 are one and the same. Hence they are not different points. Let the common point be denoted by G.

Therefore the three medians are concurrent and the point of concurrence is G.

Result:

The point of intersection of the three medians of a triangle is called the centroid of the triangle.

The position vector of the centroid G of ΔABC is $\overrightarrow{OG} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

where $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of the vertices A, B, C respectively and O is the origin of reference.

Example 2.1: If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the vectors represented by the three sides of a triangle, taken in order, then prove that $\vec{a} + \vec{b} + \vec{c} = \vec{O}$

Solution:

Let ABC be a triangle such that

$$\overrightarrow{BC} = \vec{a},\ \overrightarrow{CA} = \vec{b} \text{ and } \overrightarrow{AB} = \vec{c}$$

$$\begin{aligned} \vec{a} + \vec{b} + \vec{c} &= \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} \\ &= \overrightarrow{BA} + \overrightarrow{AB} \quad (\because \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}) \\ &= \overrightarrow{BB} = \vec{O} \end{aligned}$$

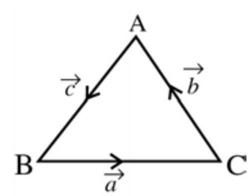


Fig. 2. 16

Example 2.2:

If $\vec{a}$ and $\vec{b}$ are the vectors determined by two adjacent sides of a regular hexagon, find the vectors determined by the other sides taken in order.

Solution:

Let ABCDEF be a regular hexagon such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$

Since AD || BC such that AD = 2.BC

$\therefore \quad \overrightarrow{AD} = 2\overrightarrow{BC} = 2\vec{b}$

In ΔABC, we have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\Rightarrow \quad \overrightarrow{AC} = \vec{a} + \vec{b}$

In ΔACD, $\quad \overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$

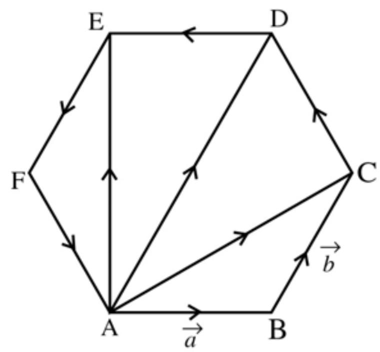


Fig. 2. 17

$$\therefore \quad \overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC} = 2\vec{b} - \left(\vec{a} + \vec{b}\right) = \vec{b} - \vec{a}$$

$$\overrightarrow{DE} = -\overrightarrow{AB} = -\vec{a}$$

$$\overrightarrow{EF} = -\overrightarrow{BC} = -\vec{b}$$

$$\overrightarrow{FA} = -\overrightarrow{CD} = -\left(\vec{b} - \vec{a}\right) = \vec{a} - \vec{b}$$

Example 2.3:

The position vectors of the points A, B, C, D are $\vec{a}$, $\vec{b}$, $2\vec{a} + 3\vec{b}$, $\vec{a} - 2\vec{b}$ respectively. Find $\overrightarrow{DB}$ and $\overrightarrow{AC}$

Solution: Given that

$$\overrightarrow{OA} = \vec{a} \quad ; \quad \overrightarrow{OB} = \vec{b} \quad ; \quad \overrightarrow{OC} = 2\vec{a} + 3\vec{b} \quad ; \quad \overrightarrow{OD} = \vec{a} - 2\vec{b}$$

$$\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD} = \vec{b} - \left(\vec{a} - 2\vec{b}\right) = \vec{b} - \vec{a} + 2\vec{b} = 3\vec{b} - \vec{a}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= \left(2\vec{a} + 3\vec{b}\right) - \vec{a}$$

$$= \vec{a} + 3\vec{b}$$

Example 2.4: Find the position vector of the points which divide the join of the points A and B whose P.Vs are $\vec{a} - 2\vec{b}$ and $2\vec{a} - \vec{b}$ internally and externally in the ratio 3 : 2

Solution:

$$\overrightarrow{OA} = \vec{a} - 2\vec{b} \quad ; \quad \overrightarrow{OB} = 2\vec{a} - \vec{b}$$

Let P divide AB internally in the ratio 3 : 2

$$\text{P.V. of P} = \frac{3\overrightarrow{OB} + 2\overrightarrow{OA}}{3+2} = \frac{3\left(2\vec{a} - \vec{b}\right) + 2\left(\vec{a} - 2\vec{b}\right)}{5}$$

$$= \frac{6\vec{a} - 3\vec{b} + 2\vec{a} - 4\vec{b}}{5} = \frac{8\vec{a} - 7\vec{b}}{5} = \frac{8}{5}\vec{a} - \frac{7}{5}\vec{b}$$

Let Q divide AB externally in the ratio 3 : 2

$$\text{P.V. of Q} = \frac{3\overrightarrow{OB} - 2\overrightarrow{OA}}{3-2} = \frac{3\left(2\vec{a} - \vec{b}\right) - 2\left(\vec{a} - 2\vec{b}\right)}{1}$$

$$= 6\vec{a} - 3\vec{b} - 2\vec{a} + 4\vec{b} = 4\vec{a} + \vec{b}$$

Example 2.5: If $\vec{a}$ and $\vec{b}$ are position vectors of points A and B respectively, then find the position vector of points of trisection of AB.

Solution:

Let P and Q be the points of trisection of AB

Let AP = PQ = QB = λ (say)

P divides AB in the ratio 1 : 2

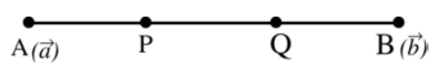


Fig. 2. 18

$$\text{P.V. of P} = \overrightarrow{OP} = \frac{1.\overrightarrow{OB} + 2.\overrightarrow{OA}}{1+2} = \frac{1.\vec{b} + 2.\vec{a}}{3} = \frac{\vec{b} + 2\vec{a}}{3}$$

Q is the mid-point of PB

$$\text{P.V. of Q} = \frac{\overrightarrow{OP} + \overrightarrow{OB}}{2} = \frac{\frac{\vec{b} + 2\vec{a}}{3} + \vec{b}}{2} = \frac{\frac{\vec{b} + 2\vec{a} + 3\vec{b}}{3}}{2} = \frac{2\vec{a} + 4\vec{b}}{6}$$

$$= \frac{\vec{a} + 2\vec{b}}{3}$$

Example 2.6: By using vectors, prove that a quadrilateral is a parallelogram if and only if the diagonals bisect each other.

Solution:

Let ABCD be a quadrilateral

First we assume that ABCD is a parallelogram

To prove that its diagonals bisect each other

Let O be the origin of reference.

$\therefore\ \overrightarrow{OA} = \vec{a}\ ,\ \overrightarrow{OB} = \vec{b}\ ,\ \overrightarrow{OC} = \vec{c}\ ,\ \overrightarrow{OD} = \vec{d}$

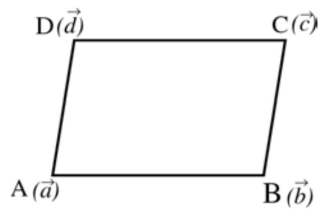


Fig. 2. 19

Since ABCD is a parallelogram $\overrightarrow{AB} = \overrightarrow{DC}$

$$\Rightarrow \quad \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD} \quad \Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$$

$$\Rightarrow \quad \vec{b} + \vec{d} = \vec{a} + \vec{c} \quad \Rightarrow \quad \frac{\vec{b} + \vec{d}}{2} = \frac{\vec{a} + \vec{c}}{2}$$

i.e. P.V. of the mid-point of BD = P.V. of the mid-point of AC. Thus, the point, which bisects AC also, bisects BD. Hence the diagonals of a parallelogram ABCD bisect each other.

Conversely suppose that ABCD is a quadrilateral such that its diagonals bisect each other. To prove that it is a parallelogram.

Let $\vec{a}\ ,\ \vec{b}\ ,\ \vec{c}\ ,\ \vec{d}$ be the position vectors of its vertices A, B, C and D respectively. Since diagonals AC and BD bisect each other.

P.V. of the mid-point of AC = P.V. of the mid-point of BD

$$\Rightarrow \quad \frac{\vec{a} + \vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2} \Rightarrow \vec{a} + \vec{c} = \vec{b} + \vec{d} \qquad \ldots (1)$$

$$\Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d} \quad \text{i.e. } \overrightarrow{AB} = \overrightarrow{DC}$$

Also (1) $\Rightarrow \quad \vec{d} - \vec{a} = \vec{c} - \vec{b} \quad$ i.e. $\overrightarrow{AD} = \overrightarrow{BC}$

Hence ABCD is a parallelogram.

Example 2.7:

In a triangle ABC if D and E are the midpoints of sides AB and AC respectively, show that $\overrightarrow{BE} + \overrightarrow{DC} = \frac{3}{2}\overrightarrow{BC}$

Solution:

For convenience we choose A as the origin.

Let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Since D and E are the mid-points of AB and AC, the position vectors of D and E are $\frac{\vec{b}}{2}$ and $\frac{\vec{c}}{2}$ respectively.

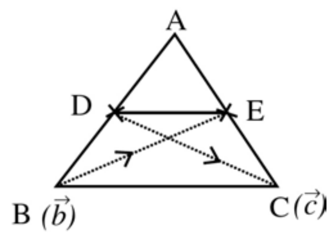


Fig. 2. 20

Now $\overrightarrow{BE}$ = P.V. of E – P.V. of B = $\frac{\vec{c}}{2} - \vec{b}$

$\overrightarrow{DC}$ = P.V. of C – P.V. of D = $\vec{c} - \frac{\vec{b}}{2}$

$\therefore\ \overrightarrow{BE} + \overrightarrow{DC} = \frac{\vec{c}}{2} - \vec{b} + \vec{c} - \frac{\vec{b}}{2} = \frac{3}{2}\vec{c} - \frac{3}{2}\vec{b}$

$= \frac{3}{2}\left(\vec{c} - \vec{b}\right)$ $= \frac{3}{2}$ [P.V. of C – P.V. of B]

$= \frac{3}{2}\overrightarrow{BC}$

Example 2.8: Prove that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.

Solution:

Let ABC be a triangle, and let O be the origin of reference. Let D and E be the midpoints of AB and AC respectively.

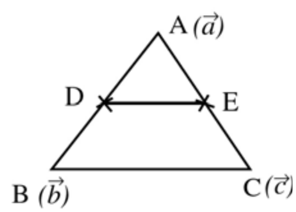


Fig. 2. 21

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$

P.V. of D = $\overrightarrow{OD} = \frac{\vec{a} + \vec{b}}{2}$

P.V. of E = $\overrightarrow{OE} = \frac{\vec{a} + \vec{c}}{2}$

Now $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = \left(\frac{\vec{a} + \vec{c}}{2}\right) - \left(\frac{\vec{a} + \vec{b}}{2}\right)$

$$= \frac{\vec{a} + \vec{c} - \vec{a} - \vec{b}}{2} = \frac{1}{2}\left(\vec{c} - \vec{b}\right) = \frac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OB}\right) = \frac{1}{2}\overrightarrow{BC}$$

$$\therefore \overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC} \quad \Rightarrow \quad DE \parallel BC$$

Also $\quad \overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC} \quad \Rightarrow \quad \left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{BC}\right| \Rightarrow DE = \frac{1}{2}BC$

Hence DE || BC and DE = $\frac{1}{2}$ BC.

Example 2.9: Using vector method, prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral taken in order form a parallelogram.

Solution:

Let ABCD be a quadrilateral and let P, Q, R, S be the mid-points of the sides AB, BC, CD and DA respectively.

Then the position vectors of P, Q, R, S are

$$\frac{\vec{a} + \vec{b}}{2}, \quad \frac{\vec{b} + \vec{c}}{2}, \quad \frac{\vec{c} + \vec{d}}{2}, \quad \frac{\vec{d} + \vec{a}}{2}$$

respectively.

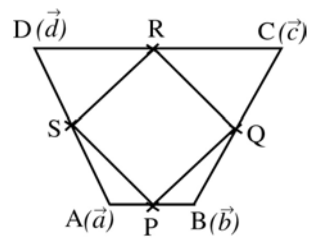


Fig. 2. 22

In order to prove that PQRS is a parallelogram it is sufficient to show that $\overrightarrow{PQ} = \overrightarrow{SR}$ and $\overrightarrow{PS} = \overrightarrow{QR}$

Now $\overrightarrow{PQ}$ = P.V. of Q – P.V. of P = $\left(\frac{\vec{b} + \vec{c}}{2}\right) - \left(\frac{\vec{a} + \vec{b}}{2}\right) = \frac{\vec{c} - \vec{a}}{2}$

$\overrightarrow{SR}$ = P.V. of R – P.V. of S = $\left(\frac{\vec{c} + \vec{d}}{2}\right) - \left(\frac{\vec{d} + \vec{a}}{2}\right) = \frac{\vec{c} - \vec{a}}{2}$

$\therefore \overrightarrow{PQ} = \overrightarrow{SR}$

$\Rightarrow$PQ || SR and PQ = SR

Similarly we can prove that PS = QR and PS || QR

Hence PQRS is a parallelogram.

Example 2.10 :

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of three distinct points A, B, C. If there exists scalars l, m, n (not all zero) such that $l\vec{a} + m\vec{b} + n\vec{c} = 0$ and $l + m + n = 0$ then show that A, B and C lie on a line.

Solution:

It is given that l, m, n are not all zero. So, let n be a non-zero scalar.

$$l\vec{a} + m\vec{b} + n\vec{c} = 0 \Rightarrow n\vec{c} = -\left(l\vec{a} + m\vec{b}\right)$$

$$\vec{c} = -\frac{\left(l\vec{a} + m\vec{b}\right)}{n} \Rightarrow \vec{c} = -\frac{\left(l\vec{a} + m\vec{b}\right)}{-(l+m)} = \frac{l\vec{a} + m\vec{b}}{l+m}$$

$\Rightarrow$ The point C divides the line joining A and B in the ratio $m : l$
Hence A, B and C lies on the same line.

Note : $\vec{a}, \vec{b}$ are collinear vectors $\Rightarrow \vec{a} = \lambda\vec{b}$ or $\vec{b} = \lambda\vec{a}$ for some scalar λ

Collinear points: If A, B, C are three points in a plane such that $\overrightarrow{AB} = \lambda\overrightarrow{BC}$ or $\overrightarrow{AB} = \lambda\overrightarrow{AC}$ (or) $\overrightarrow{BC} = \lambda\overrightarrow{AC}$ for some scalar λ, then A, B, C are collinear.

Example 2.11: Show that the points with position vectors

$\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - \vec{c}$ and $4\vec{a} - 7\vec{b} + 7\vec{c}$ are collinear.

Solution:

Let A, B, C be the points with position vectors

$\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - \vec{c}$ and $4\vec{a} - 7\vec{b} + 7\vec{c}$ respectively.

$$\overrightarrow{OA} = \vec{a} - 2\vec{b} + 3\vec{c},\ \overrightarrow{OB} = -2\vec{a} + 3\vec{b} - \vec{c},\ \overrightarrow{OC} = 4\vec{a} - 7\vec{b} + 7\vec{c}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(-2\vec{a} + 3\vec{b} - \vec{c}\right) - \left(\vec{a} - 2\vec{b} + 3\vec{c}\right)$$

$$= -2\vec{a} + 3\vec{b} - \vec{c} - \vec{a} + 2\vec{b} - 3\vec{c} = -3\vec{a} + 5\vec{b} - 4\vec{c}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \left(4\vec{a} - 7\vec{b} + 7\vec{c}\right) - \left(-2\vec{a} + 3\vec{b} - \vec{c}\right)$$

$$= 4\vec{a} - 7\vec{b} + 7\vec{c} + 2\vec{a} - 3\vec{b} + \vec{c} = 6\vec{a} - 10\vec{b} + 8\vec{c}$$

Clearly $\overrightarrow{BC} = 6\vec{a} - 10\vec{b} + 8\vec{c} = -2\left(-3\vec{a} + 5\vec{b} - 4\vec{c}\right) = -2\left(\overrightarrow{AB}\right)$

$\Rightarrow$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors but B is a point common to them. So $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear vectors. Hence A, B, C are collinear points.

EXERCISE 2.1

(1) If $\vec{a}$ and $\vec{b}$ represent two adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively of a paralleogram ABCD. Find the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

(2) If $\overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{QO} + \overrightarrow{OR}$, show that the points P, Q, R are collinear.

(3) Show that the points with position vectors

$\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} + 2\vec{c}$ and $-8\vec{a} + 13\vec{b}$ are collinear.

(4) Show that the points A, B, C with position vectors $-2\vec{a} + 3\vec{b} + 5\vec{c}$, $\vec{a} + 2\vec{b} + 3\vec{c}$ and $7\vec{a} - \vec{c}$ respectively, are collinear.

(5) If D is the mid-point of the side BC of a triangle ABC, prove that

$\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$

(6) If G is the centroid of a triangle ABC, prove that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{O}$

(7) If ABC and A′B′C′ are two triangles and G, G′ be their corresponding centroids, prove that $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 3\overrightarrow{GG'}$

(8) Prove that the sum of the vectors directed from the vertices to the mid-points of opposite sides of a triangle is zero

(9) Prove by vector method that the line segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and equal to half of their difference.

(10) Prove by vector method that the internal bisectors of the angles of a triangle are concurrent.

(11) Prove using vectors the mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals are the vertices of a parallelogram.

(12) If ABCD is a quadrilateral and E and F are the mid-points of AC and BD respectively, prove that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{EF}$

2.5 Resolution of a Vector

Theorem 2.9 (Without Proof) :

Let $\vec{a}$ and $\vec{b}$ be two non-collinear vectors and $\vec{r}$ be a vector coplanar with them. Then $\vec{r}$ can be expressed uniquely as $\vec{r} = l\vec{a} + m\vec{b}$ where l, m are scalars.

Note : We call $l\vec{a} + m\vec{b}$ as a linear combination of vectors $\vec{a}$ and $\vec{b}$, where l, m are scalars.

Rectangular resolution of a vector in two dimension

Theorem 2.10 :

If P is a point in a two dimensional plane which has coordinates (x, y) then $\overrightarrow{OP} = x\vec{i} + y\vec{j}$, where $\vec{i}$ and $\vec{j}$ are unit vectors along OX and OY respectively.

Proof:

Let $P(x, y)$ be a point in a plane with reference to OX and OY as co-ordinate axes as shown in the figure.

Draw PL perpendicular to OX.

Then OL = x and LP = y

Let $\vec{i}$, $\vec{j}$ be the unit vectors along OX and OY respectively.

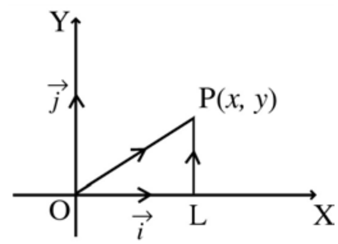


Fig. 2. 23

Then $\overrightarrow{OL} = x\vec{i}$ and $\overrightarrow{LP} = y\vec{j}$

Vectors $\overrightarrow{OL}$ and $\overrightarrow{LP}$ are known as the components of $\overrightarrow{OP}$ along x-axis and y-axis respectively.

Now by triangle law of addition

$$\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = x\vec{i} + y\vec{j} = \vec{r} \quad \text{(say)}$$

$$\therefore \quad \vec{r} = x\vec{i} + y\vec{j}$$

Now $\quad OP^2 = OL^2 + LP^2 = x^2 + y^2$

$$\Rightarrow \quad OP = \sqrt{x^2 + y^2} \Rightarrow \left|\vec{r}\right| = \sqrt{x^2 + y^2}$$

Thus, if a point P in a plane has coordinates (x, y) then

(i) $\vec{r} = \overrightarrow{OP} = x\vec{i} + y\vec{j}$

(ii) $\left|\vec{r}\right| = \left|\overrightarrow{OP}\right| = \left|x\vec{i} + y\vec{j}\right| = \sqrt{x^2 + y^2}$

(iii) The component of $\overrightarrow{OP}$ along x-axis is a vector $x\vec{i}$ and the component of $\overrightarrow{OP}$ along y-axis is a vector $y\vec{j}$

Components of a vector $\overrightarrow{AB}$ in terms of coordinates of A and B

Let A(x_1, y_1) and B(x_2, y_2) be any two points in XOY plane. Let $\vec{i}$ and $\vec{j}$ be unit vectors along OX and OY respectively.

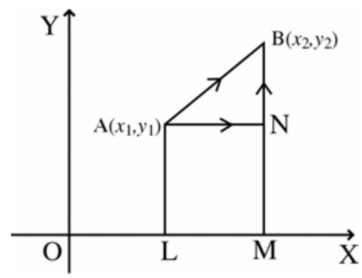


Fig. 2. 24

AN = $x_2 - x_1$,   BN = $y_2 - y_1$

$\therefore \quad \overrightarrow{AN} = (x_2 - x_1)\,\vec{i}$, $\overrightarrow{NB} = (y_2 - y_1)\,\vec{j}$

Now by triangle law of addition

$$\overrightarrow{AB} = \overrightarrow{AN} + \overrightarrow{NB} = (x_2 - x_1)\,\vec{i} + (y_2 - y_1)\,\vec{j}$$

Component of $\overrightarrow{AB}$ along x-axis $= (x_2 - x_1)\,\vec{i}$

Component of $\overrightarrow{AB}$ along y-axis $= (y_2 - y_1)\,\vec{j}$

$$AB^2 = AN^2 + NB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\Rightarrow \quad AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which gives the distance between A and B.

Addition, Subtraction, Multiplication of a vector by a scalar and equality of vectors in terms of components:

$$\text{Let} \quad \vec{a} = a_1\,\vec{i} + a_2\,\vec{j} \quad \text{and} \quad \vec{b} = b_1\,\vec{i} + b_2\,\vec{j}$$

We define

(i) $\vec{a} + \vec{b} = \left(a_1\,\vec{i} + a_2\,\vec{j}\right) + \left(b_1\,\vec{i} + b_2\,\vec{j}\right) = (a_1 + b_1)\,\vec{i} + (a_2 + b_2)\,\vec{j}$

(ii) $\vec{a} - \vec{b} = \left(a_1\,\vec{i} + a_2\,\vec{j}\right) - \left(b_1\,\vec{i} + b_2\,\vec{j}\right) = (a_1 - b_1)\,\vec{i} + (a_2 - b_2)\,\vec{j}$

(iii) $m\,\vec{a} = m\left(a_1\,\vec{i} + a_2\,\vec{j}\right) = ma_1\,\vec{i} + ma_2\,\vec{j}$ where m is a scalar

(iv) $\vec{a} = \vec{b} \Rightarrow a_1\,\vec{i} + a_2\,\vec{j} = b_1\,\vec{i} + b_2\,\vec{j} \Rightarrow a_1 = b_1$ and $a_2 = b_2$

Example 2.12: Let O be the origin and P(– 2, 4) be a point in the xy-plane. Express $\overrightarrow{OP}$ in terms of vectors $\vec{i}$ and $\vec{j}$. Also find $\left|\overrightarrow{OP}\right|$

Solution: The position vector of P, $\overrightarrow{OP} = -2\vec{i} + 4\vec{j}$

$$\left|\overrightarrow{OP}\right| = \left|-2\vec{i} + 4\vec{j}\right| = \sqrt{(-2)^2 + (4)^2} = \sqrt{4+16} = \sqrt{20}$$
$$= 2\sqrt{5}$$

Example 2.13: Find the components along the coordinates of the position vector of P(– 4, 3)

Solution:

The position vector of P = $\overrightarrow{OP} = -4\vec{i} + 3\vec{j}$

Component of $\overrightarrow{OP}$ along x-axis is $-4\vec{i}$

i.e. component of $\overrightarrow{OP}$ along x-axis is a vector of magnitude 4 and its direction is along the negative direction of x-axis.

Component of $\overrightarrow{OP}$ along y-axis is $3\vec{j}$

i.e. the component of $\overrightarrow{OP}$ along y –axis is a vector of magnitude 3, having its direction along the positive direction of y-axis.

Example 2.14: Express $\overrightarrow{AB}$ in terms of unit vectors $\vec{i}$ and $\vec{j}$, where the points are A(– 6, 3) and B(– 2, – 5). Find also $\left|\overrightarrow{AB}\right|$

Solution:

Given $\quad \overrightarrow{OA} = -6\vec{i} + 3\vec{j}$; $\overrightarrow{OB} = -2\vec{i} - 5\vec{j}$

$$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(-2\vec{i} - 5\vec{j}\right) - \left(-6\vec{i} + 3\vec{j}\right)$$
$$= 4\vec{i} - 8\vec{j}$$
$$\left|\overrightarrow{AB}\right| = \left|4\vec{i} - 8\vec{j}\right| = \sqrt{(4)^2 + (-8)^2} = \sqrt{16+64} = \sqrt{80}$$
$$= 4\sqrt{5}$$

Theorem 2.11 (Without Proof) :

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three given non-coplanar vectors then every vector $\vec{r}$ in space can be uniquely expressed as $\vec{r} = l\vec{a} + m\vec{b} + n\vec{c}$ for some scalars l, m and n

Rectangular Resolution of a vector in three dimension

Theorem 2.12:

If a point P in space has coordinate (x, y, z) then its position vector $\vec{r}$ is $x\vec{i} + y\vec{j} + z\vec{k}$ and $\left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$ where $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along OX, OY and OZ respectively.

Proof:

OX, OY, OZ are three mutually perpendicular axes. $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along OX, OY, OZ respectively. Let P be any point (x, y, z) in space and let $\overrightarrow{OP} = \vec{r}$

Draw PQ perpendicular to XOY plane and QR perpendicular to OX

Then OR $= x$; RQ $= y$; QP $= z$

$\therefore \overrightarrow{OR} = x\vec{i}$; $\overrightarrow{RQ} = y\vec{j}$; $\overrightarrow{QP} = z\vec{k}$

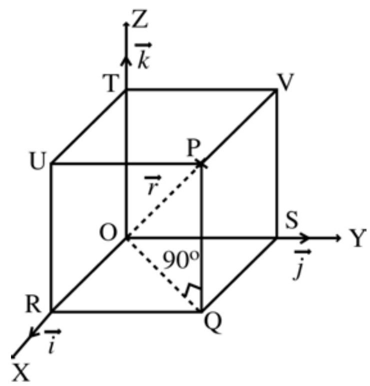


Fig. 2. 25

Now $\quad \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OR} + \overrightarrow{RQ} + \overrightarrow{QP}$

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Thus if P is a point (x, y, z) and $\vec{r}$ is the position vector of P, then $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

From the right angled triangle OQP, $\quad OP^2 = OQ^2 + QP^2$

From the right angled triangle ORQ, $\quad OQ^2 = OR^2 + RQ^2$

$$\therefore OP^2 = OR^2 + RQ^2 + QP^2 \Rightarrow OP^2 = x^2 + y^2 + z^2$$

$$\Rightarrow OP = \sqrt{x^2 + y^2 + z^2} \Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore r = \left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$$

2.6 Direction cosines and direction ratios

Let $P(x, y, z)$ be any point in space with reference to a rectangular coordinate system O (XYZ). Let α, β and γ be the angles made by OP with the positive direction of coordinate axes OX, OY, OZ respectively. Then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the **direction cosines** of $\overrightarrow{OP}$.

In the fig 2.25 $\lfloor\underline{OQP} = 90°$; $\lfloor\underline{POZ} = \gamma$ $\therefore \lfloor\underline{OPQ} = \gamma$ $(\because QP \parallel OZ)$

$$\therefore \cos\gamma = \frac{PQ}{OP} \Rightarrow \cos\gamma = \frac{z}{r} \text{ Similarly } \cos\alpha = \frac{x}{r} \text{ and } \cos\beta = \frac{y}{r}$$

$\therefore$ The direction cosines of $\overrightarrow{OP}$ are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$

Result 1: Sum of the squares of direction cosines is unity.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 = \frac{x^2 + y^2 + z^2}{r^2}$$

$$= \frac{r^2}{r^2} = 1 \qquad [\because r^2 = x^2 + y^2 + z^2]$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Result 2: Sum of the squares of direction sines is 2.

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = (1 - \cos^2\alpha) + (1 - \cos^2\beta) + (1 - \cos^2\gamma)$$

$$= 3 - [\cos^2\alpha + \cos^2\beta + \cos^2\gamma] = 3 - 1 = 2$$

$$\therefore \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

Direction ratios:

Any three numbers proportional to direction cosines of a vector are called its direction ratios. (d. r's).

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be any vector

$\Rightarrow$ Direction cosines of $\vec{r}$ are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow \cos\alpha = \frac{x}{r}$; $\cos\beta = \frac{y}{r}$; $\cos\gamma = \frac{z}{r}$ where α, β, γ be the angles made by $\vec{r}$ with the coordinate axes OX, OY, OZ respectively

$$\Rightarrow \frac{x}{\cos\alpha} = r, \quad \frac{y}{\cos\beta} = r, \quad \frac{z}{\cos\gamma} = r$$

$$\Rightarrow \frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma} = r$$

$$\Rightarrow x : y : z = \cos\alpha : \cos\beta : \cos\gamma$$

i.e. the coefficients of i, j, k in the rectangular resolution of a vector are proportional to the direction cosines of that vector.

$\therefore x, y, z$ are the direction ratios of the vector $\vec{r} = x\vec{i} + y\vec{l} + z\vec{k}$

Addition, Subtraction and Multiplication of a vector by a scalar and equality in terms of components:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be any two vectors.

Then

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\vec{i} + (a_2 - b_2)\vec{j} + (a_3 - b_3)\vec{k}$

(iii) $m\vec{a} = m\left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right)$

$= ma_1\vec{i} + ma_2\vec{j} + ma_3\vec{k}$ where m is a scalar

(iv) $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$

Distance between two points:

Let A (x_1, y_1, z_1) and B(x_2, y_2, z_2) be any two points

Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \left(x_2\vec{i} + y_2\vec{j} + z_2\vec{k}\right) - \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right)$$

$$= (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

$\therefore$ The distance between A and B is $AB = \left|\overrightarrow{AB}\right|$

$$\left|\overrightarrow{AB}\right| = \left|(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}\right|$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 2.15: Find the magnitude and direction cosines of $2\vec{i} - \vec{j} + 7\vec{k}$

Solution:

Magnitude of $2\vec{i} - \vec{j} + 7\vec{k} = \left|2\vec{i} - \vec{j} + 7\vec{k}\right| = \sqrt{(2)^2 + (-1)^2 + (7)^2}$

$= \sqrt{4 + 1 + 49} = \sqrt{54} = 3\sqrt{6}$

Direction cosines of $2\vec{i} - \vec{j} + 7\vec{k}$ are $\frac{2}{3\sqrt{6}}, -\frac{1}{3\sqrt{6}}, \frac{7}{3\sqrt{6}}$

Example 2.16: Find the unit vector in the direction of $3\vec{i} + 4\vec{j} - 12\vec{k}$

Solution: Let $\vec{a} = 3\vec{i} + 4\vec{j} - 12\vec{k}$

$$\left|\vec{a}\right| = \left|3\vec{i} + 4\vec{j} - 12\vec{k}\right| = \sqrt{(3)^2 + (4)^2 + (-12)^2}$$

$$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13$$

Unit vector in the direction of $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{\left|\vec{a}\right|} = \dfrac{3\vec{i} + 4\vec{j} - 12\vec{k}}{13}$

Example 2.17: Find the sum of the vectors $\vec{i} - \vec{j} + 2\vec{k}$ and $2\vec{i} + 3\vec{j} - 4\vec{k}$ and also find the modulus of the sum.

Solution:

Let $\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} - 4\vec{k}$

$$\vec{a} + \vec{b} = \left(\vec{i} - \vec{j} + 2\vec{k}\right) + \left(2\vec{i} + 3\vec{j} - 4\vec{k}\right) = 3\vec{i} + 2\vec{j} - 2\vec{k}$$

$$\left|\vec{a} + \vec{b}\right| = \sqrt{3^2 + 2^2 + (-2)^2} = \sqrt{9 + 4 + 4}$$

$$= \sqrt{17}$$

Example 2.18: If the position vectors of the two points A and B are $\vec{i} + 2\vec{j} - 3\vec{k}$ and $2\vec{i} - 4\vec{j} + \vec{k}$ respectively then find $\left|\overrightarrow{AB}\right|$

Solution:

If O be the origin, then $\overrightarrow{OA} = \vec{i} + 2\vec{j} - 3\vec{k}$, $\overrightarrow{OB} = 2\vec{i} - 4\vec{j} + \vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(2\vec{i} - 4\vec{j} + \vec{k}\right) - \left(\vec{i} + 2\vec{j} - 3\vec{k}\right)$$

$$= \vec{i} - 6\vec{j} + 4\vec{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(1)^2 + (-6)^2 + (4)^2} = \sqrt{53}$$

Example 2.19: Find the unit vectors parallel to the vector $-3\vec{i} + 4\vec{j}$

Solution: Let $\vec{a} = -3\vec{i} + 4\vec{j}$

$$\left|\vec{a}\right| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = 5$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{|\vec{a}|}\,\vec{a} = \frac{1}{5}\left(-3\vec{i} + 4\vec{j}\right)$$

Unit vectors parallel to $\vec{a}$ are $\pm\,\hat{a} = \pm\left(\frac{-3}{5}\,\vec{i} + \frac{4}{5}\vec{j}\right)$

Example 2.20: Find the vectors of magnitude 5 units, which are parallel to the vector $2\vec{i} - \vec{j}$

Solution: Let $\vec{a} = 2\vec{i} - \vec{j}$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{5}}\left(2\vec{i} - \vec{j}\right) = \frac{2}{\sqrt{5}}\,\vec{i} - \frac{1}{\sqrt{5}}\,\vec{j}$$

Vectors of magnitude 5 parallel to $2\vec{i} - \vec{j} = \pm 5\,\hat{a}$

$$= \pm 5\left(\frac{2}{\sqrt{5}}\,\vec{i} - \frac{1}{\sqrt{5}}\,\vec{j}\right) = \pm\left(2\sqrt{5}\,\vec{i} - \sqrt{5}\,\vec{j}\right)$$

Example 2.21: Show that the points whose position vectors $2\vec{i} + 3\vec{j} - 5\vec{k}$, $3\vec{i} + \vec{j} - 2\vec{k}$ and $6\vec{i} - 5\vec{j} + 7\vec{k}$ are collinear.

Solution: Let the points be A, B and C and O be the origin. Then

$$\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 5\vec{k}\ ;\ \overrightarrow{OB} = 3\vec{i} + \vec{j} - 2\vec{k}\ ;\ \overrightarrow{OC} = 6\vec{i} - 5\vec{j} + 7\vec{k}$$

$$\therefore\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(3\vec{i} + \vec{j} - 2\vec{k}\right) - \left(2\vec{i} + 3\vec{j} - 5\vec{k}\right)$$

$$= \vec{i} - 2\vec{j} + 3\vec{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(6\vec{i} - 5\vec{j} + 7\vec{k}\right) - \left(2\vec{i} + 3\vec{j} - 5\vec{k}\right)$$

$$\overrightarrow{AC} = 4\vec{i} - 8\vec{j} + 12\vec{k} = 4\left(\vec{i} - 2\vec{j} + 3\vec{k}\right)$$

$$= 4\,\overrightarrow{AB}$$

Hence the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. Further they have the common point A.

$\therefore$ The points A, B, C are collinear.

Example 2.22: If the position vectors of A and B are $3\vec{i} - 7\vec{j} - 7\vec{k}$ and $5\vec{i} + 4\vec{j} + 3\vec{k}$, find $\overrightarrow{AB}$ and determine its magnitude and direction cosines.

Solution:

Let O be the origin. Then

$$\overrightarrow{OA} = 3\vec{i} - 7\vec{j} - 7\vec{k},\quad \overrightarrow{OB} = 5\vec{i} + 4\vec{j} + 3\vec{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(5\vec{i} + 4\vec{j} + 3\vec{k}\right) - \left(3\vec{i} - 7\vec{j} - 7\vec{k}\right)$$

$$\overrightarrow{AB} = 2\vec{i} + 11\vec{j} + 10\vec{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(2)^2 + (11)^2 + (10)^2} = 15$$

The direction cosines are $\frac{2}{15}, \frac{11}{15}, \frac{10}{15}$

EXERCISE 2.2

(1) Find the sum of the vectors $4\vec{i} + 5\vec{j} + \vec{k}$, $-2\vec{i} + 4\vec{j} - \vec{k}$ and $3\vec{i} - 4\vec{j} + 5\vec{k}$. Find also the magnitude of the sum.

(2) If $\vec{a} = 3\vec{i} - \vec{j} - 4\vec{k}$, $\vec{b} = -2\vec{i} + 4\vec{j} - 3\vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$ find $\left|2\vec{a} - \vec{b} + 3\vec{c}\right|$

(3) The position vectors of the vertices A, B, C of a triangle ABC are respectively

$2\vec{i} + 3\vec{j} + 4\vec{k}$, $-\vec{i} + 2\vec{j} - \vec{k}$ and $3\vec{i} - 5\vec{j} + 6\vec{k}$

Find the vectors determined by the sides and calculate the length of the sides.

(4) Show that the points whose position vectors given by

(i) $-2\vec{i} + 3\vec{j} + 5\vec{k}$, $\vec{i} + 2\vec{j} + 3\vec{k}$, $7\vec{i} - \vec{k}$

(ii) $\vec{i} - 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $-7\vec{j} + 10\vec{k}$ are collinear.

(5) If the vectors $\vec{a} = 2\vec{i} - 3\vec{j}$ and $\vec{b} = -6\vec{i} + m\vec{j}$ are collinear, find the value of *m*.

(6) Find a unit vector in the direction of $\vec{i} + \sqrt{3}\,\vec{j}$

(7) Find the unit vectors parallel to the sum of $3\vec{i} - 5\vec{j} + 8\vec{k}$ and $-2\vec{j} - 2\vec{k}$

(8) Find the unit vectors parallel to $3\vec{a} - 2\vec{b} + 4\vec{c}$ where $\vec{a} = 3\vec{i} - \vec{j} - 4\vec{k}$, $\vec{b} = -2\vec{i} + 4\vec{j} - 3\vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$

(9) The vertices of a triangle have position vectors $4\vec{i} + 5\vec{j} + 6\vec{k}$, $5\vec{i} + 6\vec{j} + 4\vec{k}$, $6\vec{i} + 4\vec{j} + 5\vec{k}$. Prove that the triangle is equilateral.

(10) Show that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $3\vec{i} - 4\vec{j} - 4\vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ form a right angled triangle.

(11) Prove that the points $2\vec{i} + 3\vec{j} + 4\vec{k}$, $3\vec{i} + 4\vec{j} + 2\vec{k}$, $4\vec{i} + 2\vec{j} + 3\vec{k}$ form an equilateral triangle.

(12) If the vertices of a triangle have position vectors $\vec{i} + 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} + \vec{k}$ and $3\vec{i} + \vec{j} + 2\vec{k}$, find the position vector of its centroid.

(13) If the position vectors of P and Q are $\vec{i} + 3\vec{j} - 7\vec{k}$ and $5\vec{i} - 2\vec{j} + 4\vec{k}$, find $\overrightarrow{PQ}$ and determine its direction cosines.

(14) Show that the following vectors are coplanar

(i) $\vec{i} - 2\vec{j} + 3\vec{k}$, $-2\vec{i} + 3\vec{j} - 4\vec{k}$, $-\vec{j} + 2\vec{k}$

(ii) $5\vec{i} + 6\vec{j} + 7\vec{k}$, $7\vec{i} - 8\vec{j} + 9\vec{k}$, $3\vec{i} + 20\vec{j} + 5\vec{k}$

(15) Show that the points given by the vectors $4\vec{i} + 5\vec{j} + \vec{k}$, $-\vec{j} - \vec{k}$, $3\vec{i} + 9\vec{j} + 4\vec{k}$ and $-4\vec{i} + 4\vec{j} + 4\vec{k}$ are coplanar.

(16) Examine whether the vectors $\vec{i} + 3\vec{j} + \vec{k}$, $2\vec{i} - \vec{j} - \vec{k}$ and $7\vec{j} + 5\vec{k}$ are coplanar.

3. ALGEBRA

3.1 Partial Fractions:

Definitions:

Rational Expression: An expression of the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x) \neq 0$ are polynomials in x is called a rational expression.

The expressions $\frac{5x - 2}{x^2 + 3x + 2}$, $\frac{3x^2 + 2x - 1}{x^2 + x - 22}$ are examples for rational expressions.

Proper Fraction: A proper fraction is one in which the degree of the numerator is less than the degree of the denominator.

The expressions $\frac{3x + 1}{x^2 + 4x + 3}$, $\frac{7x^2 + 9}{x^3 + x^2 - 5}$ are examples for proper fractions.

Improper Fraction: An improper fraction is a fraction in which the degree of the numerator is greater than or equal to the degree of the denominator.

The expressions $\frac{x^3 + 5x^2 + 4}{x^2 + 2x + 3}$, $\frac{x^2 - x + 1}{x^2 + x + 3}$ are examples for improper fractions.

Partial Fraction:

Consider the sum of $\frac{7}{x - 2}$ and $\frac{5}{x - 1}$

We simplify it as follows:

$$\frac{7}{x-2} + \frac{5}{x-1} = \frac{7(x-1) + 5(x-2)}{(x-2)(x-1)} = \frac{7x - 7 + 5x - 10}{(x-2)(x-1)} = \frac{12x - 17}{(x-2)(x-1)}$$

Conversely the process of writing the given fraction $\frac{12x - 17}{(x-2)(x-1)}$ as $\frac{7}{x-2} + \frac{5}{x-1}$ is known as splitting into partial fractions (or) expressing as partial fractions.

A given proper fraction can be expressed as the sum of other simple fractions corresponding to the factors of the denominator of the given proper fraction. This process is called splitting into Partial Fractions. If the given fraction $\frac{p(x)}{q(x)}$ is improper then convert into sum of a polynomial expression and a proper rational fraction by dividing $p(x)$ by $q(x)$.

Working Rule :

Given the proper fraction $\frac{p(x)}{q(x)}$. Factorise $q(x)$ into prime factors.

Type 1: Linear factors, none of which is repeated.

If a linear factor $ax + b$ is a factor of the denominator $q(x)$ then corresponding to this factor associate a simple fraction $\frac{A}{ax + b}$, where A is a constant (A $\neq$ 0).

i.e., When the factors of the denominator of the given fraction are all linear factors none of which is repeated, we write the partial fraction as follows :

$$\frac{x + 3}{(x + 5)(2x + 1)} = \frac{A}{x + 5} + \frac{B}{2x + 1}$$ where A and B are constants to be determined.

Example 3.1: Resolve into partial fractions $\frac{3x + 7}{x^2 - 3x + 2}$

The denominator $x^2 - 3x + 2$ can be factorised into linear factors.

$$x^2 - 3x + 2 = x^2 - x - 2x + 2 = x(x - 1) - 2(x - 1) = (x - 1)(x - 2)$$

We assume $\frac{3x + 7}{x^2 - 3x + 2} = \frac{A}{x - 1} + \frac{B}{x - 2}$ where A and B are constants.

$$\Rightarrow \quad \frac{3x + 7}{x^2 - 3x + 2} = \frac{A(x - 2) + B(x - 1)}{(x - 1)(x - 2)}$$

$$\Rightarrow \quad 3x + 7 = A(x - 2) + B(x - 1) \qquad \ldots(1)$$

Equating the coefficients of like powers of x, we get

Coefficient of x : $A + B = 3$ … (2)

Constant term : $-2A - B = 7$ … (3)

Solving (2) and (3) we get

$$A = -10$$

$$B = 13$$

$$\therefore \frac{3x + 7}{x^2 - 3x + 2} = \frac{-10}{x - 1} + \frac{13}{x - 2} = \frac{13}{x - 2} - \frac{10}{x - 1}$$

Note: The constants A and B can also be found by successively giving suitable values for x.

To find A, put $x = 1$ in (1)

$$3(1) + 7 = A(1 - 2) + B(0)$$

$$10 = A(-1)$$

$$A = -10$$

To find B, put $x = 2$ in (1)

$$3(2) + 7 = A(0) + B(2 - 1)$$

$$B = 13$$

$$\therefore \frac{3x + 7}{x^2 - 3x + 2} = \frac{-10}{x - 1} + \frac{13}{x - 2}$$

$$\frac{3x + 7}{x^2 - 3x + 2} = \frac{13}{x - 2} - \frac{10}{x - 1}$$

Example: 3.2: Resolve into partial fractions $\dfrac{x + 4}{(x^2 - 4)(x + 1)}$

The denominator $(x^2 - 4)(x + 1)$ can be further factored into linear factors

i.e. $(x^2 - 4)(x + 1) = (x + 2)(x - 2)(x + 1)$

Let $\dfrac{x + 4}{(x^2 - 4)(x + 1)} = \dfrac{A}{x + 2} + \dfrac{B}{x - 2} + \dfrac{C}{x + 1}$, where A, B and C are constants to be determined.

$$\frac{x + 4}{(x^2 - 4)(x + 1)} = \frac{A(x - 2)(x + 1) + B(x + 2)(x + 1) + C(x + 2)(x - 2)}{(x + 2)(x - 2)(x + 1)}$$

$$\Rightarrow \quad x + 4 = A(x - 2)(x + 1) + B(x + 2)(x + 1) + C(x + 2)(x - 2) \quad \ldots (1)$$

To find A, put $x = -2$ in (1)

$$-2 + 4 = A(-2 - 2)(-2 + 1) + B(0) + C(0)$$

$$2 = 4A \Rightarrow \quad A = 1/2$$

To find B, put $x = 2$ in (1), we get $B = 1/2$

To find C, put $x = -1$ in (1), we get $C = -1$

$$\therefore \frac{x + 4}{(x^2 - 4)(x + 1)} = \frac{1/2}{(x + 2)} + \frac{1/2}{(x - 2)} + \frac{(-1)}{x + 1}$$

$$\Rightarrow \quad \frac{x + 4}{(x^2 - 4)(x + 1)} = \frac{1}{2(x + 2)} + \frac{1}{2(x - 2)} - \frac{1}{x + 1}$$

Type 2: Linear factors, some of which are repeated

If a linear factor $ax + b$ occurs n times as a factor of the denominator of the given fraction, then corresponding to these factors associate the sum of n simple fractions,

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \ldots + \frac{A_n}{(ax + b)^n}$$

Where $A_1, A_2, A_3, \ldots A_n$ are constants.

Example 3.3: Resolve into partial fractions $\dfrac{9}{(x-1)(x+2)^2}$

Let $\dfrac{9}{(x-1)(x+2)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$

$$\Rightarrow \quad \frac{9}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$$

$$\Rightarrow \quad 9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad \ldots (1)$$

To find A, put $x = 1$ in (1)

We get $9 = A(1+2)^2 \Rightarrow A = 1$

To find C, put $x = -2$ in (1)

We get $9 = C(-2-1) \Rightarrow C = -3$

In (1), equating the coefficient of x^2 on both sides we get

$$A + B = 0$$

$$\Rightarrow \quad 1 + B = 0 \quad \Rightarrow B = -1$$

$$\therefore \frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

Type 3: Quadratic factors, none of which is repeated

If a quadratic factor $ax^2 + bx + c$ which is not factorable into linear factors occurs only once as a factor of the denominator of the given fraction, then corresponding to this factor associate a partial fraction $\dfrac{Ax + B}{ax^2 + bx + c}$ where A and B are constants which are not both zeros.

Consider $\dfrac{2x}{(x+1)(x^2+1)}$

We can write this proper fraction in the form $\dfrac{2x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2+1}$

The first factor of the denominator $x + 1$ is of first degree, so we assume its numerator as a constant A. The second factor of the denominator $x^2 + 1$ is of 2^{nd} degree and which is not factorable into linear factors. We assume its numerator as a general first-degree expression $Bx + C$.

Example 3.4: Resolve into partial fractions $\dfrac{x^2 - 2x - 9}{(x^2 + x + 6)(x+1)}$

$$\text{Let}\quad \frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} = \frac{Ax + B}{x^2 + x + 6} + \frac{C}{x + 1}$$

$$\Rightarrow\quad \frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} = \frac{(Ax + B)(x + 1) + C(x^2 + x + 6)}{(x^2 + x + 6)(x + 1)}$$

$$\Rightarrow\quad x^2 - 2x - 9 = (Ax + B)(x + 1) + C(x^2 + x + 6) \qquad \ldots (1)$$

To find C put $x = -1$ in (1)

We get $1 + 2 - 9 = C(1 - 1 + 6) \Rightarrow C = -1$

To find B, put $x = 0$ in (1)

We get $-9 = B + 6C$

$-9 = B - 6 \quad \Rightarrow \quad B = -3$

To find A, Put $x = 1$ in (1)

$1 - 2 - 9 = (A - 3)(2) + (-1)(8) \Rightarrow -10 = 2A - 14$

$A = 2$

$$\therefore\quad \frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} = \frac{2x - 3}{x^2 + x + 6} - \frac{1}{x + 1}$$

Example 3.5: Resolve into partial fractions $\dfrac{x^2 + x + 1}{x^2 - 5x + 6}$

Solution:

Here the degree of the numerator is same as the degree of the denominator, i.e. an improper fraction.

$$\text{On division}\quad \frac{x^2 + x + 1}{x^2 - 5x + 6} = 1 + \frac{6x - 5}{x^2 - 5x + 6} \qquad \ldots (1)$$

$$\text{Let}\quad \frac{6x - 5}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3}$$

$$6x - 5 = A(x - 3) + B(x - 2)$$

By putting $x = 2$, $-A = 12 - 5 \Rightarrow A = -7$

By putting $x = 3$, $B = 18 - 5 \Rightarrow B = 13$

$$\therefore\quad \frac{x^2 + x + 1}{x^2 - 5x + 6} = -\frac{7}{x - 2} + \frac{13}{x - 3}$$

$$\therefore\ (1) \Rightarrow\quad \frac{x^2 + x + 1}{x^2 - 5x + 6} = 1 - \frac{7}{x - 2} + \frac{13}{x - 3}$$

EXERCISE 3.1

Resolve into partial fractions

(1) $\dfrac{1}{(x-1)(x+1)}$ (2) $\dfrac{7x-1}{6-5x+x^2}$ (3) $\dfrac{x^2+x+1}{(x-1)(x-2)(x-3)}$

(4) $\dfrac{1}{(x-1)(x+2)^2}$ (5) $\dfrac{x-2}{(x+2)(x-1)^2}$ (6) $\dfrac{x+1}{(x-2)^2(x+3)}$

(7) $\dfrac{x^2-6x+2}{x^2(x+2)}$ (8) $\dfrac{2x^2-5x-7}{(x-2)^3}$ (9) $\dfrac{x^2-3}{(x+2)(x^2+1)}$

(10) $\dfrac{x+2}{(x+1)(x^2+1)}$ (11) $\dfrac{7x^2-25x+6}{(x^2-2x-1)(3x-2)}$ (12) $\dfrac{x^2+x+1}{x^2+2x+1}$

3.2 Permutations:

Factorial:

The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\lfloor\underline{n}$

$$\text{i.e.} \quad n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$$
$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

Zero Factorial:

We will require zero factorial in the latter sections of this chapter. It does not make any sense to define it as the product of the integers from 1 to zero. So, we define $0! = 1$.

Deduction:

$$\begin{aligned} n! &= 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n \\ &= [1 \times 2 \times 3 \times 4 \times \ldots \times (n-1)]n \\ &= \left[(n-1)!\right] n \end{aligned}$$

Thus, $\quad n! = n\left[(n-1)!\right]$

For example,

$$8! = 8(7!)$$

3.2.1 Fundamental Principles of Counting:

In this section we shall discuss two fundamental principles viz., principle of addition and principle of multiplication. These two principles will enable us to understand permutations and combinations and form the base for permutations and combinations.

Fundamental Principle of Multiplication:

If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.

Explanation:

If the first job is performed in any one of the m ways, we can associate with this any one of the n ways of performing the second job: and thus there are n ways of performing the two jobs without considering more than one way of performing the first; and so corresponding to each of the m ways of performing the first job, we have n ways of performing the second job. Hence, the number of ways in which the two jobs can be performed is $m \times n$.

Example 3.6: In a class there are 15 boys and 20 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution :

Here the teacher is to perform two jobs :

(i) Selecting a boy among 15 boys, and

(ii) Selecting a girl among 20 girls

The first of these can be performed in 15 ways and the second in 20 ways.

Therefore by the fundamental principle of multiplication, the required number of ways is $15 \times 20 = 300$.

Fundamental Principle of Addition:

If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.

Example 3.7: In a class there are 20 boys and 10 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution:

Here the teacher is to perform either of the following two jobs :

(i) selecting a boy among 20 boys. (or)

(ii) Selecting a girl among 10 girls

The first of these can be performed in 20 ways and the second in 10 ways.

Therefore, by fundamental principle of addition either of the two jobs can be performed in $(20 + 10) = 30$ ways.

Hence, the teacher can make the selection of either a boy or a girl in 30 ways.

Example 3.8: A room has 10 doors. In how many ways can a man enter the room through one door and come out through a different door?

Solution:

Clearly, a person can enter the room through any one of the ten doors. So, there are ten ways of entering into the room.

After entering into the room, the man can come out through any one of the remaining 9 doors. So, he can come out through a different door in 9 ways.

Hence, the number of ways in which a man can enter a room through one door and come out through a different door = $10 \times 9 = 90$.

Example 3.9: How many words (with or without meaning) of three distinct letters of the English alphabets are there?

Solution:

Here we have to fill up three places by distinct letters of the English alphabets. Since there are 26 letters of the English alphabet, the first place can be filled by any of these letters. So, there are 26 ways of filling up the first place.

Now, the second place can be filled up by any of the remaining 25 letters. So, there are 25 ways of filling up the second place.

After filling up the first two places only 24 letters are left to fill up the third place. So, the third place can be filled in 24 ways.

Hence, the required number of words

$$= 26 \times 25 \times 24 = 15600$$

Example 3.10:

How many three-digit numbers can be formed by using the digits 1, 2, 3, 4, 5.

Solution :

We have to determine the total number of three digit numbers formed by using the digits 1, 2, 3, 4, 5.

Clearly, the repetition of digits is allowed.

A three digit number has three places viz. unit's, ten's and hundred's. Unit's place can be filled by any of the digits 1, 2, 3, 4, 5. So unit's place can be filled in 5 ways.

Similarly, each one of the ten's and hundred's place can be filled in 5 ways.

$\therefore$ Total number of required numbers

$$= \quad 5 \times 5 \times 5 = 125$$

Example 3.11: There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 5 each?

Solution:

Here we have to perform 6 jobs of answering 6 multiple choice questions.

Each of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences

$$= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$$

Example 3.12: How many three-digit numbers greater than 600 can be formed by using the digits 4, 5, 6, 7, 8?

Solution:

Clearly, repetition of digits is allowed. Since a three-digit number greater than 600 will have 6, 7 or 8 at hundred's place. So, hundred's place can be filled in 3 ways.

Each of the ten's and one's place can be filled in 5 ways.

Hence, total number of required numbers

$$= 3 \times 5 \times 5 = 75$$

Example 3.13: How many numbers divisible by 5 and lying between 5000 and 6000 can be formed from the digits 5, 6, 7, 8 and 9?

Solution:

Clearly, a number between 5000 and 6000 must have 5 at thousand's place. Since the number is divisible by 5 it must have 5 at unit's place.

Now, each of the remaining places (viz. Hundred's and ten's) can be filled in 5 ways.

Hence the total number of required numbers

$$= 1 \times 5 \times 5 \times 1 = 25$$

Example 3.14: How many three digit odd numbers can be formed by using the digits 4, 5, 6, 7, 8, 9 if :

(i) the repetition of digits is not allowed?

(ii) the repetition of digits is allowed?

Solution:

For a number to be odd, we must have 5, 7 or 9 at the unit's place.

So, there are 3 ways of filling the unit's place.

(i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways.

Now, four digits are left. So, hundred's place can be filled in 4 ways.

So, required number of numbers

$$= 3 \times 5 \times 4 = 60$$

(ii) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.

Hence required number of numbers = $3 \times 6 \times 6 = 108$

EXERCISE 3.2

1. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent a competition. In how many ways can the teacher make this selection?
2. Given 7 flags of different colours, how many different signals can be generated if a signal requires the use of two flags, one below the other?
3. A person wants to buy one fountain pen, one ball pen and one pencil from a stationery shop. If there are 10 fountain pen varieties, 12 ball pen varieties and 5 pencil varieties, in how many ways can he select these articles?
4. Twelve students compete in a race. In how many ways first three prizes be given?
5. From among the 36 teachers in a college, one principal, one vice-principal and the teacher-in charge are to be appointed. In how many ways this can be done?
6. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 2 each?
7. How many numbers are there between 500 and 1000 which have exactly one of their digits as 8?
8. How many five-digit number license plates can be made if

   (i) first digit cannot be zero and the repetition of digits is not allowed.

   (ii) the first digit cannot be zero, but the repetition of digits is allowed?
9. How many different numbers of six digits can be formed from the digits 2, 3, 0, 7, 9, 5 when repetition of digits is not allowed?
10. How many odd numbers less than 1000 can be formed by using the digits 0, 3, 5, 7 when repetition of digits is not allowed?
11. In how many ways can an examinee answer a set of 5 true / false type questions?

12. How many 4-digit numbers are there?
13. How many three – letter words can be formed using *a, b, c, d, e* if :
    (i) repetition is allowed (ii) repetition is not allowed?
14. A coin is tossed five times and outcomes are recorded. How many possible outcomes are there?

3.2.2. Concept of Permutations:

The word permutation means arrangement.

For example, given 3 letters *a, b, c* suppose we arrange them taking 2 at a time.

The various arrangements are *ab, ba, bc, cb, ac, ca*.

Hence the number of arrangements of 3 things taken 2 at a time is 6 and this can be written as $3P2 = 6$.

Definition:

The number of arrangements that can be made out of *n* things taking *r* at a time is called the number of permutations of *n* things taken *r* at a time.

Notation:

If *n* and *r* are positive integers such that $1 \leq r \leq n$, then the number of all permutations of *n* distinct things, taken *r* at a time is denoted by the symbol $P(n, r)$ or nPr.

We use the symbol nPr throughout our discussion. Thus nPr = Total number of permutations of *n* distinct things taken *r* at a time.

Note: In permutations the order of arrangement is taken into account; when the order is changed, a different permutation is obtained.

Example 3.15: Write down all the permutations of the vowels A, E, I, O, U in English alphabets taking 3 at a time and starting with E.

Solution: The permutations of vowels A, E, I, O, U taking three at a time and starting with E are

EAI, EIA, EIO, EOI, EOU, EUO, EAO, EOA, EIU, EUI, EAU, EUA.

Clearly there are 12 permutations.

Theorem 3.1:

Let *r* and *n* be positive integers such that $1 \leq r \leq n$.

Then the number of all permutations of *n* distinct things taken *r* at a time is given by $n(n-1)(n-2)\ldots\left(n - \overline{r-1}\right)$

i.e. $nPr = n(n-1)(n-2)\ldots\left(n - \overline{r-1}\right)$

Proof:

Let nPr denote the number of permutations of n things taken r at a time.

Clearly the total number of permutations required is same as the number of possible ways of filling up r blank spaces by n things.

$$\begin{array}{ccccc} \square & \square & \square & \cdots & \square \\ 1 & 2 & 3 & & r \end{array}$$

Let there be r blank spaces arranged in a row

The first place can be filled by any one of the n things in n ways.

If the first place is filled up by any one of the n things, there will be $(n - 1)$ things remaining. Now the second place can be filled up by any one of the $(n - 1)$ remaining things.

Here it can be filled up in $(n - 1)$ ways.

Hence the first two places can be together filled in $n(n - 1)$ ways.

Having filled up these two places, we have $(n - 2)$ things remaining with which we can fill up the third place. So the third place can be filled up by any one of these things in $(n - 2)$ ways.

Hence the first three places can be together filled in $n(n - 1)(n - 2)$ ways.

Proceeding in this way, we find that the total number of ways of filling up the r spaces is

$n(n - 1)(n - 2) \ldots$ upto r factors

i.e. $n(n - 1)(n - 2) \ldots \left(n - \overline{r-1}\right)$

$\therefore\ nPr = n(n - 1)(n - 2) \ldots \left(n - \overline{r-1}\right) = n(n - 1)(n - 2) \ldots (n - r + 1)$

Theorem 3.2:

Let r and n be positive integers such that $1 \le r \le n$. Then $nPr = \dfrac{n!}{(n-r)!}$

Proof:

$$\begin{aligned} nPr &= n(n - 1)(n - 2) \ldots \left(n - \overline{r-1}\right) \\ &= \frac{n(n - 1)(n - 2) \ldots \left(n - \overline{r-1}\right)(n - r)\left(n - \overline{r+1}\right) \ldots 2.1}{(n - r)\left(n - \overline{r+1}\right) \ldots 2.1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Theorem 3.3:

The number of all permutations of n distinct things, taken all at a time is $n!$

Proof: $\quad nPr = n(n-1)(n-2) \ldots \left(n - \overline{r-1}\right)$

By putting $r = n$, $\quad nPn = n(n-1)(n-2) \ldots \left(n - \overline{n-1}\right)$

$$= n(n-1)(n-2) \ldots \left(n - \overline{n-1}\right)$$

$$= n(n-1)(n-2) \ldots 1$$

$$= n!$$

$$\therefore\ nPn = n!$$

Remark: We have already defined $0! = 1$. This can also be derived as follows.

We know that $\quad nPr = \dfrac{n!}{(n-r)!}$

By putting $r = n$, $\quad nPn = \dfrac{n!}{(n-n)!}$

$$\Rightarrow \quad n! = \frac{n!}{0!} \quad (\because\ nPn = n!)$$

$$\Rightarrow \quad 0! = \frac{n!}{n!} = 1$$

$$0! = 1$$

Example 3.16: Evaluate $8P_3$

Solution: $\quad 8P_3 = \dfrac{8!}{(8-3)!} = \dfrac{8!}{5!} = \dfrac{(8 \times 7 \times 6) \times 5!}{5!}$

$$= 8 \times 7 \times 6$$

$$= 336$$

Example 3.17 : If $5Pr = 6P_{r-1}$, find r

Solution: $\quad 5Pr = 6P_{r-1}$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{6!}{\left(6 - \overline{r-1}\right)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{6 \times 5!}{(7-r)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{6 \times 5!}{\{(7-r)(6-r)\}(5-r)!}$$

$$\Rightarrow \quad 1 = \frac{6}{(7-r)(6-r)}$$

$\Rightarrow \quad (7 - r)(6 - r) = 6 \quad \Rightarrow \quad 42 - 7r - 6r + r^2 - 6 = 0$

$\Rightarrow \quad r^2 - 13r + 36 = 0 \quad \Rightarrow \quad (r - 9)(r - 4) = 0$

$\Rightarrow \quad r = 9 \text{ or } r = 4$

$\Rightarrow \quad r = 4 \qquad (\because\ 5Pr \text{ is meaningful for } r \leq 5)$

Example 3.18:

If $nP_4 = 360$, find the value of n.

Solution: $nP_4 = 360 \quad \Rightarrow \quad \dfrac{n\,!}{(n-4)\,!} = 6 \times 5 \times 4 \times 3$

$$\Rightarrow \quad \frac{n\,!}{(n-4)\,!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = \frac{6!}{2!}$$

$\Rightarrow \quad n! = 6!$

$\Rightarrow \quad n = 6$

Example 3.19:

If $9Pr = 3024$, find r.

Solution: $9Pr = 3024$

$\Rightarrow \quad = 9 \times 8 \times 7 \times 6 = 9P_4$

$\Rightarrow \quad r = 4$

Example 3.20:

If $(n-1)P_3 : nP_4 = 1 : 9$, find n.

Solution:

$(n-1)P_3 : nP_4 = 1 : 9$

$\Rightarrow (n-1)(n-2)(n-3) : n(n-1)(n-2)(n-3) = 1 : 9$

$\Rightarrow$ i.e. $9(n-1)(n-2)(n-3) = n(n-1)(n-2)(n-3)$

$\Rightarrow n = 9$

Example 3.21: In how many ways can five children stand in a queue?

Solution:

The number of ways in which 5 persons can stand in a queue is same as the number of arrangements of 5 different things taken all at a time.

Hence the required number of ways

$$= \ {}_5P_5 = 5! = 120$$

Example 3.22: How many different signals can be made by hoisting 6 differently coloured flags one above the other, when any number of them may be hoisted at one time?

Solution:

The signals can be made by using at a time one or two or three or four or five or six flags.

The total number of signals when r-flags are used at a time from 6 flags is equal to the number of arrangements of 6, taking r at a time i.e. $6P_r$

Hence, by the fundamental principle of addition, the total number of different signals

$$= {}_6P_1 + {}_6P_2 + {}_6P_3 + {}_6P_4 + {}_6P_5 + {}_6P_6$$

$$= 6 + (6 \times 5) + (6 \times 5 \times 4) + (6 \times 5 \times 4 \times 3) + (6 \times 5 \times 4 \times 3 \times 2) + (6 \times 5 \times 4 \times 3 \times 2 \times 1)$$

$$= 6 + 30 + 120 + 360 + 720 + 720 = 1956$$

Example 3.23: Find the number of different 4-letter words with or without meanings, that can be formed from the letters of the word 'NUMBER'

Solution:

There are 6 letters in the word 'NUMBER'.

So, the number of 4-letter words

= the number of arrangements of 6 letters taken 4 at a time

$= {}_6P_4$

$= 360$

Example 3.24: A family of 4 brothers and 3 sisters is to be arranged in a row, for a photograph. In how many ways can they be seated, if

(i) all the sisters sit together.

(ii) all the sisters are not together.

Solution :

(i) Since the 3 sisters are inseparable, consider them as one single unit.

This together with the 4 brothers make 5 persons who can be arranged among themselves in 5! ways.

In everyone of these permutations, the 3 sisters can be rearranged among themselves in 3! ways.

Hence the total number of arrangements required $= 5! \times 3! = 120 \times 6 = 720$

(ii) The number of arrangements of all the 7 persons without any restriction $=7! = 5040$

Number of arrangements in which all the sisters sit together = 720

$\therefore$ Number of arrangements required $= 5040 - 720 = 4320$

3.2.3 Permutations of objects not all distinct:

The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is $\dfrac{n!}{p!\, q!}$

Example 3.25: How many permutations of the letters of the word 'APPLE' are there?

Solution:

Here there are 5 letters, two of which are of the same kind.

The others are each of its own kind.

$\therefore$ Required number of permutations is $= \dfrac{5!}{2!\ 1!\ 1!\ 1!} = \dfrac{5!}{2!} = \dfrac{120}{2} = 60$

Example 3.26: How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

Solution:

There are 4 odd digits 1, 1, 3, 3 and 4 odd places.

So odd digits can be arranged in odd places in $\dfrac{4!}{2!\ 2!}$ ways.

The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places in $\dfrac{3!}{2!}$ ways.

Hence, the required number of numbers $= \dfrac{4!}{2!\ 2!} \times \dfrac{3!}{2!} = 6 \times 3 = 18$

Example 3.27: How many arrangements can be made with the letters of the word "MATHEMATICS"?

Solution:

There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and all other are distinct.

$\therefore$ required number of arrangements $= \dfrac{11!}{2! \times 2! \times 2!} = 4989600$

3.2.4 Permutations when objects can repeat:

The number of permutations of n different things, taken r at a time, when each may be repeated any number of times in each arrangement, is n^r

Consider the following example:

In how many ways can 2 different balls be distributed among 3 boxes?

Let A and B be the 2 balls. The different ways are

Box 1	Box 2	Box 3
A	B	
B	A	
	A	B
	B	A
A		B
B		A
		AB
AB		
	AB	

i.e. 9 ways. By formula $n^r = 3^2 = 9$ ways

Example 3.28: In how many ways can 5 different balls be distributed among 3 boxes?

Solution:

There are 5 balls and each ball can be placed in 3 ways.

So the total number of ways = $3^5 = 243$

Example: 3.29: In how many ways can 3 prizes be distributed among 4 boys, when (i) no boy gets more than one prize?

(ii) a boy may get any number of prizes?

(iii) no boy gets all the prizes?

Solution:

(i) The total number of ways is the number of arrangements of 4 taken 3 at a time.

So, the required number of ways = ${}_4P_3 = 4! = 24$

(ii) The first prize can be given away in 4 ways as it may be given to anyone of the 4 boys.

The second prize can also be given away in 4 ways, since it may be obtained by the boy who has already received a prize.

Similarly, third prize can be given away in 4 ways.

Hence, the number of ways in which all the prizes can be given away

$= 4 \times 4 \times 4 = 4^3 = 64$

(iii) Since any one of the 4 boys may get all the prizes. So, the number of ways in which a boy get all the 3 prizes = 4.

So, the number of ways in which a boy does not get all the prizes = 64–4=60

3.2.5 Circular Permutations:

We have seen that the number of permutations of n different things taken all together is $n!$, where each permutation is a different arrangement of the n things in a row, or a straight line. These permutations are called linear permutations or simply permutations.

A circular permutation is one in which the things are arranged along a circle. It is also called closed permutation.

Theorem 3.4:

The number of circular permutations of n distinct objects is $(n - 1)!$

Proof:

Let $a_1, \quad a_2, \ldots, \quad a_{n-1}, \ a_n$ be n distinct objects.

Let the total number of circular permutations be x.

Consider one of these x permutations as shown in figure.

Clearly this circular permutation provides n near permutations as given below

$$\begin{array}{llll}
a_1, a_2, a_3, & \ldots, & a_{n-1}, & a_n \\
a_2, a_3, a_4, & \ldots, & a_n, & a_1 \\
a_3, a_4, a_5, & \ldots, & a_1, & a_2 \\
\ldots \ \ldots & \ldots & \ldots & \\
\ldots \ \ldots & \ldots & \ldots & \\
a_n, a_1, a_2, & \ldots, & a_{n-2}, & a_{n-1}
\end{array}$$

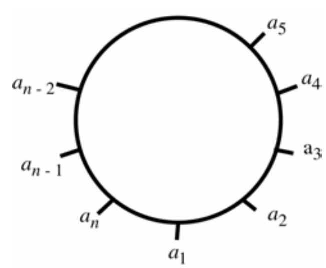


Fig. 3. 1

Thus, each circular permutation gives n linear permutations.

But there are x circular permutations.

So, total number of linear permutations is xn.

But the number of linear permutations of n distinct objects is $n!$.

$$\begin{aligned}
\therefore xn &= n! \\
\Rightarrow \quad x &= \frac{n!}{n} \\
x &= (n-1)!
\end{aligned}$$

$\therefore$ The total number of circular permutations of n distinct objects is $(n - 1)!$

Note: In the above theorem anti-clockwise and clockwise order of arrangements are considered as distinct permutations.

Difference between clockwise and anti-clockwise arrangements:

Consider the following circular permutations:

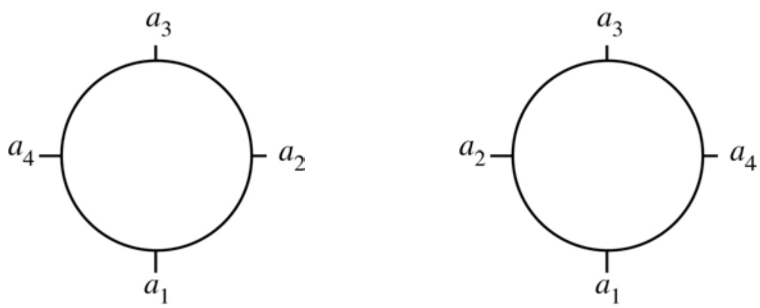


Fig. 3. 2 ***Fig. 3. 3***

We observe that in both, the order of the circular arrangement is a_1, a_2, a_3, a_4.

In fig (3.2) the order is anti-clockwise, whereas in fig. (3.3) the order is clockwise.

Thus the number of circular permutation of n things in which clockwise and anti-clockwise arrangements give rise to different permutations is $(n - 1)!$

If there are n things and if the direction is not taken into consideration, the number of circular permutations is $\frac{1}{2}\ (n - 1)!$

Example 3.30:

In how many ways 10 persons may be arranged in a (i) line (ii) circle?

Solution:

(i) The number of ways in which 10 persons can be arranged in a line $= {}_{10}P_{10} = 10!$

(ii) The number of ways in which 10 persons can be arranged in a circle $= (10 - 1)! = 9!$

Example 3.31: In how many ways can 7 identical beads be stung into a ring?

Solution: Since the arrangement is circular either clockwise arrangement or anti-clockwise arrangement may be considered.

$\therefore$ The required number of ways $= \frac{1}{2}\ (7 - 1)! = \frac{6!}{2} = 360$

Example 3.32: In how many ways can 5 gentlemen and 5 ladies sit together at a round table, so that no two ladies may be together?

Solution:

The number of ways in which 5 gentlemen may be arranged is $(5 - 1)! = 4!$

Then the ladies may be arranged among themselves in 5! ways.

Thus the total number of ways $= 4! \times 5! = 24 \times 120 = 2880$

Example 3.33: Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.

Solution:

Considering 4 particular flowers as one flower, we have five flowers, which can be strung to form a garland in 4! ways.

But 4 particular flowers can be arranged in 4! ways.

$\therefore$ Required number of ways $= 4\,! \times 4\,! = 576$

EXERCISE 3.3

1. Evaluate the following :

   (i) ${}_5P_3$ (ii) ${}_{15}P_3$ (iii) ${}_5P_5$ (iv) ${}_{25}P_{20}$ (v) ${}_9P_5$

2. If ${}_nP_4 = 20 \,.\, {}_nP_3$, find n.

3. If ${}_{10}P_r = 5040$, find the value of r.

4. If ${}_{56}P_{(r+6)} : {}_{54}P_{(r+3)} = 30800 : 1$, find r

5. If P_m stands for ${}_mP_m$, then prove that $1 + 1.P_1 + 2.P_2 + 3.P_3 + \dots$

   $+ n.P_n = (n + 1)!$

6. Prove that ${}_nP_r = {}_{(n-1)}P_r + r \,.\, {}_{(n-1)}P_{(r-1)}$.

7. Three men have 4 coats, 5 waistcoats and 6 caps. In how many ways can they wear them?

8. How many 4-letter words, with or without meaning, can be formed, out of the letters of the word, ‘LOGARITHMS’, if repetition of letters is not allowed?

9. How many 3-digit numbers are there, with distinct digits, with each digit odd?

10. Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.

11. How many different words can be formed with the letters of the word ‘MISSISSIPPI’?

12. (i) How many different words can be formed with letters of the word ‘HARYANA’?

    (ii) How many of these begin with H and end with N?

13. How many 4-digit numbers are there, when a digit may be repeated any number of times?

14. In how many ways 5 rings of different types can be worn in 4 fingers?

15. In how many ways can 8 students are seated in a (i) line (ii) circle?

16. In how many ways can a garland of 20 similar flowers are made?

3.3 Combinations:

The word combination means selection. Suppose we are asked to make a selection of any two things from three things *a*, *b* and *c*, the different selections are *ab*, *bc*, *ac*.

Here there is no reference to the order in which they are selected.

i.e. *ab* and *ba* denote the same selection. These selections are called combinations.

Definition:

A selection of any *r* things out of *n* things is called a combination of *n* things *r* at a time.

Notation:

The number of all combinations of *n* objects, taken *r* at a time is generally denoted by ${}_nC_r$ or $C(n,r)$ or $\binom{n}{r}$. We use the symbol ${}_nC_r$ throughout our discussion.

$$\text{Thus } {}_nC_r = \begin{cases}\text{Number of ways of selecting} \\ r \text{ objects from } n \text{ objects}\end{cases}$$

Difference between Permutation and Combination:

1. In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

i.e. in a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential.

2. Usually the number of permutation exceeds the number of combinations.
3. Each combination corresponds to many permutations.

Combinations of *n* different things taken *r* at a time:

Theorem 3.5:

The number of all combinations of *n* distinct objects, taken *r* at a time is given by ${}_nC_r = \dfrac{n!}{(n-r)!\,r!}$

Proof: Let the number of combinations of *n* distinct objects, taken *r* at a time be denoted by ${}_nC_r$.

Each of these combinations contains *r* things and all these things are permuted among themselves.

$\therefore$ The number of permutations obtained is $r\,!$

Hence from all the ${}_nC_r$ combinations we get ${}_nC_r \times r!$ permutations.

But this gives all the permutations of n things taken r at a time i.e. ${}_nP_r$.

Hence, $\qquad {}_nC_r \, . \, r\,! \;=\; {}_nP_r$

$$\therefore\ {}_nC_r = \frac{{}_nP_r}{r!}$$

$$= \frac{n!}{(n-r)!\; r!} \qquad \left(\because\ {}_nP_r = \frac{n!}{(n-r)!}\right)$$

Properties

(1) ${}_nC_n = 1$ $\qquad$ (2) ${}_nC_0 = 1$ $\qquad$ (3) ${}_nC_r = {}_nC_{n-r}$ $\qquad 0 \le r \le n$

Proof:

(1) We know that $\qquad {}_nC_r = \dfrac{n!}{(n-r)!\; r!}$

Putting $r = n$, we have $\qquad {}_nC_n = \dfrac{n!}{(n-n)!\; n!} = \dfrac{n\,!}{0!\; n!}$

$= 1$

(2) Putting $r = 0$, we have

$${}_nC_0 = \frac{n!}{(n-0)\,!\; 0!} = \frac{n\,!}{n!} = 1$$

(3) We have $\qquad {}_nC_{n-r} = \dfrac{n!}{(n-r)!\,\big(n-\overline{n-r}\big)!} = \dfrac{n!}{(n-r)!\; r!}$

$= {}_nC_r$

Note: The above property can be restated as follows :

If x and y are non-negative integers such that $x + y = n$, then ${}_nC_x = {}_nC_y$

(4) If n and r are positive integers such that $r \le n$,

then ${}_nC_r + {}_nC_{(r-1)} = {}_{(n+1)}C_r$

Proof: We have

$${}_nC_r + {}_nC_{(r-1)} = \frac{n!}{(n-r)!\; r!} + \frac{n!}{\big(n-\overline{r-1}\big)!\;(r-1)!}$$

$$= \frac{n!}{(n-r)!\; r!} + \frac{n!}{(n-r+1)\,!\;(r-1)!}$$

$$= \frac{n!}{(n-r)\,!\; r\{(r-1)!\}} + \frac{n!}{(n-r+1)\;\{(n-r)!\;(r-1)!\}}$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left\{\frac{1}{r} + \frac{1}{n-r+1}\right\}$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left\{\frac{n-r+1+r}{r(n-r+1)}\right\}$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left\{\frac{n+1}{r(n-r+1)}\right\}$$

$$= \frac{(n+1)\,\{n!\}}{(n-r+1)\,(n-r)!\,r(r-1)!}$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)!}{(n+1-r)!\,r!}$$

$$= {}_{(n+1)}\mathrm{C}_{\mathrm{r}}$$

(5) If n and r are positive integers such that $1 \leq r \leq n$,

then ${}_{n}\mathrm{C}_{r} = \frac{n}{r}\ {}_{(n-1)}\mathrm{C}_{(r-1)}$

Proof:

$${}_{n}\mathrm{C}_{r} = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n(n-1)!}{[(n-1)-(r-1)]!\,r\,(r-1)!}$$

$$= \frac{n}{r} \quad \frac{(n-1)!}{[(n-1)-(r-1)]!\,(r-1)!}$$

$$= \frac{n}{r} \quad {}_{(n-1)}\mathrm{C}_{(r-1)}$$

(6)If $1 \leq r \leq n$, then $n \,.\, {}_{(n-1)}\mathrm{C}_{(r-1)} = (n-r+1) \,.\, {}_{n}\mathrm{C}_{(r-1)}$

Proof:

We have $n \,.\, {}_{(n-1)}\mathrm{C}_{(r-1)} = n\left\{\frac{(n-1)!}{[(n-1)-(r-1)]!\,(r-1)!}\right\}$

$$= \frac{n!}{(n-r)!\,(r-1)!}$$

$$= \frac{(n-r+1)n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$= (n - r + 1)\left[\frac{n!}{(n-r+1)!\,(r-1)!}\right]$$

$$= (n - r + 1)\left[\frac{n!}{(n-r+1)!\,(r-1)!}\right]$$

$$= (n - r + 1)\,.\,{}_nC_{(r-1)}$$

(7) For any positive integers x and y,

$${}_nC_x = {}_nC_y \Rightarrow x = y \text{ or } x + y = n$$

Proof: We have ${}_nC_x = {}_nC_y$

$\Rightarrow$ ${}_nC_x = {}_nC_y = {}_nC_{(n-y)}$ $[\because {}_nC_y = {}_nC_{(n-y)}]$

$\Rightarrow x = y$ or $x = n - y$

$\Rightarrow x = y$ or $x + y = n$

Note: If ${}_nC_x = {}_nC_y$ and $x \neq y$, then $x + y = n$

Example 3.34: Evaluate the following :

(i) ${}_6C_3$ (ii) $\sum_{r=1}^{5} {}_5C_r$

Solution:

(i) $${}_6C_3 = \frac{{}_6P_3}{3!} = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$$

(ii) $$\sum_{r=1}^{5} {}_5C_r = {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5$$

$$= 5 + 10 + 10 + 5 + 1 = 31$$

Example 3.35: If ${}_nC_4 = {}_nC_6$, find ${}_{12}C_n$

Solution:

$${}_nC_4 = {}_nC_6 \Rightarrow n = 4 + 6 = 10$$

Now $${}_{12}C_n = {}_{12}C_{10}$$

$$= {}_{12}C_{(12-10)} = {}_{12}C_2 = \frac{12 \times 11}{1 \times 2}$$

$$= 66$$

Example 3.36: If ${}_{15}C_r : {}_{15}C_{(r-1)} = 11 : 5$, find r

Solution:

$$^{15}C_r : {}^{15}C_{(r-1)} = 11 : 5 \quad \Rightarrow \quad \frac{^{15}C_r}{^{15}C_{(r-1)}} = \frac{11}{5}$$

$$\Rightarrow \quad \frac{\dfrac{15!}{r!(15-r)!}}{\dfrac{15!}{(r-1)!\,(15-r+1)!}} = \frac{11}{5}$$

$$\Rightarrow \quad \frac{15!}{r!\,(15-r)!} \times \frac{(r-1)!\,(16-r)!}{15!} = \frac{11}{5}$$

$$\Rightarrow \quad \frac{(r-1)!\,(16-r)\,\{(15-r)!\}}{r(r-1)!\,(15-r)!} = \frac{11}{5}$$

$$\Rightarrow \quad \frac{16-r}{r} = \frac{11}{5}$$

$$\Rightarrow \quad 5(16-r) = 11r \quad \Rightarrow \quad 80 = 16r$$

$$\Rightarrow \quad r = 5$$

Example 3.37: Show that the product of r consecutive integers is divisible by r!

Solution:

Let the r consecutive integers be $n + 1, n + 2, n + 3, \ldots, n + r$

$$\begin{aligned} \text{Hence their product} &= (n+1)\,(n+2)\,(n+3)\ldots(n+r) \\ &= \frac{1.2.3\ldots n.\,(n+1)\,(n+2)\ldots(n+r)}{1.2.3\ldots n} \\ &= \frac{(n+r)!}{n!} \end{aligned}$$

$$\therefore \frac{\text{their product}}{r!} = \frac{(n+r)!}{n!\,r!}$$

$$= {}^{(n+r)}C_r \text{ which is an integer.}$$

$\therefore$ The product of r consecutive integers is divisible by r!

Example 3.38: Let r and n be positive integers such that $1 \le r \le n$. Then prove the following :

$$\frac{^nC_r}{^nC_{(r-1)}} = \frac{n-r+1}{r}$$

Solution: $$\frac{^nC_r}{^nC_{(r-1)}} = \frac{\dfrac{n!}{r!(n-r)!}}{\dfrac{n!}{(r-1)!\,(n-r+1)!}}$$

$$= \frac{n!}{r!\,(n-r)!} \times \frac{(r-1)!\,(n-r+1)!}{n!}$$

$$= \frac{(r-1)!\,(n-r+1)\,\{(n-r)!\}}{r(r-1)!\,(n-r)!}$$

$$= \frac{n-r+1}{r}$$

Example 3.39 : If ${}_nP_r = {}_nP_{(r+1)}$ and ${}_nC_r = {}_nC_{(r-1)}$, find the values of n and r

Solution:

$${}_nP_r = {}_nP_{(r+1)} \Rightarrow \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$$

$$\Rightarrow \frac{1}{(n-r)\,(n-r-1)!} = \frac{1}{(n-r-1)!}$$

$$\Rightarrow n-r = 1 \qquad \ldots (1)$$

$${}_nC_r = {}_nC_{(r-1)} \Rightarrow \frac{n!}{r!\,(n-r)!} = \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$\Rightarrow \frac{n!}{r(r-1)!\,(n-r)!} = \frac{n!}{(r-1)!\,(n-r+1)\,\{(n-r)!\}}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{n-r+1}$$

$$\Rightarrow n-r+1 = r$$

$$\Rightarrow n-2r = -1 \qquad \ldots (2)$$

Solving (1) and (2) we get $n = 3$ and $r = 2$

EXERCISE 3.4

1. Evaluate the following:

(i) ${}_{10}C_8$ (ii) ${}_{100}C_{98}$ (iii) ${}_{75}C_{75}$

2. If ${}_nC_{10} = {}_nC_{12}$, find ${}_{23}C_n$

3. If ${}_8C_r - {}_7C_3 = {}_7C_2$, find r

4. If ${}_{16}C_4 = {}_{16}C_{r+2}$, find ${}_rC_2$

5. Find n if (i) $2 \,.\, {}_nC_3 = \frac{20}{3}\ {}_nC_2$ (ii) ${}_nC_{(n-4)} = 70$

6. If ${}_{(n+2)}C_8 : {}_{(n-2)}P_4 = 57 : 16$, find n.

7. If ${}_{28}C_{2r} : {}_{24}C_{(2r-4)} = 225 : 11$, find r.

Practical problems on Combinations

Example 3.40: From a group of 15 cricket players, a team of 11 players is to be chosen. In how many ways this can be done?

Solution:

There are 15 players in a group. We have to select 11 players from the group.

$\therefore$ The required number of ways = ${}_{15}C_{11}$

$$ {}_{15}C_{11} \quad = \quad \frac{15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4} = 1365 \text{ ways} $$

Example 3.41: How many different teams of 8, consisting of 5 boys and 3 girls can be made from 25 boys and 10 girls?

Solution:

5 boys out of 25 boys can be selected in ${}_{25}C_5$ ways.

3 girls out of 10 girls can be selected in ${}_{10}C_3$ ways.

$\therefore$ The required number of teams = ${}_{25}C_5 \times {}_{10}C_3 = 6375600$

Example 3.42: How many triangles can be formed by joining the vertices of a hexagon?

Solution:

There are 6 vertices of a hexagon.

One triangle is formed by selecting a group of 3 vertices from given 6 vertices.

This can be done in ${}_6C_3$ ways.

$$ \therefore \text{ Number of triangles } = {}_6C_3 = \frac{6!}{3!\,3!} = 20 $$

Example 3.43:

A class contains 12 boys and 10 girls. From the class 10 students are to be chosen for a competition under the condition that atleast 4 boys and atleast 4 girls must be represented. The 2 girls who won the prizes last year should be included. In how many ways can the selection are made?

Solution:

There are 12 boys and 10 girls. From these we have to select 10 students.

Since two girls who won the prizes last year are to be included in every selection.

So, we have to select 8 students from 12 boys and 8 girls, choosing atleast 4 boys and atleast 2 girls. The selection can be formed by choosing

(i) 6 boys and 2 girls

(ii) 5 boys and 3 girls

(iii) 4 boys and 4 girls

$$
\begin{aligned}
\therefore \text{ Required number of ways} &= ({}_{12}C_6 \times {}_{8}C_2) + ({}_{12}C_5 \times {}_{8}C_3) + ({}_{12}C_4 \times {}_{8}C_4) \\
&= (924 \times 28) + (792 \times 56) + (495 \times 70) \\
&= 25872 + 44352 + 34650 \\
&= 104874
\end{aligned}
$$

Example 3.44: How many diagonals are there in a polygon?

Solution: A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon.

$$\left.\begin{array}{c}\text{Number of line segments obtained by} \\ \text{joining the vertices of a } n \text{ sided} \\ \text{polygon taken two at a time}\end{array}\right\} = \text{Number of ways of selecting 2 out of } n$$

$$= {}_{n}C_2 = \frac{n(n-1)}{2}$$

Out of these lines, n lines are the sides of the polygon.

$$
\begin{aligned}
\therefore \text{ Number of diagonals of the polygon} &= \frac{n(n-1)}{2} - n \\
&= \frac{n(n-3)}{2}
\end{aligned}
$$

Example 3.45 How many different sections of 4 books can be made from 10 different books, if (i) there is no restriction

(ii) two particular books are always selected;

(iii) two particular books are never selected?

Solution:

(i) The total number of ways of selecting 4 books out of 10 = ${}_{10}C_4 = \frac{10!}{4!\,6!} = 210$

(ii) If two particular books are always selected.

This means two books are selected out of the remaining 8 books

$$\therefore \text{ Required number of ways} = {}_{8}C_2 = \frac{8!}{2!\,6!} = 28$$

(iii) If two particular books are never selected

This means four books are selected out of the remaining 8 books.

$$\therefore \text{ Required number of ways} = {}_{8}C_4 = \frac{8!}{4!\,4!} = 70$$

Example 3.46:

In how many ways players for a cricket team can be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers? Assume that the team requires 5 batsmen, 3 all-rounder, 2 bowlers and 1 wicket keeper.

Solution:

The selection of team is divided into 4 phases:

(i) Selection of 5 batsmen out of 10. This can be done in ${}_{10}C_5$ ways.

(ii) Selection of 3 all-rounders out of 5. This can be done in ${}_{5}C_3$ ways.

(iii) Selection of 2 bowlers out of 8. This can be done in ${}_{8}C_2$ ways.

(iv) Selection of one wicket keeper out of 2. This can be done in ${}_{2}C_1$ ways.

$\therefore$ The team can be selected in ${}_{10}C_5 \times {}_{5}C_3 \times {}_{8}C_2 \times {}_{2}C_1$ ways

$= 252 \times 10 \times 28 \times 2$ ways

$= 141120$ ways

Example 3.47: Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. How many

(i) straight lines (ii) triangles can be formed by joining them?

Solution:

(i) Number of straight lines formed joining the 18 points,
taking 2 at a time $= {}_{18}C_2 = 153$

Number of straight lines formed by joining the 5 points,
taking 2 at a time $= {}_{5}C_2 = 10$

But 5 collinear points, when joined pairwise give only one line.

$\therefore$ Required number of straight lines $= 153 - 10 + 1 = 144$

(ii) Number of triangles formed by joining the 18 points,
taken 3 at a time $= {}_{18}C_3 = 816$

Number of triangles formed by joining the 5 points,
taken 3 at a time $= {}_{5}C_3 = 10$

But 5 collinear points cannot form a triangle when taken 3 at a time.

$\therefore$ Required number of triangles $= 816 - 10 = 806$

EXERCISE 3.5

1. If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?

2. In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 3 men and 2 women?
3. How many triangles can be obtained by joining 12 points, five of, which are collinear?
4. A box contains 5 different red and 6 different white balls. In how many ways 6 balls be selected so that there are atleast two balls of each colour?
5. In how many ways can a cricket team of eleven be chosen out of a batch of 15 players if
   (i) there is no restriction on the selection
   (ii) a particular player is always chosen;
   (iii) a particular player is never chosen?
6. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. In how many ways can he choose the 7 questions.
7. There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the
   (i) the number of straight lines obtained from the pairs of these points
   (ii) number of triangles that can be formed with the vertices as these points.
8. In how many ways can 21 identical books on Tamil and 19 identical books on English be placed in a row on a shelf so that two books on English may not be together?
9. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen?

3.4 Mathematical Induction:

Introduction:

The name 'Mathematical induction' in the sense in which we have given here, was first used by the English Mathematician Augustus De-Morgan (1809 – 1871) in his article on 'Induction Mathematics' in 1938. However the originator of the Principle of Induction was Italian Mathematician Francesco Mau Rolycus (1494 – 1575). The Indian Mathematician Bhaskara (1153 A.D) had also used traces of 'Mathematical Induction' in his writings.

"Induction is the process of inferring a general statement from the truth of particular cases".

For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 7 + 3$ and so on.

From these cases one may make a general statement "every even integer except 2 can be expressed as a sum of two prime numbers. There are hundreds of particular cases where this is known to be true. But we cannot conclude that this statement is true unless it is proved. Such a statement inferred from particular cases is called a conjecture. A conjecture remains a conjecture until it is proved or disproved.

Let the conjecture be a statement involving natural numbers. Then a method to prove a general statement after it is known to be true in some particular cases is the principle of mathematical induction.

Mathematical induction is a principle by which one can conclude that a statement is true for all positive integers, after proving certain related propositions.

The Principle of Mathematical Induction:

Corresponding to each positive integer n let there be a statement or proposition $P(n)$.

If (i) $P(1)$ is true,

and (ii) $P(k + 1)$ is true whenever $P(k)$ is true,

then $P(n)$ is true for all positive integers n.

We shall not prove this principle here, but we shall illustrate it by some examples.

Working rules for using principle of mathematical induction:

Step (1) : Show that the result is true for $n = 1$.

Step (2) : Assume the validity of the result for n equal to some arbitrary but fixed natural number, say k.

Step (3) : Show that the result is also true for $n = k + 1$.

Step (4) : Conclude that the result holds for all natural numbers.

Example 3.48: Prove by mathematical induction $n^2 + n$ is even.

Solution: Let $P(n)$ denote the statement "$n^2 + n$ is even"

Step (1):

$$\begin{aligned} \text{Put } n &= 1 \\ n^2 + n &= 1^2 + 1 \\ &= 2, \text{ which is even} \end{aligned}$$

$\therefore$ $P(1)$ is true

Step (2):

Let us assume that the statement be true for $n = k$

(i.e.) assume P(k) be true.

(i.e.) assume "$k^2 + k$ is even" be true … (1)

Step (3):

To prove P(k + 1) is true.

(i.e.) to prove $(k + 1)^2 + (k + 1)$ is even

$$\begin{aligned}\text{Consider}\quad (k + 1)^2 + (k + 1) &= k^2 + 2k + 1 + k + 1\\ &= k^2 + 2k + k + 2\\ &= (k^2 + k) + 2(k + 1)\\ &= \text{an even number} + 2(k + 1), \text{ from (1)}\\ &= \text{sum of two even numbers}\\ &= \text{an even number}\end{aligned}$$

$\therefore$ P(k + 1) is true.

Thus if P(k) is true, then P(k + 1) is also true.

Step (4):

$\therefore$ By the principle of Mathematical induction, P(n) is true for all $n \in$N.

i.e. $n^2 + n$ is even for all $n \in$N.

Example 3.49: Prove by Mathematical induction $1 + 2 + 3 + \ldots + n = \dfrac{n(n + 1)}{2}$, $n \in$N

Solution: Let P(n) denote the statement : "$1 + 2 + 3 + \ldots + n = \dfrac{n(n + 1)}{2}$ "

Put $n = 1$

P(1) is the statement :

$$1 = \frac{1\,(1 + 1)}{2}$$

$$1 = \frac{1(2)}{2}$$

$$1 = 1$$

$\therefore$ P(1) is true

Now assume that the statement be true for $n = k$.

(i.e.) assume P(k) be true.

(i.e.) assume $1 + 2 + 3 + \ldots + k = \dfrac{k(k + 1)}{2}$ … (1) be true

To prove P(k + 1) is true

(i.e.) to prove $1 + 2 + 3 + \ldots + k + (k + 1) = \dfrac{(k + 1)\,(k + 2)}{2}$ is true,

$$[1 + 2 + 3 + \ldots + k] + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) \qquad \text{from (1)}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$\therefore$ P(k + 1) is true.

Thus if P(k) is true, then P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all n$\in$N

$$\therefore\ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \quad \text{for all } n \in N$$

Example 3.50: Prove by Mathematical induction

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{for all } n \in N$$

Solution:

Let P(n) denote the statement "$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$"

Put $n = 1$

P(1) is the statement : $1^2 = \frac{1(1+1)[2(1)+1]}{6}$

$$1 = \frac{1(2)(3)}{6}$$

$$1 = 1$$

$\therefore$ P(1) is true.

Now assume that the statement be true for $n = k$.

(i.e.) assume P(k) be true.

(i.e.) $\quad 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ $\quad\quad$ … (1)

To prove : P(k + 1) is true

(i.e.) to prove: $1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ is true.

$$[1^2 + 2^2 + 3^2 + \ldots + k^2] + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore$ P(k + 1) is true

Thus if P(k) is true, then P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all n$\in$N

(i.e.) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all n$\in$N

Example 3.51: Prove by Mathematical induction

$$1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}, \ n \in \text{N}.$$

Solution:

Let P(n) denote the statement "$1.2+2.3 + 3.4 +\ldots+ n(n + 1)= \frac{n(n+1)(n+2)}{3}$"

Put $n = 1$

P(1) is the statement : $1(1+1) = \frac{1(1+1)(1+2)}{3}$

$$1(2) = \frac{1(2)(3)}{3}$$

$$2 = \frac{2(3)}{3}$$

$$2 = 2$$

$\therefore$ P(1) is true.

Now assume that the statement be true for $n = k$.

(i.e.) assume P(k) be true

(i.e.) assume $1.2 + 2.3 + 3.4 +\ldots+ k(k+1) = \frac{k(k+1)(k+2)}{3}$ be true

To prove : P(k + 1) is true

i.e. to prove :

$$1.2 + 2.3 + 3.4 + \ldots,+ k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

Consider $1.2 + 2.3 + 3.4 + \ldots,+ k(k+1) + (k+1)(k+2)$

$$= \left[1.2 + 2.3 + \ldots + k(k+1)\right] + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$\therefore$ P(k + 1) is true

Thus if P(k) is true, P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all $n \in$ N.

$$1.2 + 2.3 + 3.4 + \ldots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$$

Example 3.52: Prove by Mathematical induction $2^{3n} - 1$ is divisible by 7, for all natural numbers n.

Solution:

Let P(n) denote the statement "$2^{3n} - 1$ is divisible by 7"

Put $n = 1$

Then P(1) is the statement : $\quad 2^{3(1)} - 1 = 2^3 - 1$

$$= 8 - 1$$

$$= 7, \text{ which is divisible by } 7$$

$\therefore$ P(1) is true

Now assume that the statement be true for $n = k$

(i.e.) assume P(k) be true. (i.e.) "$2^{3k} - 1$ is divisible by 7" be true

Now to prove P(k + 1) is true. (i.e.) to prove $2^{3(k+1)} - 1$ is divisible by 7

Consider

$$\begin{aligned} 2^{3(k+1)} - 1 &= 2^{3k+3} - 1 \\ &= 2^{3k} . 2^3 - 1 = 2^{3k} . 8 - 1 \\ &= 2^{3k} . 8 - 1 + 8 - 8 \qquad \text{(add and subtract 8)} \\ &= (2^{3k} - 1)\, 8 + 8 - 1 \\ &= (2^{3k} - 1)\, 8 + 7 = \text{a multiple of } 7 + 7 \\ &= \text{a multiple of } 7 \end{aligned}$$

$\therefore$ $2^{3(k+1)} - 1$ is divisible by 7

$\therefore$ P(k + 1) is true

Thus if P(k) is true, then P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all $n \in$ N

$\therefore$ $2^{3n} - 1$ is divisible by 7 for all natural numbers n.

Example 3.53: Prove by Mathematical induction that $a^n - b^n$ is divisible by (a–b) for all $n \in$ N

Solution: Let P(n) denote the statement "$a^n - b^n$ is divisible by $a - b$".

Put $n = 1$

Then P(1) is the statement : $a^1 - b^1 = a - b$ is divisible by $a - b$

$\therefore$ P(1) is true.

Now assume that the statement be true for $n = k$.

(i.e.) assume P(k) be true. (i.e.) $a^k - b^k$ is divisible by $(a - b)$ be true.

$$\Rightarrow \quad \frac{a^k - b^k}{a - b} = c \text{ (say) where } c \in \mathbb{N}$$

$$\Rightarrow \quad a^k - b^k = c(a - b)$$

$$\Rightarrow \quad a^k = b^k + c(a - b) \qquad \dots (1)$$

Now to prove P($k + 1$) is true. (i.e.) to prove : $a^{k+1} - b^{k+1}$ is divisible by $a - b$

$$\begin{aligned} \text{Consider} \quad a^{k+1} - b^{k+1} &= a^k \,.\, a - b^k\, b \\ &= \left[b^k + c(a - b)\right] a - b^k\, b \\ &= b^k a + ac(a - b) - b^k\, b \\ &= b^k(a - b) + ac\,(a - b) \\ &= (a - b)\,(b^k + ac) \text{ is divisible by } (a - b) \end{aligned}$$

$\therefore$ P($k + 1$) is true.

By the principle of Mathematical induction, P(n) is true for all $n \in \mathbb{N}$

$\therefore$ $a^n - b^n$ is divisible by $a - b$ for all $n \in \mathbb{N}$

EXERCISE 3.6

Prove the following by the principle of Mathematical Induction.

(1) $(2n + 1)(2n - 1)$ is an odd number for all $n \in \mathbb{N}$

(2) $2 + 4 + 6 + 8 + \dots + 2n = n(n + 1)$

(3) $1 + 3 + 5 + \dots + (2n - 1) = n^2$

(4) $1 + 4 + 7 + \dots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

(5) $4 + 8 + 12 + \dots + 4n = 2n(n + 1)$

(6) $1^3 + 2^3 + 3^3 + \dots + n^3 = \dfrac{n^2(n + 1)^2}{4}$

(7) $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \dots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

(8) In the arithmetic progression $a, a + d, a + 2d, \dots$
the n^{th} term is $a + (n - 1)d$

(9) $5^{2n} - 1$ is divisible by 24 for all $n \in N$

(10) $10^{2n-1} + 1$ is divisible by 11.

(11) $n(n + 1)(n + 2)$ is divisible by 6 where n is a natural number.

(12) The sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for all $n \in N$

(13) $7^{2n} + 16n - 1$ is divisible by 64

(14) $2^n > n$ for all $n \in N$

3.5 Binomial Theorem:

Introduction:

A BINOMIAL is an algebraic expression of two terms which are connected by the operation '+' (or) '–'

For example, $x + 2y$, $x - y$, $x^3 + 4y$, $a + b$ etc.. are binomials.

Expansion of Binomials with positive Integral Index:

We have already learnt how to multiply a binomial by itself. Finding squares and cubes of a binomial by actual multiplication is not difficult.

But the process of finding the expansion of binomials with higher powers such as $(x + a)^{10}$, $(x + a)^{17}$, $(x + a)^{25}$ etc becomes more difficult. Therefore we look for a general formula which will help us in finding the expansion of binomials with higher powers.

We know that

$(x + a)^1 = x + a = {}_1C_0\, x^1 a^0 + {}_1C_1\, x^0 a^1$

$(x + a)^2 = x^2 + 2ax + a^2 = {}_2C_0 x^2 a^0 + {}_2C_1 x^1 a^1 + {}_2C_2 x^0 a^2$

$(x + a)^3 = x^3 + 3x^2 a + 3xa^2 + a^3 = {}_3C_0 x^3 a^0 + {}_3C_1 x^2 a^1 + {}_3C_2 x^1 a^2 + {}_3C_3 x^0 a^3$

$(x+a)^4 = x^4 + 4x^3 + 6x^2 a^2 + 4xa^3 + a^4 = {}_4C_0 x^4 a^0 + {}_4C_1 x^3 a^1 + {}_4C_2 x^2 a^2 + {}_4C_3 x^1 a^3 + {}_4C_4 x^0 a^4$

For $n = 1, 2, 3, 4$ the expansion of $(x + a)^n$ has been expressed in a very systematic manner in terms of combinatorial coefficients. The above expressions suggest the conjecture that $(x + a)^n$ should be expressible in the form,

$(x + a)^n = nC_0\, x^n a^0 + nC_1 x^{n-1} a^1 + \dots + nC_{n-1}\, x^1\, a^{n-1} + nC_n\, x^0 a^n$

In fact, this conjecture is proved to be true and we establish it by using the principle of mathematical induction.

Theorem 3.6: (Binomial theorem for a Positive Integral Index)

Statement: For any natural number n

$$(x + a)^n = nC_0\, x^n a^0 + nC_1 x^{n-1} a^1 + \dots + nC_r\, x^{n-r}\, a^r + \dots + nC_{n-1}\, x^1\, a^{n-1} + nC_n\, x^0 a^n$$

Proof:

We shall prove the theorem by the principle of mathematical induction.

Let P(n) denote the statement :

$$(x + a)^n = nC_0\, x^n a^0 + nC_1 x^{n-1} a^1 + \dots + nC_r\, x^{n-r}\, a^r + \dots + nC_{n-1}\, x^1\, a^{n-1} + nC_n\, x^0 a^n$$

Step (1) :

Put $n = 1$

Then P(1) is the statement : $(x + a)^1 = {}_1C_0 x^1\, a^0 + {}_1C_1\, x^{1-1}\, a^1$

$x + a = x + a$

$\therefore$ P(1) is true

Step (2):

Now assume that the statement be true for $n = k$

(i.e.) assume P(k) be true.

$$(x + a)^k = kC_0 x^k\, a^0 + kC_1 x^{k-1} a^1 + kC_2\, x^{k-2} a^2 + \dots + kC_r\, x^{k-r} a^r + \dots + {}_kC_k\, x^0 a^k$$

be true … (1)

Step (3):

Now to prove P(k + 1) is true

(i.e.) **To prove:**

$$(x + a)^{K+1} = {}_{(k+1)}C_0 x^{k+1} + {}_{(k+1)}C_1 x^{(k+1)-1} a^1 + {}_{(k+1)}C_2 x^{(k+1)-2}\, a^2 + \dots + {}_{(k+1)}C_r x^{(k+1)-r}\, a^r + \dots + {}_{(k+1)}C_{(k+1)}\, a^{k+1}$$

Consider $(x + a)^{k+1} = (x + a)^k (x + a)$

$$= [kC_0 x^k + kC_1 x^{k-1} a^1 + kC_2\, x^{k-2} a^2 + \dots + kC_{(r-1)}\, x^{k-(r-1)}\, a^{(r-1)} + kC_r\, x^{k-r} a^r + \dots + kC_k a^k]\, (x + a)$$

$$= [{}^kC_0x^{k+1} + {}^kC_1x^k a^1 + {}^kC_2x^{k-1}a^2 + \dots + {}^kC_{r-1}\, x^{k-r+2}\, a^{r-1}$$
$$+ \ {}^kC_r x^{k-r+1}a^r + \dots + {}^kC_k x a^k]$$
$$+ [{}^kC_0x^k a + {}^kC_1x^{k-1}a^2 + {}^kC_2\, x^{k-2}a^3 + \dots + {}^kC_{r-1}\, x^{k-r+1}\, a^r$$
$$+ {}^kC_r\, x^{k-r}a^{r+1} + \dots + {}^kC_k a^{k+1}]$$
$$(x+a)^{k+1} = {}^kC_0x^{k+1} + ({}^kC_1 + {}^kC_0)\, x^k.a + ({}^kC_2 + {}^kC_1)\, x^{k-1}\, a^2$$
$$+ \dots + ({}^kC_r + {}^kC_{r-1})\, x^{k-r+1}a^r + \dots + {}^kC_k\, a^{k+1} \quad \dots (2)$$

We know that ${}^kC_r + {}^kC_{r-1} = {}^{(k+1)}C_r$

Put $r = 1, 2, 3, \dots$ etc.

$${}^kC_1 + {}^kC_0 = {}^{(k+1)}C_1$$
$${}^kC_2 + {}^kC_1 = {}^{(k+1)}C_2$$
$${}^kC_r + {}^kC_{r-1} = {}^{(k+1)}C_r \qquad \text{for } 1 \le r \le k$$
$${}^kC_0 = 1 = {}^{(k+1)}C_0$$
$${}^kC_k = 1 = {}^{(k+1)}C_{(k+1)}$$

$\therefore$ (2) becomes

$$(x+a)^{k+1} = {}^{(k+1)}C_0\, x^{k+1} + {}^{(k+1)}C_1\, x^k a + {}^{(k+1)}C_2\, x^{k-1}a^2$$
$$+ \dots + {}^{(k+1)}C_r\, x^{k+1-r}a^r + \dots + {}^{(k+1)}C_{(k+1)}\, a^{k+1}$$

$\therefore$ P(k + 1) is true

Thus if P(k) is true, P(k + 1) is true.

$\therefore$ By the principle of mathematical induction P(n) is true for all $n \in N$

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1x^{n-1}a^1 + \dots + {}^nC_r\, x^{n-r}\, a^r + \dots$$
$$+ {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0 a^n \qquad \text{for all } n \in N$$

Some observations:

1. In the expansion

   $(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1x^{n-1}a^1 + \dots + {}^nC_r\, x^{n-r}\, a^r + \dots$

   $+ {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0 a^n$, the general term is ${}^nC_r\, x^{n-r}\, a^r$.

   Since this is nothing but the $(r+1)^{th}$ term, it is denoted by T_{r+1}

   i.e. $\boldsymbol{T_{r+1} = {}^nC_r\, x^{n-r}\, a^r}$.

2. The $(n+1)^{th}$ term is $T_{n+1} = {}^nC_n\, x^{n-n}\, a^n = {}^nC_n\, a^n$, the last term.

Thus there are (n + 1) terms in the expansion of $(x+a)^n$

3. The degree of x in each term decreases while that of "a" increases such that the sum of the powers in each term is equal to n.

$$\boxed{\text{We can write } (x+a)^n = \sum_{r=0}^{n} nC_r\, x^{n-r} a^r}$$

4. nC_0, nC_1, nC_2, …, nC_r, … , nC_n are called binomial coefficients. They are also written as C_0, C_1 , C_2, … , C_n.
5. From the relation $nC_r = nC_{n-r}$, we see that the coefficients of terms equidistant from the beginning and the end are equal.
6. The binomial coefficients of the various terms of the expansion of $(x+a)^n$ for $n = 1, 2, 3, \ldots$ form a pattern.

Binomials	**Binomial coefficients**
$(x+a)^0$	1
$(x+a)^1$	1 1
$(x+a)^2$	1 2 1
$(x+a)^3$	1 3 3 1
$(x+a)^4$	1 4 6 4 1
$(x+a)^5$	1 5 10 10 5 1

This arrangement of the binomial coefficients is known as Pascal's triangle after the French mathematician Blaise Pascal (1623 – 1662). The numbers in any row can be obtained by the following rule. The first and last numbers are 1 each. The other numbers are obtained by adding the left and right numbers in the previous row.

1, $1 + 4 = 5$, $4 + 6 = 10$, $6 + 4 = 10$, $4 + 1 = 5$, 1

Some Particular Expansions:

In the expansion

$$(x+a)^n = nC_0\, x^n a^0 + nC_1 x^{n-1} a^1 + \ldots + nC_r\, x^{n-r} a^r + \ldots$$
$$+ nC_{n-1}\, x^1 a^{n-1} + nC_n\, x^0 a^n \quad \ldots (1)$$

1. If we put $-a$ in the place of a we get

$$\therefore\ (x-a)^n = nC_0 x^n - nC_1\, x^{n-1} a^1 + nC_2 x^{n-2}\, a^2 - \ldots$$
$$+ (-1)^r\, nC_r\, x^{n-r} a^r + \ldots + (-1)^n\, nC_n\, a^n$$

We note that the signs of the terms are positive and negative alternatively.

2. If we put 1 in the place of a in (1) we get,

$$(1 + x)^n = 1 + nC_1x + nC_2x^2 + \ldots + nC_rx^r + \ldots + nC_nx^n \quad \ldots (2)$$

3. If we put $-x$ in the place of x in (2) we get

$$(1 - x)^n = 1 - nC_1\, x + nC_2x^2 - \ldots + (-1)^r\, nC_rx^r + \ldots + (-1)^n\, nC_nx^n$$

Middle Term:

The number of terms in the expansion of $(x + a)^n$ depends upon the index n. The index is either even (or) odd. Let us find the middle terms.

Case (i) : n is even

The number of terms in the expansion is $(n + 1)$, which is odd.

Therefore, there is only one middle term and it is given by $T_{\frac{n}{2}+1}$

Case (ii) : n is odd

The number of terms in the expansion is $(n + 1)$, which is even.

Therefore, there are two middle terms and they are given by $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

Particular Terms:

Sometimes a particular term satisfying certain conditions is required in the binomial expansion of $(x + a)^n$. This can be done by expanding $(x + a)^n$ and then locating the required term. Generally this becomes a tedious task, when the index n is large. In such cases, we begin by evaluating the general term T_{r+1} and then finding the values of r by assuming T_{r+1} to be the required term.

To get the term independent of x, we put the power of x equal to zero and get the value of r for which the term is independent of x. Putting this value of r in T_{r+1}, we get the term independent of x.

***Example 3.54*:**Find the expansion of : (i) $(2x + 3y)^5$ (ii) $\left(2x^2 - \frac{3}{x}\right)^4$

Solution:

(i) $(2x + 3y)^5 = {}_5C_0\,(2x)^5\,(3y)^o + {}_5C_1\,(2x)^4\,(3y)^1 + {}_5C_2\,(2x)^3\,(3y)^2$

$+ {}_5C_3\,(2x)^2\,(3y)^3 + {}_5C_4\,(2x)^1\,(3y)^4 + {}_5C_5\,(2x)^0\,(3y)^5$

$= 1(32)x^5\,(1) + 5(16x^4)\,(3y) + 10(8x^3)\,(9y^2)$

$$+ 10(4x^2)(27y^3) + 5(2x)(81y^4) + (1)(1)(243y^5)$$

$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$$

(ii) $\left(2x^2 - \frac{3}{x}\right)^4 = {}_4C_0 (2x^2)^4 \left(-\frac{3}{x}\right)^0 + {}_4C_1 (2x^2)^3 \left(-\frac{3}{x}\right)^1$

$$+ {}_4C_2 (2x^2)^2 \left(-\frac{3}{x}\right)^2 + {}_4C_3 (2x^2)^1 \left(-\frac{3}{x}\right)^3 + {}_4C_4 (2x^2)^0 \left(-\frac{3}{x}\right)^4$$

$$= (1)\, 16x^8(1) + 4(8x^6)\left(-\frac{3}{x}\right) + 6(4x^4)\left(\frac{9}{x^2}\right) + 4(2x^2)\left(-\frac{27}{x^3}\right)$$

$$+ (1)(1)\left(\frac{81}{x^4}\right)$$

$$= 16x^8 - 96x^5 + 216x^2 - \frac{216}{x} + \frac{81}{x^4}$$

Example 3.55: Using binomial theorem, find the 7^{th} power of 11.

Solution:

$11^7 = (1 + 10)^7$

$$= {}_7C_0 (1)^7 (10)^0 + {}_7C_1 (1)^6 (10)^1 + {}_7C_2 (1)^5 (10)^2 + {}_7C_3 (1)^4 (10)^3 + {}_7C_4 (1)^3 (10)^4$$

$$+ {}_7C_5 (1)^2 (10)^5 + {}_7C_6 (1)^1 (10)^6 + {}_7C_7 (1)^0 (10)^7$$

$$= 1 + 70 + \frac{7 \times 6}{1 \times 2} 10^2 + \frac{7 \times 6 \times 5}{1 \times 2 \times 3} 10^3 + \frac{7 \times 6 \times 5}{1 \times 2 \times 3} 10^4 + \frac{7 \times 6}{1 \times 2} 10^5 + 7(10)^6 + 10^7$$

$$= 1 + 70 + 2100 + 35000 + 350000 + 2100000 + 7000000 + 10000000$$

$$= 19487171$$

Example 3.56: Find the coefficient of x^5 in the expansion of $\left(x + \frac{1}{x^3}\right)^{17}$

Solution:

In the expansion of $\left(x + \frac{1}{x^3}\right)^{17}$, the general term is

$$T_{r+1} = {}_{17}C_r\, x^{17-r} \left(\frac{1}{x^3}\right)^r$$

$$= {}_{17}C_r x^{17-4r}$$

Let T_{r+1} be the term containing x^5

then, $17 - 4r = 5 \Rightarrow r = 3$

$$\therefore\ T_{r+1} = T_{3+1}$$
$$= {}_{17}C_3\, x^{17-4(3)} = 680x^5$$
$$\therefore \text{ coefficient of } x^5 = 680$$

Example 3.57: Find the constant term in the expansion of $\left(\sqrt{x} - \dfrac{2}{x^2}\right)^{10}$

Solution:

In the expansion of $\left(\sqrt{x} - \dfrac{2}{x^2}\right)^{10}$

$$T_{r+1} = {}_{10}C_r \left(\sqrt{x}\right)^{10-r} \left(\frac{-2}{x^2}\right)^r$$

$$= {}_{10}C_r\, x^{\frac{10-r}{2}} \frac{(-2)^r}{x^{2r}} = {}_{10}C_r\, (-2)^r\, x^{\frac{10-r}{2} - 2r}$$

$$= {}_{10}C_r\, (-2)^r\, x^{\frac{10-5r}{2}}$$

Let T_{r+1} be the constant term

Then, $\dfrac{10-5r}{2} = 0 \Rightarrow r = 2$

$$\therefore \text{ The constant term } = {}_{10}C_2\, (-2)^2\, x^{\frac{10-5(2)}{2}}$$
$$= \frac{10 \times 9}{1 \times 2} \times 4 \times x^0$$
$$= 180$$

Example 3.58: If $n \in$ N, in the expansion of $(1 + x)^n$ prove the following :

(i) Sum of the binomial coefficients $= 2^n$

(ii) Sum of the coefficients of odd terms = Sum of the coefficients of even terms = 2^{n-1}

Solution: The coefficients nC_0, nC_1, nC_2, … , nC_n in the expansion of $(1 + x)^n$ are called the binomial coefficients, we write them as C_0, C_1, C_2, … C_n,

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_rx^r + \ldots + C_nx^n$$

It is an identity in x and so it is true for all values of x.

Putting $x = 1$ we get

$$2^n = C_0 + C_1 + C_2 + \ldots + C_n \qquad \ldots (1)$$

$$\text{put } x = -1$$

$$0 = C_0 - C_1 + C_2 - C_3 + \ldots (-1)^n C_n$$

$$\Rightarrow C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots$$

It is enough to prove that

$$C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$$

Let $\quad C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = k \quad \ldots (2)$

From (1), $\quad C_0 + C_1 + C_2 + \ldots + C_n = 2^n$

$$2k = 2^n \qquad \text{From (2)}$$

$$k = 2^{n-1}$$

From (2), $\quad C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

EXERCISE 3.7

(1) Expand the following by using binomial theorem

(i) $(3a + 5b)^5$ (ii) $(a - 2b)^5$ (iii) $(2x - 3x^2)^5$

(iv) $\left(x + \frac{1}{y}\right)^{11}$ (v) $(x^2 + 2y^3)^6$ (vi) $\left(x\sqrt{y} + y\sqrt{x}\right)^4$

(2) Evaluate the following:

(i) $\left(\sqrt{2} + 1\right)^5 + \left(\sqrt{2} - 1\right)^5$ (ii) $\left(\sqrt{3} + 1\right)^5 - \left(\sqrt{3} - 1\right)^5$

(iii) $\left(1 + \sqrt{5}\right)^5 + \left(1 - \sqrt{5}\right)^5$ (iv) $\left(2\sqrt{a} + 3\right)^6 + \left(2\sqrt{a} - 3\right)^6$

(v) $\left(2 + \sqrt{3}\right)^7 - \left(2 - \sqrt{3}\right)^7$

(3) Using Binomial theorem find the value of $(101)^3$ and $(99)^3$.

(4) Using Binomial theorem find the value of $(0.998)^3$.

(5) Find the middle term in the expansion of

(i) $\left(3x - \frac{2x^2}{3}\right)^8$ (ii) $\left(\frac{b}{x} + \frac{x}{b}\right)^{16}$

(iii) $\left(\frac{a}{x} - \sqrt{x}\right)^{16}$ (iv) $(x - 2y)^{13}$ (v) $\left(x + \frac{2}{x^2}\right)^{17}$

(6) Show that the middle term of

(i) $(1+x)^{2n}$ is $\dfrac{1.3.5.7 \ldots (2n-1)2^n x^n}{n!}$

(ii) $\left(x + \dfrac{1}{2x}\right)^{2n}$ is $\dfrac{1.3.5. \ldots (2n-1)}{n!}$

(iii) $\left(x - \dfrac{1}{x}\right)^{2n}$ is $\dfrac{(-1)^n . 1.3.5.7. \ldots (2n-1)}{n!} \; 2^n$

(7) Find the coefficient of x^5 in the expansion of $\left(x - \dfrac{1}{x}\right)^{11}$

(8) Find the term independent of x (constant term) in the expansion of

(i) $\left(2x^2 + \dfrac{1}{x}\right)^{12}$ (ii) $\left(\dfrac{4x^2}{3} - \dfrac{3}{2x}\right)^9$ (iii) $\left(9x - \dfrac{b}{cx^2}\right)^{17}$

(9) In the expansion of $(1+x)^{20}$, the coefficient of r^{th} and $(r+1)^{th}$ terms are in the ratio 1 : 6, find the value of r.

(10) If the coefficients of 5^{th}, 6^{th} and 7^{th} terms in the expansion of $(1+x)^n$ are in A.P., find n.

4. SEQUENCE AND SERIES

4.1 Introduction

We hear statements such as "a sequence of events", "a series of tests before the board examination", "a cricket test match series". In all these statements the words "sequence" and "series" are used in the same sense. They are used to suggest a succession of things or events arranged in some order. In mathematics these words have special technical meanings. The word 'sequence' is used as in the common use of the term to convey the idea of a set of things in order, but the word "series" is used in a different sense.

Let us consider the following example.

A rabbit and a frog are jumping on the same direction. When they started they were one metre apart. The rabbit is jumping on the frog in order to catch it. At the same time the frog is jumping forward half of the earlier distance to avoid the catch. The jumping process is going on. Can the rabbit catch the frog?

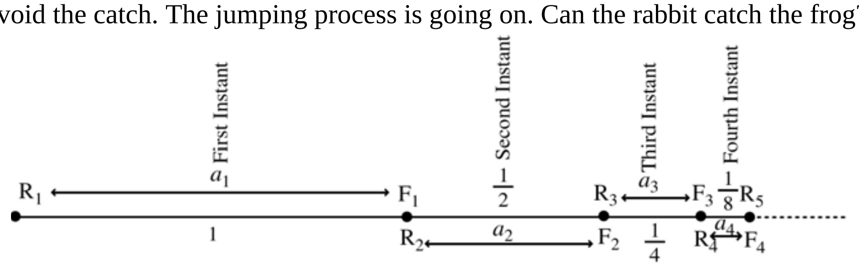


Fig. 4. 1

Let $a_1, a_2, a_3, a_4 \ldots$ be the distances between the rabbit and the frog at the first, second, third, fourth instants etc,. The distance between the rabbit and the frog at the first instant is 1 metre.

$$\therefore\ a_1 = 1\ ;\ a_2 = \frac{1}{2}\ ;\ a_3 = \frac{1}{4} = \frac{1}{2^2}\ ;\ a_4 = \frac{1}{8} = \frac{1}{2^3}$$

Here $a_1, a_2, a_3 \ldots$ form a sequence. There is a pattern behind the arrangement of $a_1, a_2, a_3 \ldots$ Now a_n has the meaning,

(i.e.) a_n is the distance between the rabbit and the frog at the n^{th} instant

Further $a_n = \dfrac{1}{2^{n-1}}$. When a_n becomes 0 the rabbit will catch the frog.

As $n \to \infty$, $a_n \to 0$

i.e. the distance between the frog and the rabbit is zero when $n \to \infty$

At this stage the rabbit will catch the frog.

This example suggests that for each natural number there is a unique real number.

i.e.

$$\begin{array}{ccccc} 1 & 2 & 3 & \ldots & n \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ a_1 & a_2 & a_3 & \ldots & a_n \\ = 1 & = \frac{1}{2} = \frac{1}{2^1} & = \frac{1}{4} = \frac{1}{2^2} & \ldots & = \frac{1}{2^{n-1}} \end{array}$$

Consider the following list of numbers

(a) 8, 15, 22, 29, …… (b) 6, 18, 54, 162, ……

In the list (a) the first number is 8, the 2nd number is 15, the 3rd number is 22, and so on. Each number in the list is obtained by adding 7 to the previous number.

In the list (*b*) the first number is 6, the 2 nd number is 18, the 3rd number is 54 etc. Each number in the list is obtained by multiplying the previous number by 3.

In these examples we observe the following:

(i) A rule by which the elements are written (pattern).

(ii) An ordering among the elements (order).

Thus a sequence means an arrangement of numbers in a definite order according to some rule.

4.2 Sequence

A sequence is a function from the set of natural numbers to the set of real numbers.

If the sequence is denoted by the letter a, then the image of $n \in$ N under the sequence a is $a(n) = a_n$.

Since the domain for every sequence is the set of natural numbers, the images of 1, 2, 3, … n … under the sequence a are denoted by $a_1, a_2, a_3 \ldots a_n$, … respectively. Here $a_1, a_2, a_3 \ldots a_n$, … form the sequence.

"A sequence is represented by its range".

Recursive formula

A sequence may be described by specifying its first few terms and a formula to determine the other terms of the sequence in terms of its preceding terms. Such a formula is called as **recursive** formula.

For example, 1, 4, 5, 9, 14, …, is a sequence because each term (except the first two) is obtained by taking the sum of preceding two terms. The corresponding recursive formula is $a_{n+2} = a_n + a_{n+1}$, $n \geq 1$ here $a_1=1$, $a_2= 4$

Terms of a sequence:

The various numbers occurring in a sequence are called its terms. We denote the terms of a sequence by $a_1, a_2, a_3, \ldots , a_n, \ldots$, the subscript denote the position of the term. The n^{th} term is called the general term of the sequence.

For example, in the sequence 1, 3, 5, 7, … $2n - 1$, …

the 1^{st} term is 1, 2^{nd} term is 3, … … and n^{th} term is $2n - 1$

Consider the following electrical circuit in which the resistors are indicated with saw-toothed lines.

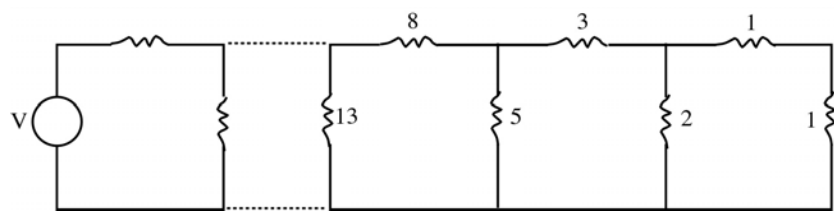


Fig. 4. 2

If all the resistors in the circuit are 1 ohm with a current of 1 ampere then the voltage across the resistors are 1, 1, 2, 3, 5, 8, 13, 21, …

In this sequence there is no fixed pattern. But we can generate the terms of the sequence recursively using a relation. Every number after the second is obtained by the sum of the previous two terms.

i.e.

$$
\begin{aligned}
V_1 &= 1 \\
V_2 &= 1 \\
V_3 &= V_2 + V_1 \\
V_4 &= V_3 + V_2 \\
V_5 &= V_4 + V_3 \\
&\;\vdots \\
V_n &= V_{n-1} + V_{n-2} \\
&\;\vdots
\end{aligned}
$$

Thus the above sequence is given by the rule:

$$V_1 = 1$$
$$V2 = 1$$
$$Vn = Vn - 1 + Vn - 2 \quad ; \quad n \geq 3$$

This sequence is called Fibonacci sequence. The numbers occurring in this sequence are called Fibonacci numbers named after the Italian Mathematician Leonardo Fibonacci.

Example 4.1:

Find the 7^{th} term of the sequence whose n^{th} term is $(-1)^{n+1}\left(\frac{n+1}{n}\right)$

Solution:

Given $\quad a_n = (-1)^{n+1}\left(\frac{n+1}{n}\right)$

substituting $\quad n = 7$, we get

$$a_7 = (-1)^{7+1}\left(\frac{8}{7}\right) = \frac{8}{7}$$

4.3 Series

For a finite sequence 1, 3, 5, 7, 9 the familiar operation of addition gives the symbol 1 + 3 + 5 + 7 + 9 which has the value 25.

If we consider the infinite sequence 1, 3, 5, 7, … then the symbol 1 + 3 + 5 + 7 + … has no definite value, because when we add more and more terms the value steadily increases. 1 + 3 + 5 + 7 + 9 + … is called an infinite series. Thus a series is obtained by adding the terms of a sequence.

If $a_1, a_2, a_3, \ldots a_n \ldots$ is an infinite sequence then $a_1 + a_2 + \ldots + a_n + \ldots$ is called an infinite series. It is also denoted by $\sum\limits_{k=1}^{\infty} a_k$

If $S_n = a_1 + a_2 + \ldots + a_n$ then S_n is called the n^{th} partial sum of the series $\sum\limits_{k=1}^{\infty} a_k$

Example 4.2 Find the n^{th} partial sum of the series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$

Solution:

$$S_n = \frac{1}{2^1} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$$

$$\text{and } S_{n+1} = \frac{1}{2^1} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$S_{n+1} = S_n + \frac{1}{2^{n+1}} \qquad \ldots (1)$$

Also we can write S_{n+1} as

$$S_{n+1} = \frac{1}{2^1} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}\right]$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}\right)\right]$$

$$S_{n+1} = \frac{1}{2}\,[1 + S_n] \qquad \ldots (2)$$

From (1) and (2) $\quad S_n + \dfrac{1}{2^{n+1}} = \dfrac{1}{2}\,[1 + S_n]$

$$2S_n + \frac{1}{2^n} = 1 + S_n$$

$$\therefore\ S_n = 1 - \frac{1}{2^n}$$

Note: This can be obtained by using the idea of geometric series also. We know that the sum to n terms of a geometric series is $\quad S_n = \dfrac{a(1 - r^n)}{(1 - r)}$

Here $a = \dfrac{1}{2}$, $\quad n = n, \quad r = \dfrac{1}{2}$ (< 1)

$$S_n = \frac{\frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}$$

EXERCISE 4.1

(1) Write the first 5 terms of each of the following sequences:

(i) $a_n = (-1)^{n-1}\, 5^{n+1}$ $\quad$ (ii) $a_n = \dfrac{n(n^2 + 5)}{4}$ $\quad$ (iii) $a_n = -11n + 10$

(iv) $a_n = \dfrac{n + 1}{n + 2}$ $\quad$ (v) $a_n = \dfrac{1 - (-1)^n}{3}$ $\quad$ (vi) $a_n = \dfrac{n^2}{3^n}$

(2) Find the indicated terms of the following sequences whose n^{th} term is

(i) $a_n = 2 + \frac{1}{n}$; a_5 , a_7 (ii) $a_n = \cos\left(\frac{n\pi}{2}\right)$; a_4 , a_5

(iii) $a_n = \frac{(n+1)^2}{n}$; a_7 , a_{10} (iv) $a_n = (-1)^{n-1}\, 2^{n+1}$, a_5 , a_8

(3) Find the first 6 terms of the sequence whose general term is

$$a_n = \begin{cases} n^2 - 1 & \text{if } n \text{ is odd} \\ \frac{n^2+1}{2} & \text{if } n \text{ is even} \end{cases}$$

(4) Write the first five terms of the sequence given by

(i) $a_1 = a_2 = 2,\ \ a_n = a_{n-1} - 1,\ \ n > 2$

(ii) $a_1 = 1,\ a_2 = 2,\ a_n = a_{n-1} + a_{n-2},\ n > 2$

(iii) $a_1 = 1,\ a_n = na_{n-1}\ ,\ n \geq 2$

(iv) $a_1 = a_2 = 1,\ a_n = 2a_{n-1} + 3a_{n-2},\ n > 2$

(5) Find the n^{th} partial sum of the series $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$

(6) Find the sum of first n terms of the series $\sum\limits_{n=1}^{\infty} 5^n$

(7) Find the sum of 101^{th} terms to 200^{th} term of the series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$

4.4 Some special types of sequences and their series

(1) Arithmetic progression:

An arithmetic progression (abbreviated as A.P) is a sequence of numbers in which each term, except the first, is obtained by adding a fixed number to the immediately preceding term. This fixed number is called the common difference, which is generally denoted by *d*.

For example, 1, 3, 5, 7, … is an A.P with common difference 2.

(2) Arithmetic series:

The series whose terms are in A.P is called an arithmetic series.

For example, 1 + 3 + 5 + 7 + … is an arithmetic series.

(3) Geometric progression

A geometric progression (abbreviated as G.P.) is a sequence of numbers in which the first term is non-zero and each term, except the first is obtained by multiplying the term immediately preceeding it by a fixed non-zero number. This fixed number is called the common ratio and it is denoted by the letter 'r'.

The general form of a G.P. is $a, ar, ar^2, \ldots$, with $a \neq 0$ and $r \neq 0$, the first term is 'a'

(4) Geometric series:

The series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ is called a geometric series because the terms of the series are in G.P. Note that the geometric series is finite or infinite according as the corresponding G.P. consists of finite (or) infinite number of terms.

(5) Harmonic progression:

A sequence of non-zero numbers is said to be in harmonic progression (abbreviated as H.P.) if their reciprocals are in A.P.

The general form of H.P is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$, where $a \neq 0$.

n^{th} term of H.P. is $T_n = \frac{1}{a + (n-1)d}$

For example the sequences $1, \frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \ldots$ is a H.P., since their reciprocals $1, 5, 9, 13, \ldots$ are in A.P.

Note: There is no general formula for the sum to n terms of a H.P. as we have for A.P. and G.P.

Example 4.3 If the 5^{th} and 12^{th} terms of a H.P. are 12 and 5 respectively, find the 15^{th} term.

Solution:

$$T_n = \frac{1}{a + (n-1)d}$$

Given $$T_5 = 12 \Rightarrow \frac{1}{a + (5-1)d} = 12 \Rightarrow \frac{1}{a + 4d} = 12$$

$$a + 4d = \frac{1}{12} \qquad \ldots (1)$$

and $$T_{12} = 5 \Rightarrow \frac{1}{a + (12-1)d} = 5 \Rightarrow \frac{1}{a + 11d} = 5$$

$$\Rightarrow \quad a + 11d = \frac{1}{5} \qquad \ldots (2)$$

$$(2) - (1) \qquad 7d = \frac{7}{60} \quad \Rightarrow \quad d = \frac{1}{60}$$

$$(1) \quad \Rightarrow \quad a + 4\left(\frac{1}{60}\right) = \frac{1}{12}$$

$$a + \frac{4}{60} = \frac{1}{12} \quad \Rightarrow \quad a = \frac{1}{12} - \frac{4}{60}$$

$$a = \frac{1}{60}$$

$$\therefore \; T_{15} = \frac{1}{a + (15-1)d} = \frac{1}{\frac{1}{60} + 14 \times \frac{1}{60}}$$

$$= \frac{1}{\frac{15}{60}} = \frac{60}{15}$$

$$T_{15} = 4$$

4.5 Means of Progressions

4.5.1 Arithmetic mean

A is called the arithmetic mean of the numbers a and b if and only if a, A, b are in A.P. If A is the A.M between a and b then a, A, b are in A.P

$$\Rightarrow \qquad A - a = b - A$$

$$\Rightarrow \qquad 2A = a + b$$

$$\Rightarrow \qquad \boxed{A = \frac{a+b}{2}}$$

$A_1, A_2, \dots, A_n$ are called n arithmetic means between two given numbers a and b if and only if $a, A_1, A_2, \dots A_n, b$ are in A.P.

Example 4.4 : Find the n arithmetic means between a and b and find their sum.

Solution:

Let $A_1, A_2, \dots, A_n$ be the n A.Ms between a and b. Then by the definition of A.Ms $\quad a, A_1, A_2, \dots, A_n, b$ are in A.P

Let the common difference be d.

$\therefore \; A_1 = a + d, \; A_2 = a + 2d, A_3 = a + 3d, \dots, A_n = a + nd$ and $b = a + (n+1)d$

$$\Rightarrow \quad (n+1)d = b - a$$

$$\therefore \; d = \frac{b-a}{n+1}$$

$$\therefore \; A_1 = a + \frac{b-a}{n+1} \;;\quad A_2 = a + \frac{2(b-a)}{n+1} \;\dots\; A_n = a + \frac{n(b-a)}{n+1}$$

Sum of n A.Ms between a and b is

$$A_1 + A_2 + \ldots + A_n = \left[a + \frac{b-a}{n+1}\right] + \left[a + \frac{2(b-a)}{n+1}\right] + \ldots + \left[a + \frac{n(b-a)}{n+1}\right]$$

$$= na + \frac{(b-a)}{n+1}[1 + 2 + \ldots + n]$$

$$= na + \frac{(b-a)}{(n+1)} \cdot \frac{n(n+1)}{2} = na + \frac{n(b-a)}{2}$$

$$= \frac{2na + nb - na}{2} = \frac{na + nb}{2} = n\left(\frac{a+b}{2}\right)$$

Example 4.5: Prove that the sum of n arithmetic means between two numbers is n times the single A.M between them

Solution:

Let $A_1, A_2, \ldots, A_n$ be the n A.Ms between a and b.

From the example (4.4)

$$A_1 + A_2 + A_3 + \ldots + A_n = n\left(\frac{a+b}{2}\right) = n \times (\text{A.M between } a \text{ and } b)$$

$$= n \text{ (single A.M between } a \text{ and } b)$$

Example 4.6: Insert four A.Ms between – 1 and 14.

Solution:

Let A_1, A_2, A_3, A_4 be the four A.Ms between – 1 and 14.

By the definition – 1, A_1, A_2, A_3, A_4, 14 are in A.P. Let d be the common difference.

$\therefore$ $A_1 = -1 + d$, ; $A_2 = -1 + 2d$; $A_3 = -1 + 3d$, ; $A_4 = -1 + 4d$; $14 = -1+5d$

$\therefore$ $d = 3$

$\therefore$ $A_1 = -1 + 3 = 2$; $A_2 = -1+2 \times 3 = 5$; $A_3 = -1+3\times3 = 8$; $A_4 = -1 + 12 = 11$

$\therefore$ The four A.Ms are 2, 5, 8 and 11.

4.5.2 Geometric Mean

G is called the geometric mean of the numbers a and b if and only if a, G, b are in G.P.

$$\Rightarrow \quad \frac{G}{a} = \frac{b}{G} = r$$

$$\Rightarrow \quad G^2 = ab$$

$$G = \pm\sqrt{ab}$$

Note:

(1) If a and b are positive then G = $+\sqrt{ab}$

(2) If a and b are negative then G = $-\sqrt{ab}$

(3) If a and b are opposite sign then their G.M is not real and it is discarded since we are dealing with real sequences.

i.e. If a and b are opposite in signs, then G.M between them does not exist.

Example 4.7: Find n geometric means between two given numbers a and b and find their product.

Solution:

Let $G_1, G_2, \ldots, G_n$ be n geometric means between a and b.

By definition $a, G_1, G_2, \ldots, G_n, b$ are in G.P. Let r be the common ratio.

Then $G_1 = ar$, $G_2 = ar^2, \ldots, G_n = ar^n$ and $b = ar^{n+1}$

$$r^{n+1} = \frac{b}{a} \qquad \therefore\ r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$\Rightarrow \quad G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \ \ldots\ G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

The product is

$$\begin{aligned} G_1 . G_2 . G_3 . G_n &= a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} . a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \ldots a\left(\frac{b}{a}\right)^{\frac{n}{n+1}} \\ &= a^n\left[\left(\frac{b}{a}\right)^{\frac{1+2+\ldots+n}{n+1}}\right] \\ &= a^n\left[\left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}\right] = a^n\left(\frac{b}{a}\right)^{\frac{n}{2}} \\ &= (ab)^{\frac{n}{2}} \end{aligned}$$

Example 4.8: Find 5 geometric means between 576 and 9.

Solution:

Let G_1, G_2, G_3, G_4, G_5 be 5 G.Ms between $a = 576$ and $b = 9$

Let the common ratio be r

$G_1 = 576r$, $G_2 = 576r^2$, $G_3 = 576r^3$, $G_4 = 576r^4$, $G_5 = 576r^5$, $9 = 576r^6$

$$\Rightarrow \quad r^6 = \frac{9}{576} \Rightarrow r = \left(\frac{9}{576}\right)^{\frac{1}{6}} = \left(\frac{1}{64}\right)^{\frac{1}{6}}$$

$$r = \frac{1}{2}$$

$$\therefore \quad G_1 = 576r = 576 \times \frac{1}{2} = 288 \qquad G_2 = 576r^2 = 576 \times \frac{1}{4} = 144$$

$$G_3 = 576r^3 = 576 \times \frac{1}{8} = 72 \qquad G_4 = 576r^4 = 576 \times \frac{1}{16} = 36$$

$$G_5 = 576r^5 = 576 \times \frac{1}{32} = 18$$

Hence 288, 144, 72, 36, 18 are the required G.Ms between 576 and 9.

Example 4.9: If b is the A.M of a and c ($a \neq c$) and ($b - a$) is the G.M of a and $c - a$, show that $a : b : c = 1 : 3 : 5$

Solution:

Given b is the A.M of a and c

$\therefore$ a, b, c are in A.P. Let the common difference be d

$$\therefore b = a + d \qquad \ldots (1)$$

$$c = a + 2d \qquad \ldots (2)$$

Given ($b - a$) is the G.M of a and ($c - a$)

$$\therefore (b - a)^2 = a(c - a)$$

$$d^2 = a(2d) \qquad \text{From (1) and (2)}$$

$$\Rightarrow \quad d = 2a \qquad [\because d \neq 0]$$

$\therefore$ $b = a + d$

$b = a + 2a$

$\boxed{b = 3a}$

$c = a + 2d$

$c = a + 2(2a)$

$\boxed{c = 5a}$

$$\therefore a : b : c = a : 3a : 5a$$

$$= 1 : 3 : 5$$

4.5.3 Harmonic mean

H is called the harmonic mean between a and b if a, H, b are in H.P

If a, H, b are in H.P then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P

$$\Rightarrow \quad \frac{1}{H} = \frac{\frac{1}{a} + \frac{1}{b}}{2} \quad ; \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

$$\boxed{H = \frac{2ab}{a+b}}$$

This H is single H.M between a and b

Definition:

$H_1, H_2, \ldots H_n$ are called n harmonic means between a and b if $a, H_1, H_2, \ldots H_n, b$ are in H.P.

Relation between A.M., G.M. and H.M.

Example 4.10: If a, b are two different positive numbers then prove that

(i) A.M., G.M., H.M. are in G.P. (ii) A.M > G.M > H.M

Proof:

$$\text{A.M.} = \frac{a+b}{2} \quad ; \quad \text{G.M.} = \sqrt{ab} \quad ; \quad \text{H.M.} = \frac{2ab}{a+b}$$

(i)
$$\frac{\text{G.M}}{\text{A.M}} = \frac{\sqrt{ab}}{\frac{a+b}{2}} = \frac{2\sqrt{ab}}{a+b} \qquad \ldots (1)$$

$$\frac{\text{H.M}}{\text{G.M}} = \frac{\frac{2ab}{a+b}}{\sqrt{ab}} = \frac{2\sqrt{ab}}{a+b} \qquad \ldots (2)$$

From (1) and (2)

$$\frac{\text{G.M}}{\text{A.M}} = \frac{\text{H.M}}{\text{G.M}}$$

$\therefore$ A.M, G.M, H.M are in G.P

(ii)
$$\text{A.M} - \text{G.M} = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} > 0 \qquad \because a > 0 \,;\, b > 0 \,;\, a \neq b$$

$$\text{A.M} > \text{G.M} \qquad \ldots (1)$$

$$\text{G.M} - \text{H.M} = \sqrt{ab} - \frac{2ab}{a+b}$$

$$= \frac{\sqrt{ab}\,(a+b) - 2ab}{a+b} = \frac{\sqrt{ab}\left[a+b-2\sqrt{ab}\right]}{a+b}$$

$$= \frac{\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)^2}{a+b} > 0$$

$$\therefore \quad \text{G.M} > \text{H.M} \qquad \ldots (2)$$

From (1) and (2) A.M. > G.M > H.M

EXERCISE 4.2

(1) (i) Find five arithmetic means between 1 and 19

(ii) Find six arithmetic means between 3 and 17

(2) Find the single A.M between

(i) 7 and 13 (ii) 5 and – 3 (iii) $(p + q)$ and $(p - q)$

(3) If b is the G.M of a and c and x is the A.M of a and b and y is the A.M of b and c, prove that $\frac{a}{x} + \frac{c}{y} = 2$

(4) The first and second terms of a H.P are $\frac{1}{3}$ and $\frac{1}{5}$ respectively, find the 9^{th} term.

(5) If a, b, c are in H.P., prove that $\frac{b + a}{b - a} + \frac{b + c}{b - c} = 2$

(6) The difference between two positive numbers is 18, and 4 times their G.M is equal to 5 times their H.M. Find the numbers.

(7) If the A.M between two numbers is 1, prove that their H.M is the square of their G.M.

(8) If a, b, c are in A.P. and a, mb, c are in G.P then prove that a, m^2b, c are in H.P

(9) If the p^{th} and q^{th} terms of a H.P are q and p respectively, show that $(pq)^{th}$ term is 1.

(10) Three numbers form a H.P. The sum of the numbers is 11 and the sum of the reciprocals is one. Find the numbers.

4.6 Some special types of series

4.6.1 Binomial series

Binomial Theorem for a Rational Index:

In the previous chapter we have already seen the Binomial expansion for a positive integral index n. (power is a positive integer)

$$(x + a)^n = x^n + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \ldots + nC_r x^{n-r} a^r + \ldots + nC_n a^n$$

A particular form is

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots + x^n$$

When n is a positive integer the number of terms in the expansion is $(n+1)$ and so the series is a finite series. But when it is not a positive integer, the series does not terminate and it is an infinite series.

Theorem (without proof)

For any rational number n other than positive integer

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \dots \dots$$

provided $| x | < |$.

Here we require the condition that $| x |$ should be less than 1.

To see this, put $x = 1$ and $n = -1$ in the above formula for $(1 + x)^n$

The left side of the formula $= (1 + 1)^{-1} = \frac{1}{2}$,

while the right side $= 1 + (-1)(1) + \frac{(-1)(-2)}{2} 1^2 + \dots$

$= 1 - 1 + 1 - 1 + \dots$

Thus the two sides are not equal. This is because, $x = 1$ doesn't satisfy $| x | < 1$.

This extra condition $| x | < 1$ is unnecessary, if n is a positive integer.

Differences between the Binomial theorem for a positive integral index and for a rational index:

1. If $n \in N$, then $(1 + x)^n$ is defined for all values of x and if n is a rational number other than the natural number, then $(1 + x)^n$ is defined only when $| x | < |$.
2. If $n \in N$, then the expansion of $(1 + x)^n$ contains only $n + 1$ terms. If n is a rational number other than natural number, then the expansion of $(1 + x)^n$ contains infinitely many terms.

Some particular expansions

We know that , when n is a rational index,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots \qquad (1)$$

Replacing x by $-x$, we get

$$(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \dots \qquad (2)$$

Replacing n by $-n$ in (1) we get

$$(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots \qquad (3)$$

Replacing x by $-x$ in (3), we get

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots \qquad (4)$$

Note :

(1) If the exponent is negative then the value of the factors in the numerators are increasing uniformly by 1

(2) If the exponent is positive then the value of the factors in the numerators are decreasing uniformly by 1

(3) If the signs of x and n are same then all the terms in the expansion are positive.

(4) If the signs of x and n are different, then the terms alternate in sign

Special cases

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

General term:

For a rational number n and $|x| < 1$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots$$

In this expansion

$$\text{First term } T_1 = T_{0+1} = 1$$

$$\text{Second term } T_2 = T_{1+1} = nx = \frac{n}{1}x^1$$

$$\text{Third term } T_3 = T_{2+1} = \frac{n(n-1)}{1.2}x^2$$

$$\text{Fourth term } T_4 = T_{3+1} = \frac{n(n-1)(n-2)}{1.2.3}x^3 \text{ etc.}$$

$$(r+1)^{\text{th}} \text{ term : } T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{1.2.3\ldots r}x^r$$

The general term is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots r \text{ factors}}{r!}x^r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

Example 4.11: Write the first four terms in the expansions of

(i) $(1+4x)^{-5}$ where $|x| < \frac{1}{4}$ (ii) $(1-x^2)^{-4}$ where $|x| < |$

Solution: (i) $|4x| = 4|x| < 4\left(\frac{1}{4}\right) = 1 \quad \therefore \quad |4x| < 1$

$\therefore (1 + 4x)^{-5}$ can be expanded by Binomial theorem.

$$(1 + 4x)^{-5} = 1 - (5)(4x) + \frac{(5)(5+1)}{1.2}(4x)^2 - \frac{(5)(5+1)(5+2)}{1.2.3}(4x)^3 + \ldots$$

$$= 1 - 20x + 15(16x^2) - 35(64x^3) + \ldots$$

$$= 1 - 20x + 240x^2 - 2240x^3 + \ldots$$

(ii) $(1 - x^2)^{-4}$ can be expanded by Binomial theorem since $|x^2| < 1$

$$= 1 + (4)(x^2) + \frac{(4)(4+1)}{1.2}(x^2)^2 + \frac{(4)(4+1)(4+2)}{1.2.3}(x^2)^3 + \ldots$$

$$= 1 + 4x^2 + 10x^4 + 20x^6 + \ldots$$

Example 4.12: Find the expansion of $\frac{1}{(2+x)^4}$ where $|x| < 2$ upto the fourth term.

Solution:

$$\frac{1}{(2+x)^4} = (2+x)^{-4} = 2^{-4}\left(1 + \frac{x}{2}\right)^{-4} \qquad |x| < 2 \Rightarrow \left|\frac{x}{2}\right| < 1$$

$$= \frac{1}{16}\left[1 - (4)\left(\frac{x}{2}\right) + \frac{(4)(4+1)}{1.2}\left(\frac{x}{2}\right)^2 - \frac{(4)(4+1)(4+2)}{1.2.3}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= \frac{1}{16}\left[1 - 2x + \frac{(4)(5)}{2}\left(\frac{x^2}{4}\right) - \frac{(4)(5)(6)}{1.2.3}\frac{x^3}{8} + \ldots\right]$$

$$= \frac{1}{16} - \frac{x}{8} + \frac{5}{32}x^2 - \frac{5}{32}x^3 + \ldots$$

Example 4.13: Show that $(1+x)^n = 2^n\left[1 - n\left(\frac{1-x}{1+x}\right) + n\left(\frac{n+1}{2!}\right)\left(\frac{1-x}{1+x}\right)^2 + \ldots\right]$

Solution: Let $y = \frac{1-x}{1+x}$

$$\text{R.H.S} = 2^n\left[1 - ny + \frac{n(n+1)}{2!}y^2 + \ldots\right] = 2^n[1+y]^{-n}$$

$$= 2^n\left[1 + \frac{1-x}{1+x}\right]^{-n} = 2^n\left[\frac{1+x+1-x}{1+x}\right]^{-n}$$

$$= 2^n\left[\frac{2}{1+x}\right]^{-n} = 2^n\left[\frac{1+x}{2}\right]^n = (1+x)^n = \text{L.H.S.}$$

Approximation by using Binomial series

Example 4.14: Find the value of $\sqrt[3]{126}$ correct to two decimal places.
Solution:

$$\sqrt[3]{126} = (126)^{\frac{1}{3}} = (125+1)^{\frac{1}{3}}$$

$$= \left[125\left(1+\frac{1}{125}\right)\right]^{\frac{1}{3}} = (125)^{\frac{1}{3}}\left(1+\frac{1}{125}\right)^{\frac{1}{3}}$$

$$= 5\left[1+\frac{1}{3}\cdot\frac{1}{125}+\ldots\right] \qquad \because \frac{1}{125} < 1$$

$$= 5\left[1+\frac{1}{3}(0.008)\right] \text{ by neglecting other terms}$$

$$= 5[1+0.002666]$$

$$= 5.01 \text{ (correct to 2 decimal places)}$$

Example 4.15: If x is large and positive show that $\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3} = \frac{1}{x^2}$ (app.)

Solution: Since x is large, $\frac{1}{x}$ is small and hence $\left|\frac{1}{x}\right| < 1$

$$\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3} = (x^3+6)^{\frac{1}{3}} - (x^3+3)^{\frac{1}{3}} = x\left(1+\frac{6}{x^3}\right)^{\frac{1}{3}} - x\left(1+\frac{3}{x^3}\right)^{\frac{1}{3}}$$

$$= x\left[1+\frac{1}{3}\cdot\frac{6}{x^3}+\ldots\right] - x\left[1+\frac{1}{3}\cdot\frac{3}{x^3}+\ldots\right]$$

$$= \left[x+\frac{2}{x^2}+\ldots\right] - \left[x+\frac{1}{x^2}+\ldots\right] = \frac{2}{x^2} - \frac{1}{x^2} + \ldots$$

$$= \frac{1}{x^2} \quad \text{(approximately)}$$

Example 4.16: In the expansion $(1-2x)^{-\frac{1}{2}}$, find the coefficient of x^8.
Solution: We know that

$$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\ldots+\frac{n(n+1)\ldots(n+r-1)}{r!}x^r+\ldots$$

General term $\quad T_{r+1} = \frac{n(n+1)\ldots(n+r-1)}{r!}\, x^r$

Take $\quad n = \frac{1}{2}$ and replace x by $2x$.

$$T_{r+1} = \frac{\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(\frac{2r-1}{2}\right)}{r!}(2x)^r = \frac{1.3.5\ldots(2r-1)}{r!\;2^r}\,2^r x^r$$

$$\therefore \text{ coefficient of } x^r = \frac{1.3.5\ldots(2r-1)}{r!}$$

$$\text{put } r = 8$$

$$\therefore \text{ coefficient of } x^8 = \frac{1.3.5.7.9.11.13.15}{8!}$$

4.6.2. Exponential series

Exponential theorem (without proof)

For all real values of x,

$$\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots\right)^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

But $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$

$\therefore$ For all real values of x, $\quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Thus we have the following results:

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots$$

$$\frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots$$

4.6.3 Logarithmic Series:

If $-1 < x \leq 1$ then $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

This series is called the logarithmic series.

The other forms of logarithmic series are as follows:

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

$$\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

EXERCISE 4.3

(1) Write the first four terms in the expansions of the following:

(i) $\dfrac{1}{(2+x)^4}$ where $|x| > 2$ (ii) $\dfrac{1}{\sqrt[3]{6-3x}}$ where $|x| < 2$

(2) Evaluate the following:

(i) $\sqrt[3]{1003}$ correct to 2 places of decimals

(ii) $\dfrac{1}{\sqrt[3]{128}}$ correct to 2 places of decimals

(3) If x is so small show that $\sqrt{\dfrac{1-x}{1+x}} = 1 - x + \dfrac{x^2}{2}$ (app.)

(4) If x is so large prove that $\sqrt{x^2+25} - \sqrt{x^2+9} = \dfrac{8}{x}$ nearly.

(5) Find the 5th term in the expansion of $(1-2x^3)^{\frac{11}{2}}$

(6) Find the $(r+1)$th term in the expansion of $(1-x)^{-4}$

(7) Show that $x^n = 1 + n\left(1 - \dfrac{1}{x}\right) + \dfrac{n(n+1)}{1.2}\left(1 - \dfrac{1}{x}\right)^2 + \dots$

5. ANALYTICAL GEOMETRY

Introduction

'Geometry' is the study of points, lines, curves, surfaces etc and their properties. Geometry is based upon axioms and it was laid by the famous Greek Mathematician Euclid about 300 B.C. In the 17th century A.D., the methods of Algebra were applied in the study of Geometry and thereby 'Analytical Geometry' emerged out. The renowned French philosopher and Mathematician Rene Descartes (1596 – 1650) showed how the methods of Algebra could be applied to the study of Geometry. He thus became the founder of Analytical Geometry (also called as Cartesian Geometry, from the latinized form of his name Cartesius). To bring a relationship between Algebra and Geometry, Descartes introduces basic algebraic entity 'number' to the basic geometric concept of 'point'. This relationship is called 'system of coordinates'. Descartes relates the position of a point with its distance from fixed lines and its direction. This chapter is a continuation of the study of the concepts of Analytical Geometry to which the students had been introduced in earlier classes.

5.1 Locus

The path traced by a point when it moves according to specified geometrical conditions is called the **locus** of the point. For example, the locus of a point $P(x_1, y_1)$ whose distance from a fixed point C (h, k) is constant 'a', is a circle (fig. 5.1). The fixed point 'C' is called the centre and the fixed distance 'a' is called the radius of the circle.

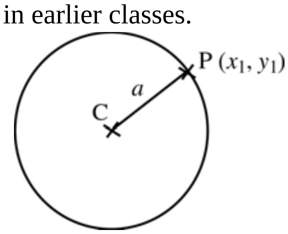


Fig. 5. 1

Example 5.1: A point in the plane moves so that its distance from (0, 1) is twice its distance from the x-axis. Find its locus.

Solution:

Let A(0, 1) be the given point. Let $P(x_1, y_1)$ be any point on the locus. Let B be the foot of the perpendicular from $P(x_1,y_1)$ to the x-axis. Thus PB = y_1.

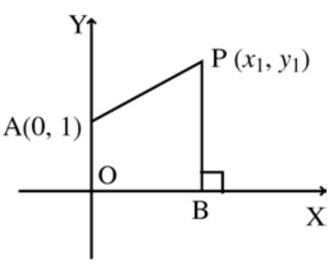


Fig. 5. 2

Given that PA = 2PB

$$\therefore\ PA^2 = 4PB^2$$

i.e. $\quad (x_1 - 0)^2 + (y_1 - 1)^2 = 4y_1^2$

i.e. $$x_1^2 + y_1^2 - 2y_1 + 1 = 4y_1^2$$

i.e. $$x_1^2 - 3y_1^2 - 2y_1 + 1 = 0$$

$\therefore$ The locus of (x_1, y_1) is $x^2 - 3y^2 - 2y + 1 = 0$

Example 5.2: Find the locus of the point which is equidistant from $(-1, 1)$ and $(4, -2)$.

Solution:

Let $A(-1, 1)$ and $B(4, -2)$ be the given points.

Let $P(x_1,y_1)$ be any point on the locus. Given that PA = PB

$$\therefore\ PA^2 = PB^2$$

i.e. $$(x_1 + 1)^2 + (y_1 - 1)^2 = (x_1 - 4)^2 + (y_1 + 2)^2$$

i.e. $$x_1^2 + 2x_1 + 1 + y_1^2 - 2y_1 + 1 = x_1^2 - 8x_1 + 16 + y_1^2 + 4y_1 + 4$$

i.e. $$10x_1 - 6y_1 - 18 = 0 \quad \text{i.e. } 5x_1 - 3y_1 - 9 = 0$$

$\therefore$ The locus of the point (x_1, y_1) is $5x - 3y - 9 = 0$

Example 5.3: If A and B are the two points $(-2, 3)$ and $(4, -5)$, find the equation of the locus of a point such that $PA^2 - PB^2 = 20$.

Solution:

$A(-2, 3)$ and $B(4, -5)$ are the two given points. Let $P(x_1, y_1)$ be any point on the locus. Given that $PA^2 - PB^2 = 20$.

$$(x_1 + 2)^2 + (y_1 - 3)^2 - \left[(x_1 - 4)^2 + (y_1 + 5)^2\right] = 20$$

$$x_1^2 + 4x_1 + 4 + y_1^2 - 6y_1 + 9 - [x_1^2 - 8x_1 + 16 + y_1^2 + 10y_1 + 25] = 20$$

$$12x_1 - 16y_1 - 48 = 0$$

i.e. $$3x_1 - 4y_1 - 12 = 0$$

The locus of (x_1,y_1) is $3x - 4y - 12 = 0$

Example 5.4: Find a point on x-axis which is equidistant from the points $(7, -6)$ and $(3, 4)$.

Solution:

Let $P(x_1, y_1)$ be the required point. Since P lies on x-axis, $y_1 = 0$. Given that $A(7, -6)$ and $B(3, 4)$ are equidistant from P.

i.e. $$PA = PB \Rightarrow PA^2 = PB^2$$

$$\Rightarrow \quad (x_1 - 7)^2 + (0 + 6)^2 = (x_1 - 3)^2 + (0 - 4)^2$$

$\Rightarrow \qquad x_1^2 - 14x_1 + 49 + 36 = x_1^2 - 6x_1 + 9 + 16$

$\Rightarrow \qquad 8x_1 = 60 \quad \therefore\ x_1 = 15/2$

Thus the required point is $\left(\frac{15}{2}, 0\right)$

EXERCISE 5.1

(1) A point moves so that it is always at a distance of 6 units from the point (1, – 4). Find its locus.

(2) Find the equation of the locus of the point which are equidistant from (1, 4) and (– 2, 3).

(3) If the point P($5t - 4$, $t + 1$) lies on the line $7x - 4y + 1 = 0$, find
(i) the value of t (ii) the co-ordinates of P.

(4) The distance of a point from the origin is five times its distance from the y-axis. Find the equation of the locus.

(5) Show that the equation of the locus of a point which moves such that its distance from the points (1, 2) and (0, – 1) are in the ratio 2 : 1 is $3x^2 + 3y^2 + 2x + 12y - 1 = 0$.

(6) A point P moves such that P and the points (2, 3), (1, 5) are always collinear. Show that the equation of the locus of P is $2x + y - 7 = 0$.

(7) A and B are two points (1, 0) and (– 2, 3). Find the equation of the locus of a point such that (i) $PA^2 + PB^2 = 10$ (ii) PA = 4PB.

5.2 Straight lines

5.2.1 Introduction

A straight line is the simplest geometrical curve. Every straight line is associated with an equation. To determine the equation of a straight line, two conditions are required. We have derived the equation of a straight line in different forms in the earlier classes. They are

(1) Slope-intercept form:

i.e. $y = mx + c$ where 'm' is the slope of the straight line and 'c' is the y intercept.

(2) Point-slope form:

i.e. $y - y_1 = m(x - x_1)$ where 'm' is the slope and (x_1, y_1) is the given point.

(3) Two point form:

i.e $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are the two given points.

(4) Intercept form:

i.e. $\frac{x}{a} + \frac{y}{b} = 1$ where 'a' and 'b' are x and y intercepts respectively.

In this section we shall derive and discuss other forms of equation of a straight line.

5.2.2 Normal form:

Equation of a straight line in terms of the length of the perpendicular p from the origin to the line and the angle α which the perpendicular makes with x-axis.

Let R and N be the points where the straight line cuts the x and y axes respectively.

Draw the perpendicular OL to RN.

Let OL = p and $\angle XOL = \alpha$.

Now OR and ON are the x and y intercepts respectively.

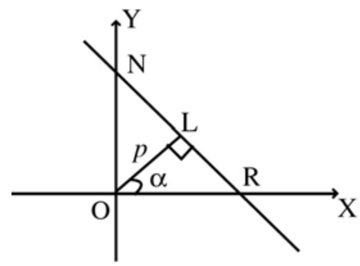


Fig. 5. 3

The equation of the straight line is $\frac{x}{OR} + \frac{y}{ON} = 1$ …(1)

From the right angled triangle OLR, $\sec\alpha = \frac{OR}{OL}$ $\quad\therefore$ OR = $p\sec\alpha$

From the right angled triangle OLN, $\operatorname{cosec}\alpha = \sec(90-\alpha) = \frac{ON}{OL}$

$\therefore$ ON = $p \operatorname{cosec}\alpha$

Substituting the values of OR and ON in equation (1),

we get, $\frac{x}{p\sec\alpha} + \frac{y}{p\operatorname{cosec}\alpha} = 1$ $\quad$ i.e. $\frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$

i.e. $x\cos\alpha + y\sin\alpha = p$ is the required equation of the straight line.

5.2.3 Parametric form

Definition: If two variables, say x and y, are functions of a third variable, say 'θ', then the functions expressing x and y in terms of θ are called the parametric representations of x and y. The variable θ is called the parameter of the function.

Equation of a straight line passing through the point (x_1, y_1) and making an angle θ with x-axis. (parametric form)

Let Q (x_1, y_1) be the given point and P(x, y) be any point on the required straight line. Assume that PQ = r.

It is given that

$\angle PTR = \theta$. But $\angle PQM = \angle PTR$

$\therefore \angle PQM = \theta$

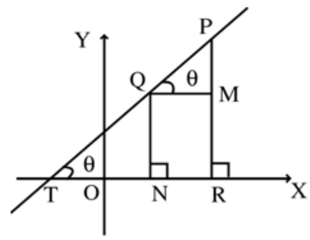


Fig. 5. 4

In the right angled triangle PQM,

… (1)

$$\cos\theta = \frac{QM}{PQ} = \frac{NR}{r} = \frac{OR - ON}{r} = \frac{x - x_1}{r}$$

$$\therefore \frac{x - x_1}{\cos\theta} = r$$

$$\text{Similarly } \sin\theta = \frac{PM}{PQ} = \frac{PR - MR}{r} = \frac{y - y_1}{r}$$

$$\therefore \frac{y - y_1}{\sin\theta} = r \qquad \dots (2)$$

From (1) and (2), $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$ which is the required equation.

Any point on this line can be taken as $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ where r is the algebraic distance. Here r is the parameter.

5.2.4 General form

The equation $ax + by + c = 0$ will always represent a straight line.

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be any three points on the locus represented by the equation $ax + by + c = 0$. Then

$$ax_1 + by_1 + c = 0 \qquad \dots (1)$$

$$ax_2 + by_2 + c = 0 \qquad \dots (2)$$

$$ax_3 + by_3 + c = 0 \qquad \dots (3)$$

(1) × $(y_2 - y_3)$ + (2) × $(y_3 - y_1)$ + (3) × $(y_1 - y_2)$ gives

$a\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] = 0$

Since $a \neq 0$, $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

That is (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear and hence they lie on a straight line.

Thus the equation $ax + by + c = 0$ represents a straight line.

5.2.5. Perpendicular distance from a point to a straight line

The length of the perpendicular from the point (x_1, y_1) to the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

Let the given line $ax + by + c = 0$ … (1) be represented by AB.

Let $P(x_1, y_1)$ be the given point.

Draw PD perpendicular to AB. Note that PD is the required distance.

Draw OM parallel to PD. Let OM = p

Assume that $\angle MOB = \alpha$.

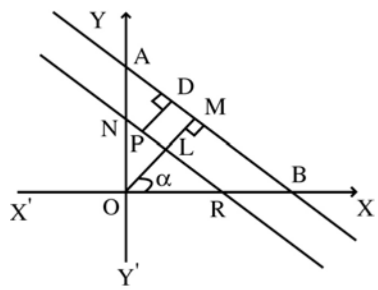


Fig. 5. 5

From 5.2.2, the equation of the straight line AB is

$x \cos\alpha + y \sin\alpha - p = 0$ … (2)

Now equations (1) and (2) are representing the same straight line. Hence their corresponding coefficients are proportional.

$$\therefore \frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{-p}{c}$$

$$\cos\alpha = -\frac{ap}{c}, \quad \sin\alpha = -\frac{pb}{c}$$

We know that $\sin^2\alpha + \cos^2\alpha = 1$

$$\frac{p^2b^2}{c^2} + \frac{p^2a^2}{c^2} = 1 \quad \text{i.e.} \quad p^2a^2 + p^2b^2 = c^2$$

$$p^2(a^2 + b^2) = c^2 \quad \text{i.e} \quad p^2 = \frac{c^2}{a^2 + b^2}$$

$$p = \pm \frac{c}{\sqrt{a^2 + b^2}}$$

Hence $\cos\alpha = \mp \dfrac{a}{\sqrt{a^2 + b^2}}$, $\sin\alpha = \mp \dfrac{b}{\sqrt{a^2 + b^2}}$

Suppose OL = p', the equation of the straight line NR is

$x \cos\alpha + y \sin\alpha - p' = 0$

since P(x_1, y_1) is a point on NR

$x_1 \cos\alpha + y_1 \sin\alpha - p' = 0$

i.e. OL = $p' = x_1 \cos\alpha + y_1 \sin\alpha$

From the figure, the required distance

$$\begin{aligned} PD &= LM = OM - OL = p - p' \\ &= p - x_1 \cos\alpha - y_1 \sin\alpha \\ &= \pm \frac{c}{\sqrt{a^2+b^2}} \pm \frac{x_1 \,.\, a}{\sqrt{a^2+b^2}} \pm \frac{y_1 \,.\, b}{\sqrt{a^2+b^2}} = \pm \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}} \end{aligned}$$

The required distance $= \left| \dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$

Corollary:

The length of the perpendicular from the origin to $ax + by + c = 0$ is $\left| \dfrac{c}{\sqrt{a^2+b^2}} \right|$

Note: The general equation of the straight line is $ax+by+c = 0$ i.e. $y = -\dfrac{a}{b}x - \dfrac{c}{a}$

This is of the form $y = mx + c$.

$\therefore\ m = -\dfrac{a}{b}$ i.e. slope $= -\dfrac{\text{co-efficient of } x}{\text{co-efficient of } y}$

Example 5.5: Determine the equation of the straight line whose slope is 2 and y-intercept is 7.

Solution:

The slope – intercept form is $y = mx + c$ Here $m = 2$, $c = 7$

$\therefore$ The required equation of the straight line is $y = 2x + 7$

Example 5.6: Determine the equation of the straight line passing through (– 1, 2) and having slope $\dfrac{2}{7}$

Solution:

The point-slope form is $y - y_1 = m(x - x_1)$.

$$\text{Here} \quad (x_1, y_1) = (-1, 2) \text{ and } m = \frac{2}{7}$$

$$\therefore \quad y - 2 = \frac{2}{7}(x+1) \quad \text{i.e.} \quad 7y - 14 = 2x + 2$$

$2x - 7y + 16 = 0$ is the equation of the straight line.

Example 5.7:

Determine the equation of the straight line passing through the points (1, 2) and (3, – 4).

Solution:

The equation of a straight line passing through two points is $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (3, -4)$.

Substituting the above, the required line is $\dfrac{y - 2}{2 + 4} = \dfrac{x - 1}{1 - 3}$

$\Rightarrow \quad \dfrac{y - 2}{6} = \dfrac{x - 1}{-2} \quad \Rightarrow \quad \dfrac{y - 2}{3} = \dfrac{x - 1}{-1}$

$\Rightarrow \quad y - 2 = -3(x - 1) \Rightarrow y - 2 = -3x + 3$

$\Rightarrow \quad 3x + y = 5$ is the required equation of the straight line.

Example 5.8: Find the equation of the straight line passing through the point (1, 2) and making intercepts on the co-ordinate axes which are in the ratio 2 : 3.

Solution:

The intercept form is $\dfrac{x}{a} + \dfrac{y}{b} = 1 \quad \ldots (1)$

The intercepts are in the ratio 2 : 3 $\quad \therefore\ a = 2k, \quad b = 3k$.

(1) becomes $\dfrac{x}{2k} + \dfrac{y}{3k} = 1$ i.e. $3x + 2y = 6k$

Since (1, 2) lies on the above straight line, $3 + 4 = 6k$ i.e. $6k = 7$

Hence the required equation of the straight line is $3x + 2y = 7$

Example 5.9: Find the length of the perpendicular from (2, – 3) to the line $2x - y + 9 = 0$

Solution:

The perpendicular distance from (x_1, y_1) to the straight line $ax + by + c = 0$ is given by $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

$\therefore$ The length of the perpendicular from (2, –3) to the straight line $2x - y + 9 = 0$ is $\left|\dfrac{2(2) - (-3) + 9}{\sqrt{(2)^2 + (-1)^2}}\right| = \dfrac{16}{\sqrt{5}}$ units.

Example 5.10: Find the co-ordinates of the points on the straight line $y = x + 1$ which are at a distance of 5 units from the straight line $4x - 3y + 20 = 0$

Solution: Let (x_1, y_1) be a point on $y = x + 1$

$\therefore y_1 = x_1 + 1$ … (1)

The length of the perpendicular from (x_1, y_1) to the straight line

$4x - 3y + 20 = 0$ is $\left|\dfrac{4x_1 - 3y_1 + 20}{\sqrt{4^2 + (-3)^2}}\right| = \pm\left(\dfrac{4x_1 - 3y_1 + 20}{5}\right)$

But the length of the perpendicular is given as 5.

$$\therefore \pm\left(\frac{4x_1 - 3y_1 + 20}{5}\right) = 5$$

$$\therefore 4x_1 - 3y_1 + 20 = \pm 25$$

Considering the positive sign, $4x_1 - 3y_1 + 20 = 25$

$\Rightarrow$ $4x_1 - 3y_1 = 5$ … (2)

Considering the negative sign, $4x_1 - 3y_1 + 20 = -25$

$\Rightarrow$ $4x_1 - 3y_1 = -45$ … (3)

Solving (1) and (2), we get $x_1 = 8$, $y_1 = 9$

Solving (1) and (3), we get $x_1 = -42$, $y_1 = -41$.

$\therefore$ The co-ordinates of the required points are (8, 9) and (– 42, – 41).

Example 5.11: Find the equation of the straight line, if the perpendicular from the origin makes an angle of 120° with x-axis and the length of the perpendicular from the origin is 6 units.

Solution:

The normal form of a straight line is $x \cos\alpha + y \sin\alpha = p$

Here $\alpha = 120°$, $p = 6$ $\quad\therefore x \cos 120° + y \sin 120° = 6$

$$\Rightarrow \quad x\left(-\frac{1}{2}\right) + y\left(\frac{\sqrt{3}}{2}\right) = 6 \Rightarrow -x + \sqrt{3}\,y = 12$$

$$\Rightarrow \quad x - \sqrt{3}\,y + 12 = 0$$

$\therefore$ The required equation of the straight line is $x - \sqrt{3}\,y + 12 = 0$

Example 5.12: Find the points on y-axis whose perpendicular distance from the straight line $4x - 3y - 12 = 0$ is 3.

Solution:

Any point on y-axis will have x co-ordinate as 0.

Let the point on y-axis be $P(0, y_1)$.

The given straight line is $4x - 3y - 12 = 0$ … (1)

The perpendicular distance from the point P to the given straight line is

$$\left|\frac{-3y_1 - 12}{\sqrt{4^2 + (-3)^2}}\right| = \left|\frac{3y_1 + 12}{5}\right|$$

But the perpendicular distance is 3.

$$\text{i.e.}\quad \left|\frac{3y_1 + 12}{5}\right| = 3 \quad \Rightarrow \quad 3y_1 + 12 = \pm 15$$

$$\begin{aligned} 3y_1 + 12 &= 15 \quad \text{or} \quad 3y_1 + 12 = -15 \\ 3y_1 &= 3 \quad \text{or} \quad 3y_1 = -27 \\ y_1 &= 1 \quad \text{or} \quad y_1 = -9 \end{aligned}$$

Thus the required points are (0, 1) and (0, – 9).

EXERCISE 5.2

(1) Determine the equation of the straight line passing through the point (– 1, – 2) and having slope $\frac{4}{7}$

(2) Determine the equation of the line with slope 3 and y-intercept 4.

(3) A straight line makes an angle of 45° with x-axis and passes through the point (3, – 3). Find its equation.

(4) Find the equation of the straight line joining the points (3, 6) and (2, – 5).

(5) Find the equation of the straight line passing through the point (2, 2) and having intercepts whose sum is 9.

(6) Find the equation of the straight line whose intercept on the x-axis is 3 times its intercept on the y-axis and which passes through the point (– 1, 3).

(7) Find the equations of the medians of the triangle formed by the points (2, 4), (4, 6) and (– 6, – 10).

(8) Find the length of the perpendicular from (3, 2) to the straight line $3x + 2y + 1=0$.

(9) The portion of a straight line between the axes is bisected at the point (– 3, 2). Find its equation.

(10) Find the equation of the diagonals of a quadrilateral whose vertices are (1, 2), (– 2, – 1), (3, 6) and (6, 8).

(11) Find the equation of the straight line, which cut off intercepts on the axes whose sum and product are 1 and – 6 respectively.

(12) Find the intercepts made by the line $7x + 3y - 6 = 0$ on the co-ordinate axis.

(13) What are the points on x-axis whose perpendicular distance from the straight line $\frac{x}{3} + \frac{y}{4} = 1$ is 4?

(14) Find the distance of the line $4x - y = 0$ from the point (4, 1) measured along the straight line making an angle of 135° with the positive direction of the x-axis.

5.3. Family of straight lines

In the previous section, we studied about a single straight line. In this section we will discuss the profile about more than one straight line, which lie on a plane.

5.3.1 Angle between two straight lines

Let $l_1 : y = m_1x + c_1$ and

$l_2 : y = m_2x + c_2$ be the two intersecting lines and assume that P be the point of intersection of the two straight lines which makes angle θ_1 and θ_2 with the positive direction of x-axis. Then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. Let θ be the angle between the two straight lines.

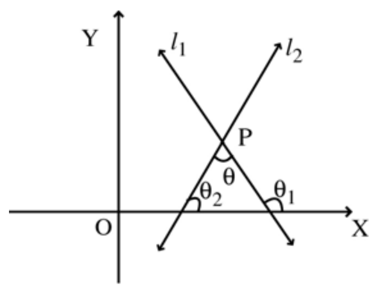


Fig. 5. 6

From the figure (5.6), $\theta_1 = \theta + \theta_2$

$$\therefore\ \theta = \theta_1 - \theta_2$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1.\tan\theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Note that $\frac{m_1 - m_2}{1 + m_1 m_2}$ is either positive or negative. As convention we consider the acute angle as the angle between any two straight lines and hence we consider only the positive value (absolute value) of $\tan\theta$.

Hence $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \therefore\ \theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Corollary (1) : If the two straight lines are parallel, then their slopes are equal.

Proof:

Since the two straight lines are parallel, $\theta = 0$. $\therefore$ $\tan\theta = 0$

$$\Rightarrow \quad \frac{m_1 - m_2}{1 + m_1 m_2} = 0 \quad \Rightarrow m_1 - m_2 = 0$$

$$\text{i.e.} \quad m_1 = m_2$$

$\therefore$ If the straight lines are parallel, then the slopes are equal.

Note : If the slopes are equal, then the straight lines are parallel.

Corollary (2): If the two straight lines are perpendicular then the product of their slopes is –1.

Proof:

Since the two straight lines are perpendicular, $\theta = 90°$.

$$\therefore \tan\theta = \tan 90° = \infty \quad \Rightarrow \frac{m_1 - m_2}{1 + m_1 m_2} = \infty$$

This is possible only if the denominator is zero.

i.e. $1 + m_1 m_2 = 0$ i.e. $m_1 m_2 = -1$

$\therefore$ If the two straight lines are perpendicular then the product of their slopes is – 1.

Note (1): If the product of the slopes is – 1, then the straight lines are perpendicular.

(2): Corollary (2) is applicable only if both the slopes m_1 and m_2 are finite. It fails when the straight lines are co-ordinate axes or parallel to axes.

Corollary (3): If the straight lines are parallel, then the coefficients of x and y are proportional in their equations. In particular, the equations of two parallel straight lines differ only by the constant term.

Proof:

Let the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be parallel.

Slope of $a_1x + b_1y + c_1 = 0$ is $m_1 = -\frac{a_1}{b_1}$; Slope of $a_2x + b_2y + c_2 = 0$ is $m_2 = -\frac{a_2}{b_2}$

Since the straight lines are parallel, $m_1 = m_2$.

$$\text{i.e.} \quad -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \quad \Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

i.e. coefficients of x and y are proportional

Let $\qquad \dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = \lambda$ (say)

$\therefore\ a_2 = a_1\lambda,\ \ b_2 = b_1\lambda$

The second equation $a_2x + b_2y + c_2 = 0$ can be written as

$$\lambda a_1 x + \lambda\, b_1 y + c_2 = 0$$

i.e. $\qquad a_1x + b_1y + \dfrac{c_2}{\lambda} = 0$ i.e. $a_1x + b_1y + k = 0$ where $k = \dfrac{c_2}{\lambda}$

i.e. If $a_1x + b_1y + c_1 = 0$ is a straight line then a line parallel to it is $a_1x + b_1y + k = 0$

$\therefore$ Equations of parallel straight lines differ by the constant term.

Note (1): In the previous section, we established a formula to find the distance between the origin and the straight line. i.e. distance = $\left|\dfrac{c}{\sqrt{a^2+b^2}}\right|$

We can find out the distance between two parallel straight lines

$ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ by using the formula $d = \dfrac{|\,c_1 - c_2\,|}{\sqrt{a^2+b^2}}$.

This is obtained by using the above result. Note that, we took $|\,c_1 - c_2\,|$ since $c_2 > c_1$ or $c_1 > c_2$

Note (2): To apply the above formula, write the equations of the parallel straight lines in the standard form $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$.

Corollary (4): The equation of the straight line perpendicular to the straight line $ax + by + c = 0$ is of the form $bx - ay + k = 0$ for some k.

Proof:

Let the straight lines $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$ be perpendicular.

Slope of $ax + by + c = 0$ is $m_1 = -\dfrac{a}{b}$

Slope $a_1x + b_1y + c_1 = 0$ is $m_2 = -\dfrac{a_1}{b_1}$

Since the straight lines are perpendicular, $m_1\, m_2 = -1$

i.e. $\qquad \left(-\dfrac{a_1}{b_1}\right)\left(-\dfrac{a}{b}\right) = -1$ i.e. $aa_1 = -bb_1$

i.e. $\frac{a_1}{b} = -\frac{b_1}{a} = \lambda$ (say) $\therefore a_1 = b\lambda$ and $b_1 = -a\lambda$

The second equation $a_1x + b_1y + c_1 = 0$ can be written as $b\lambda x - a\lambda y + c_1 = 0$

i.e. $bx - ay + \frac{c_1}{\lambda} = 0$

i.e. $bx - ay + k = 0$ where $k = \frac{c_1}{\lambda}$

A straight line perpendicular to $ax + by + c = 0$ is given by $bx - ay + k = 0$ for some k.

Note: To find the point of intersection of two straight lines, solve the simultaneous equations of the straight lines.

5.3.3 The condition for the three straight lines to be concurrent

Let the three straight lines be given by

$$a_1x + b_1y + c_1 = 0 \quad \ldots (1)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots (2)$$

$$a_3x + b_3y + c_3 = 0 \quad \ldots (3)$$

If the three straight lines are concurrent, then the point of intersection of any two straight lines lies on the third straight line.

Solving the equation (1) and (2), the coordinates of the point of intersection is

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \; y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

substituting the values of x and y in the equation (3)

$$a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

i.e. $a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$

i.e. $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$

i.e. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ is the condition for the three straight lines to be concurrent.

5.3.4 Equation of a straight line passing through the intersection of the two given straight lines

Let $\quad a_1x + b_1y + c_1 = 0 \qquad \ldots (1)$

$\quad a_2x + b_2y + c_2 = 0 \qquad \ldots (2)$

be the equations of the two given straight lines.

Consider the equation $a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0 \qquad \ldots (3)$

where λ is a constant

Equation (3) is of degree one in x and y and therefore (refer 5.2.4) it represents a straight line. Let (x_1, y_1) be the point of intersection of (1) and (2)

$\therefore$ $a_1x_1 + b_1y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$

$\therefore$ $a_1x_1 + b_1y_1 + c_1 + \lambda (a_2x_1 + b_2y_1 + c_2) = 0$

$\therefore$ Value of (x_1, y_1) satisfies equation (3) also.

Hence $a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0$ represents a straight line passing through the intersection of the straight lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_2y + c_2 = 0$

Example 5.13: Find the angle between the straight lines $3x - 2y + 9 = 0$ and $2x + y - 9 = 0$.

Solution:

Slope of the straight line $3x - 2y + 9 = 0$ is $m_1 = \frac{3}{2}$ $\left[\because y = \frac{3}{2}x + \frac{9}{2}\right]$

Slope of the straight line $2x + y - 9 = 0$ is $m_2 = -2$ [$\because$ $y = -2x + 9$]

Suppose 'θ' is the angle between the given lines, then

$$\theta = \tan^{-1} \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \tan^{-1} \left|\frac{\frac{3}{2} + 2}{1 + \frac{3}{2}(-2)}\right| = \tan^{-1} \left|\frac{\frac{7}{2}}{\frac{2-6}{2}}\right|$$

$$= \tan^{-1} \left|-\frac{7}{4}\right| = \tan^{-1} \left(\frac{7}{4}\right)$$

Example 5.14: Show that the straight lines $2x + y - 9 = 0$ and $2x + y - 10 = 0$ are parallel.

Solution:

Slope of the straight line $2x + y - 9 = 0$ is $m_1 = -2$

Slope of the straight line $2x + y - 10 = 0$ is $m_2 = -2$ $\quad \therefore\ m_1 = m_2$

$\therefore$ The given straight lines are parallel.

Example 5.15: Show that the two straight lines whose equations are

$x + 2y + 5 = 0$ and $2x + 4y - 5 = 0$ are parallel.

Solution:

The two given equations are

$$x + 2y + 5 = 0 \qquad \ldots (1)$$

$$2x + 4y - 5 = 0 \qquad \ldots (2)$$

The coefficients of x and y are proportional since $\frac{1}{2} = \frac{2}{4}$ and therefore they are parallel.

Note : This can also be done by writing the equation(2) as $x + 2y - 5/2 = 0$

Now the two equations differ by constant alone. $\therefore$ They are parallel.

Example 5.16: Find the distance between the parallel lines $2x + 3y - 6=0$ and $2x + 3y + 7 = 0$.

Solution:

The distance between the parallel lines is $\left|\dfrac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

Here $c_1 = -6$, $c_2 = 7$, $a = 2$, $b = 3$

The required distance is $\left|\dfrac{-6-7}{\sqrt{2^2 + 3^2}}\right| = \left|\dfrac{-13}{\sqrt{13}}\right| = \sqrt{13}$ units.

Example 5.17: Show that the straight lines $2x + 3y - 9 = 0$ and $3x - 2y + 10 = 0$ are at right angles.

Solution:

Slope of the straight line $\quad 2x + 3y - 9 = 0$ is $m_1 = -\dfrac{2}{3}$

Slope of the straight line $\quad 3x - 2y + 10 = 0$ is $m_2 = \dfrac{3}{2}$

$$\therefore\ m_1 m_2 = -\frac{2}{3} \cdot \frac{3}{2} = -1$$

$\therefore$ The two straight lines are at right angles.

Example 5.18: Find the equation of the straight line parallel to $3x + 2y = 9$ and which passes through the point (3, – 3).

Solution:

The straight line parallel to $3x + 2y - 9 = 0$ is of the form

$$3x + 2y + k = 0 \qquad \ldots (1)$$

The point (3, – 3) satisfies the equation (1)

Hence $9 - 6 + k = 0$ i.e. $k = -3$

$\therefore$ $3x + 2y - 3 = 0$ is the equation of the required straight line.

Example 5. 19: Find the equation of the straight line perpendicular to the straight line $3x + 4y + 28 = 0$ and passing through the point (– 1, 4).

Solution:

The equation of any straight line perpendicular to $3x + 4y + 28 = 0$ is of the form

$$4x - 3y + k = 0$$

The point (– 1, 4) lies on the straight line $4x - 3y + k = 0$

$$\therefore\ -4 - 12 + k = 0 \quad \Rightarrow \quad k = 16$$

$\therefore$ The equation of the required straight line is $4x - 3y + 16 = 0$

Example 5. 20: Show that the triangle formed by straight lines

$4x - 3y - 18 = 0$, $3x - 4y + 16 = 0$ and $x + y - 2 = 0$ is isosceles.

Solution:

Slope of the straight line $4x - 3y - 18 = 0$ is $m_1 = \frac{4}{3}$

Slope of the straight line $3x - 4y + 16 = 0$ is $m_2 = \frac{3}{4}$

Slope of the straight line $x + y - 2 = 0$ is $m_3 = -1$

Let 'α' be the angle between the straight lines $4x - 3y - 18 = 0$ and $3x - 4y + 16 = 0$

Using the formula, $\theta = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$ we get

$$\alpha = \tan^{-1} \left| \frac{\frac{4}{3} - \frac{3}{4}}{1 + \frac{4}{3}\ \frac{3}{4}} \right| = \tan^{-1} \left| \frac{\frac{16 - 9}{12}}{2} \right|$$

$$= \tan^{-1} \left| \frac{7}{24} \right| = \tan^{-1} \left(\frac{7}{24} \right)$$

Let 'β' be the angle between the straight lines $3x - 4y + 16 = 0$ and $x + y - 2 = 0$

$$\therefore\ \beta = \tan^{-1}\left|\frac{\frac{3}{4}+1}{1+\frac{3}{4}(-1)}\right| = \tan^{-1}\left|\frac{7/4}{1/4}\right|$$

$$= \tan^{-1}(7)$$

Let 'γ' be the angle between the straight lines $x+y-2 = 0$ and $4x - 3y - 18 = 0$

$$\therefore\ \gamma = \tan^{-1}\left|\frac{-1-\frac{4}{3}}{1+(-1)\left(\frac{4}{3}\right)}\right| = \tan^{-1}\left|\frac{-\frac{7}{3}}{-\frac{1}{3}}\right|$$

$$= \tan^{-1}(7)$$

Therefore $\beta = \gamma$ ∴ The triangle is isosceles.

Example 5.21: Find the point of intersection of the straight lines

$$5x + 4y - 13 = 0 \text{ and } 3x + y - 5 = 0$$

Solution:

To find the point of intersection, solve the given equations.

Let (x_1, y_1) be the point of intersection. Then (x_1, y_1) lies on both the straight lines.

$$\therefore\quad 5x_1 + 4y_1 = 13 \qquad \ldots (1)$$

$$3x_1 + y_1 = 5 \qquad \ldots (2)$$

$(2) \times 4 \Rightarrow$ $$12x_1 + 4y_1 = 20 \qquad \ldots (3)$$

$(1) - (3) \Rightarrow$ $$-7x_1 = -7 \quad \therefore\ x_1 = 1$$

Substituting $x_1 = 1$ in equation (1), we get $5 + 4y_1 = 13$

$$4y_1 = 8 \qquad \therefore\ y_1 = 2$$

The point of intersection is (1, 2).

Example 5.22: Find the equation of the straight line passing through the intersection of the straight lines $2x + y = 8$ and $3x - y = 2$ and through the point $(2, -3)$

Solution:

The equation of the straight line passing through the intersection of the given lines is

$2x + y - 8 + \lambda(3x - y - 2) = 0$ … (1)

$(2, -3)$ lies on the equation (1) and hence $4 - 3 - 8 + \lambda(6 + 3 - 2) = 0$

$$\therefore \quad \lambda = 1$$

$$\therefore \quad (1) \quad \Rightarrow \quad 2x + y - 8 + 3x - y - 2 = 0 \quad \Rightarrow \quad 5x - 10 = 0$$

$x = 2$ is the equation of the required straight line.

Example 5.23: Find the equation of the straight line passing through the intersection of the straight lines $2x + y = 8$ and $3x - 2y + 7 = 0$ and parallel to $4x + y - 11 = 0$

Solution:

Let (x_1, y_1) be the point of intersection of the given straight lines

$$2x_1 + y_1 = 8 \qquad \ldots (1)$$

$$3x_1 - 2y_1 = -7 \qquad \ldots (2)$$

$$(1) \times 2 \quad \Rightarrow \quad 4x_1 + 2y_1 = 16 \qquad \ldots (3)$$

$$(2) + (3) \quad \Rightarrow \quad \therefore x_1 = \frac{9}{7} \quad y_1 = \frac{38}{7} \quad \therefore (x_1, y_1) = \left(\frac{9}{7}, \frac{38}{7}\right)$$

The straight line parallel to $4x + y - 11 = 0$ is of the form $4x + y + k = 0$

But it passes through $\left(\frac{9}{7}, \frac{38}{7}\right)$

$$\therefore \frac{36}{7} + \frac{38}{7} + k = 0 \quad \therefore k = -\frac{74}{7}$$

$$4x + y - \frac{74}{7} = 0$$

$28x + 7y - 74 = 0$ is the equation of the required straight line.

Example 5.24:

Find the equation of the straight line which passes through the intersection of the straight lines $5x - 6y = 1$ and $3x + 2y + 5 = 0$ and is perpendicular to the straight line $3x - 5y + 11 = 0$

Solution:

The straight line passing through the intersection of the given straight lines is

$$5x - 6y - 1 + \lambda (3x + 2y + 5) = 0 \qquad \ldots (1)$$

$$(5 + 3\lambda)x + (-6 + 2\lambda)y + (-1 + 5\lambda) = 0$$

This straight line is perpendicular to $3x - 5y + 11 = 0$

Product of the slopes of the perpendicular straight lines is -1 i.e. $m_1 m_2 = -1$

$$\Rightarrow \quad -\left(\frac{5 + 3\lambda}{-6 + 2\lambda}\right)\left(\frac{3}{5}\right) = -1$$

$$15 + 9\lambda = -30 + 10\lambda \quad \therefore \lambda = 45$$

(1) $\Rightarrow$ $5x - 6y - 1 + 45\,(3x + 2y + 5) = 0$ i.e. $140x + 84y + 224 = 0$

i.e. $5x + 3y + 8 = 0$ is the equation of the required straight line.

Example 5.25: Show that the straight lines $3x + 4y = 13$; $2x - 7y + 1 = 0$ and $5x - y = 14$ are concurrent.

Solution:

Let (x_1, y_1) be the point of intersection of the first two straight lines

$$3x_1 + 4y_1 = 13 \quad \ldots (1)$$

$$2x_1 - 7y_1 = -1 \quad \ldots (2)$$

(1) × 7 $\Rightarrow$ $21x_1 + 28y_1 = 91$ … (3)

(2) × 4 $\Rightarrow$ $8x_1 - 28y_1 = -4$ … (4)

(3) + (4) $\Rightarrow$ $29x_1 = 87 \Rightarrow x_1 = 3$

(1) $\Rightarrow$ $9 + 4y_1 = 13 \Rightarrow y_1 = 1$

The point of intersection of the first two straight lines is (3, 1).

Substitute this value in the equation $5x - y = 14$

$$\text{L.H.S.} = 5x - y$$

$$= 15 - 1 = 14 = \text{R.H.S.}$$

i.e. The point (3, 1) satisfies the third equation.

Hence the three straight lines are concurrent.

Example 5.26: Find the co-ordinates of orthocentre of the triangle formed by the straight lines

$x - y - 5 = 0$, $2x - y - 8 = 0$ and $3x - y - 9 = 0$

Solution:

Let the equations of sides AB, BC and CA of a ΔABC be represented by

$$x - y - 5 = 0 \quad \ldots (1)$$

$$2x - y - 8 = 0 \quad \ldots (2)$$

$$3x - y - 9 = 0 \quad \ldots (3)$$

Solving (1) and (3), we get A as (2, – 3)

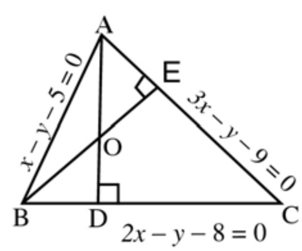


Fig. 5. 7

The equation of the straight line BC is $2x - y - 8 = 0$. The straight line perpendicular to it is of the form

$$x + 2y + k = 0 \quad \ldots (4)$$

A(2, – 3) satisfies the equation (4) $\therefore$ $2 - 6 + k = 0$ $\Rightarrow$ $k = 4$

The equation of AD is $x + 2y = -4$ … (5)

Solving the equations (1) and (2), we get B as (3, – 2)

The straight line perpendicular to $3x - y - 9 = 0$ is of the form

$$x + 3y + k = 0$$

But B(3, – 2) lies on this straight line $\therefore$ $3 - 6 + k = 0$ $\Rightarrow$ $k = 3$

$\therefore$ The equation of BE is $x + 3y = -3$ … (6)

Solving (5) and (6), we get the orthocentre O as (– 6, 1).

Example 5.27: For what values of 'a', the three straight lines $3x + y + 2 = 0$, $2x - y + 3 = 0$ and $x + ay - 3 = 0$ are concurrent?

Solution:

Let (x_1, y_1) be the point of concurrency. This point satisfies the first two equations.

$$\therefore \quad 3x_1 + y_1 + 2 = 0 \qquad \dots (1)$$

$$2x_1 - y_1 + 3 = 0 \qquad \dots (2)$$

Solving (1) and (2) we get (– 1, 1) as the point of intersection. Since it is a point of concurrency, it lies on $x + ay - 3 = 0$.

$$\therefore \quad -1 + a - 3 = 0$$

$$\text{i.e.} \quad a = 4$$

EXERCISE 5.3

(1) Find the angle between the straight lines $2x + y = 4$ and $x + 3y = 5$

(2) Show that the straight lines $2x + y = 5$ and $x - 2y = 4$ are at right angles.

(3) Find the equation of the straight line passing through the point (1, – 2) and parallel to the straight line $3x + 2y - 7 = 0$

(4) Find the equation of the straight line passing through the point (2, 1) and perpendicular to the straight line $x + y = 9$

(5) Find the point of intersection of the straight lines $5x + 4y - 13 = 0$ and $3x + y - 5 = 0$

(6) If the two straight lines $2x - 3y + 9 = 0$, $6x + ky + 4 = 0$ are parallel, find k

(7) Find the distance between the parallel lines $2x + y - 9 = 0$ and $4x + 2y + 7 = 0$

(8) Find the values of p for which the straight lines $8px + (2 - 3p)\,y + 1 = 0$ and $px + 8y - 7 = 0$ are perpendicular to each other.

(9) Find the equation of the straight line which passes through the intersection of the straight lines $2x + y = 8$ and $3x - 2y + 7 = 0$ and is parallel to the straight line $4x + y - 11 = 0$

(10) Find the equation of the straight line passing through intersection of the straight lines $5x - 6y = 1$ and $3x + 2y + 5 = 0$ and perpendicular to the straight line $3x - 5y + 11 = 0$

(11) Find the equation of the straight line joining $(4, -3)$ and the intersection of the straight lines $2x - y + 7 = 0$ and $x + y - 1 = 0$

(12) Find the equation of the straight line joining the point of the intersection of the straight lines $3x + 2y + 1 = 0$ and $x + y = 3$ to the point of intersection of the straight lines $y - x = 1$ and $2x + y + 2 = 0$

(13) Show that the angle between $3x + 2y = 0$ and $4x - y = 0$ is equal to the angle between $2x + y = 0$ and $9x + 32y = 41$

(14) Show that the triangle whose sides are $y = 2x + 7$, $x - 3y - 6 = 0$ and $x + 2y = 8$ is right angled. Find its other angles.

(15) Show that the straight lines $3x + y + 4 = 0$, $3x + 4y - 15 = 0$ and $24x - 7y - 3 = 0$ form an isosceles triangle.

(16) Show that the straight lines $3x + 4y=13$; $2x - 7y + 1 = 0$ and $5x - y = 14$ are concurrent.

(17) Find 'a' so that the straight lines $x - 6y + a = 0$, $2x + 3y + 4 = 0$ and $x + 4y + 1 = 0$ may be concurrent.

(18) Find the value of 'a' for which the straight lines $x + y - 4 = 0$, $3x + 2 = 0$ and $x - y + 3a = 0$ are concurrent.

(19) Find the co-ordinates of the orthocentre of the triangle whose vertices are the points $(-2, -1)$, $(6, -1)$ and $(2, 5)$

(20) If $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent, show that $a^3 + b^3 + c^3 = 3abc$

(21) Find the co-ordinates of the orthocentre of the triangle formed by the straight lines $x + y - 1 = 0$, $x + 2y - 4 = 0$ and $x + 3y - 9 = 0$

(22) The equation of the sides of a triangle are $x + 2y = 0$, $4x + 3y = 5$ and $3x + y = 0$. Find the co-ordinates of the orthocentre of the triangle.

5.4 Pair of straight lines

5.4.1 Combined equation of the pair of straight lines

We know that any equation of first degree in x and y represents a straight line. Let $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ be the individual equations of any two straight lines. Then their combined equation is

$$(l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0$$

$$l_1l_2x^2 + (l_1m_2 + l_2m_1)\, xy + m_1m_2y^2 + (l_1n_2 + l_2n_1)x + (m_1n_2 + m_2n_1)y + n_1n_2 = 0$$

Hence the equation of a pair of straight lines may be taken in the form

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, where a, b, c, f, g, h are constants.

5.4.2 Pair of straight lines passing through the origin

The homogeneous equation $ax^2 + 2hxy + by^2 = 0$ of second degree in x and y represents a pair of straight lines passing through the origin.

Considering $ax^2 + 2hxy + by^2 = 0$ as a quadratic equation in x, we get

$$x = \frac{-2hy \pm \sqrt{4h^2y^2 - 4aby^2}}{2a}$$

$$= \left[\frac{-2h \pm 2\sqrt{h^2 - ab}}{2a}\right] y = \frac{-h \pm \sqrt{h^2 - ab}}{a}\ y$$

$$\therefore \quad ax = \left(-h \pm \sqrt{h^2 - ab}\right) y$$

i.e. $ax + \left(h + \sqrt{h^2 - ab}\right) y = 0$ and $ax + \left(h - \sqrt{h^2 - ab}\right) y = 0$ are the two straight lines, each passing through the origin. Hence $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines intersecting at the origin.

Note : The straight lines are (1) real and distinct if $h^2 > ab$

(2) coincident if $h^2 = ab$

(3) imaginary if $h^2 < ab$

Sum and product of the slopes of pair of straight lines

The homogeneous equation $ax^2 + 2hxy + by^2 = 0$ of second degree in x and y represents a pair of straight lines passing through the origin.

Let $y = m_1x$ and $y = m_2x$ be the two straight lines passing through the origin. Therefore the combined equation is $(y - m_1x)\,(y - m_2x) = 0$

$$\Rightarrow \; m_1m_2\, x^2 - (m_1 + m_2)\, xy + y^2 = 0$$

This equation also represents a pair of straight lines passing through the origin.

Equating the co-efficients of like terms in the above equations, we get

$$\frac{m_1m_2}{a} = -\frac{(m_1 + m_2)}{2h} = \frac{1}{b}$$

$$\therefore\ m_1\, m_2 = \frac{a}{b}\ ; \text{ i.e. Product of the slopes} = \frac{a}{b}$$

$$m_1 + m_2 = -\frac{2h}{b} \quad \text{i.e. Sum of the slopes} = -\frac{2h}{b}$$

5.4.3 Angle between pair of straight lines passing through the origin

The equation of the pair of straight lines passing through the origin is

$$ax^2 + 2hxy + by^2 = 0 \qquad \ldots (1)$$

$$m_1 + m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1 m_2 = \frac{a}{b}$$

Let 'θ' be the angle between the pair of straight lines.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\left[\pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}\right]\right|$$

$$= \left|\left[\pm\frac{\sqrt{\dfrac{4h^2}{b^2} - \dfrac{4a}{b}}}{1 + \dfrac{a}{b}}\right]\right| = \left|\left[\pm\frac{\sqrt{\dfrac{4h^2 - 4ab}{b^2}}}{\dfrac{a+b}{b}}\right]\right|$$

$$\tan\theta = \left|\left[\frac{\pm 2\sqrt{h^2 - ab}}{a + b}\right]\right|$$

$$\theta = \tan^{-1}\left|\left[\frac{\pm 2\sqrt{h^2 - ab}}{a + b}\right]\right| \quad \text{i.e.} \quad \theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

It is conventional to take θ to be acute.

Corollary (1):

If 'θ' is the angle between the pair of straight lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

then $\theta = \tan^{-1}\left|\dfrac{2\sqrt{h^2 - ab}}{a + b}\right|$

It is same as the angle between the pair of straight lines

$ax^2 + 2hxy + by^2 = 0$ passing through the origin.

Corollary (2): If the straight lines are parallel, then $h^2 = ab$

[since $\theta = 0°$, $\tan\theta = 0$]

Corollary (3): If the straight lines are perpendicular then

coeff. of x^2 + coeff. of $y^2 = 0$ [since $\theta = 90°$, $\tan\theta = \infty$]

The condition for a general second degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Assume that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ … (1)

represents a pair of straight lines. Treating this equation as a quadratic in x, this can be written as $ax^2 + 2(hy + g)x + (by^2 + 2fy + c) = 0$

By solving for x,we get

$$x = \frac{-(hy+g) \pm \sqrt{(hy+g)^2 - a(by^2 + 2fy + c)}}{a}$$

$$\Rightarrow \quad ax + hy + g = \pm\sqrt{(hy+g)^2 - a(by^2+2fy+c)}$$

$$= \pm\sqrt{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)}$$

Now in order that each of these equations may be of the first degree in x and y, the expression in the R.H.S should be a perfect square. This is possible only if the discriminant of this quadratic in 'y' under the radical or within the root is zero.

$\therefore (h^2 - ab)(g^2 - ac) = (gh - af)^2$

Simplifying this we get $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ which is the required condition.

Example 5.28: Find the angle between the straight lines $x^2 + 4xy + 3y^2 = 0$

Solution:

Here $a = 1$, $2h = 4$, $b = 3$

If 'θ' is the angle between the given straight lines, then

$$\theta = \tan^{-1}\left|\left[\frac{2\sqrt{h^2 - ab}}{a+b}\right]\right| = \tan^{-1}\left|\left[\frac{2\sqrt{4-3}}{4}\right]\right| = \tan^{-1}\left(\frac{1}{2}\right)$$

Example 5.29: The slope of one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ is thrice that of the other, show that $3h^2 = 4ab$

Solution:

Let 'm_1' and 'm_2' be the slopes of pair of straight lines.

Then $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$

It is given that $m_2 = 3m_1$

$$\therefore m_1 + 3m_1 = -\frac{2h}{b} \quad \Rightarrow \quad m_1 = -\frac{h}{2b}$$

But $\quad m_1 \,.\, 3m_1 = \dfrac{a}{b} \quad \Rightarrow \; 3m_1^2 = \dfrac{a}{b} \;\Rightarrow\; 3\left(\dfrac{-h}{2b}\right)^2 = \dfrac{a}{b}$

$$\Rightarrow \qquad \frac{3h^2}{4b^2} = \frac{a}{b}$$

$$\Rightarrow \qquad 3h^2 = 4ab$$

Example 5.30: Show that $x^2 - y^2 + x - 3y - 2 = 0$ represents a pair of straight lines. Also find the angle between them.

Solution:

The given equation is

$$x^2 - y^2 + x - 3y - 2 = 0 \qquad \ldots (1)$$

Comparing this with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get $a = 1$, $h = 0$, $b = -1$, $g = \frac{1}{2}$, $f = -\frac{3}{2}$, $c = -2$. Condition for the given equation to represent a pair of straight lines is $abc+2fgh-af^2-bg^2-ch^2 = 0$

$$abc+2fgh-af^2-bg^2-ch^2 = (1)(-1)(-2)+2\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)(0)-(1)\left(\frac{9}{4}\right)- -1)\left(\frac{1}{4}\right)-(2)\,(0)$$

$$= 2 - \frac{9}{4} + \frac{1}{4} = \frac{8-9+1}{4}$$

$$= 0$$

Hence the given equation represents a pair of straight lines.

Since $a + b = 1 - 1 = 0$, the angle between the straight lines is 90°.

Example 5.31: Show that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines and also find the separate equation of the straight lines.

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 3$, $b = 2$, $h = \frac{7}{2}$, $g = \frac{5}{2}$, $f = \frac{5}{2}$, $c = 2$. The condition for the given equation to represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$abc+2fgh-af^2-bg^2-ch^2 = (3)\,(2)\,(2) + 2\left(\frac{5}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right) - 3\left(\frac{25}{4}\right) -2\left(\frac{25}{4}\right) -2\left(\frac{49}{4}\right)$$

$$= 12 + \frac{175}{4} - \frac{75}{4} - \frac{50}{4} - \frac{98}{4} = 0$$

Hence the given equation represents a pair of straight lines.

Now, factorising the second degree terms

we get $3x^2 + 7xy + 2y^2 = (x + 2y)(3x + y)$

Let $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = (x + 2y + l)(3x + y + m)$

Comparing the coefficient of x, $3l + m = 5$;

Comparing the coefficient of y, $l + 2m = 5$

Solving these two equations, we get $l = 1$, $m = 2$

$\therefore$ The separate equations are $x + 2y + 1 = 0$ and $3x + y + 2 = 0$

Example 5.32: Show that the equation $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines and find the distance between them.

Solution:

The given equation is $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$

Here $a = 4$, $h = 2$, $b = 1$; $ab - h^2 = 4(1) - 2^2 = 4 - 4 = 0$

$\therefore$ The given equation represents a pair of parallel straight lines.

Now $4x^2 + 4xy + y^2 = (2x + y)^2$

$\therefore$ $4x^2 + 4xy + y^2 - 6x - 3y - 4 = (2x + y + l)(2x + y + m)$

Comparing the coefficient of x, $2l + 2m = -6$ i.e. $l + m = -3$ … (1)

Comparing the constant term, $lm = -4$ … (2)

$$\therefore \quad l + \left(\frac{-4}{l}\right) = -3 \quad \Rightarrow l^2 + 3l - 4 = 0$$

$$\text{i.e.} \quad (l + 4)(l - 1) = 0 \quad \Rightarrow \quad l = -4, 1$$

$$\text{Now} \quad lm = -4 \quad \Rightarrow \quad m = 1, -4$$

$\therefore$ The separate equations are $2x + y - 4 = 0$ and $2x + y + 1 = 0$

The distance between them is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} = \left|\dfrac{-4 - 1}{\sqrt{2^2 + 1^2}}\right| = \sqrt{5}$ units

Example 5.33: Find the combined equation of the straight lines whose separate equations are $x + 2y - 3 = 0$ and $3x - y + 4 = 0$

Solution:

The combined equation of the given straight lines is

$$(x + 2y - 3)(3x - y + 4) = 0$$

i.e. $3x^2 + 6xy - 9x - xy - 2y^2 + 3y + 4x + 8y - 12 = 0$

i.e. $3x^2 + 5xy - 2y^2 - 5x + 11y - 12 = 0$ is the required combined equation.

EXERCISE 5.4

(1) If the equation $ax^2 + 3xy - 2y^2 - 5x + 5y + c = 0$ represents a pair of perpendicular straight lines, find a and c.

(2) Find the angle between the pair of straight lines given by

$(a^2 - 3b^2)\, x^2 + 8ab\, xy + (b^2 - 3a^2)y^2 = 0$

(3) Show that if one of the angles between pair of straight lines

$ax^2 + 2hxy + by^2 = 0$ is $60°$ then $(a + 3b)\,(3a + b) = 4h^2$

(4) Show that $9x^2 + 24xy + 16y^2 + 21x + 28y + 6 = 0$ represents a pair of parallel straight lines and find the distance between them.

(5) The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other, show that $8h^2 = 9ab$.

(6) Find the combined equation of the straight lines through the origin, one of which is parallel to and the other is perpendicular to the straight line $2x + y + 1 = 0$

(7) Find the combined equation of the straight lines whose separate equations are $x + 2y - 3 = 0$ and $3x + y + 5 = 0$

(8) Find k such that the equation $12x^2 + 7xy - 12y^2 - x + 7y + k = 0$ represents a pair of straight lines. Find the separate equations of the straight lines and also the angle between them.

(9) If the equation $12x^2 - 10xy + 2y^2 + 14x - 5y + c = 0$ represents a pair of straight lines, find the value of c. Find the separate equations of the straight lines and also the angle between them.

(10) For what value of k does $12x^2 + 7xy + ky^2 + 13x - y + 3 = 0$ represents a pair of straight lines? Also write the separate equations.

(11) Show that $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ represents a pair of straight lines and the angle between them is $\tan^{-1}\left(\frac{2}{11}\right)$

5.5 Circle

Definition: A circle is the locus of a point which moves in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

5.5.1 The equation of a circle when the centre and radius are given

Let C(h, k) be the centre and r be the radius of the circle. Let P(x, y) be any point on the circle

CP = $r \Rightarrow$ CP$^2 = r^2 \Rightarrow (x-h)^2 + (y-k)^2 = r^2$

is the required equation of the circle.

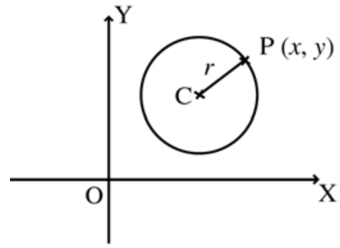


Fig. 5.8

Note :

If the centre of the circle is at the origin, i.e. $(h, k) = (0, 0)$ then the equation of the circle is $x^2 + y^2 = r^2$.

5.5.2 The equation of a circle if the end points of a diameter are given

Let A(x_1, y_1) and B(x_2, y_2) be the end points of a diameter. Let P(x, y) be any point on the circle.

The angle in a semi circle is a right angle.

$\therefore$ PA is perpendicular to PB

$\therefore$ (Slope of PA) (Slope of PB) = – 1

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$

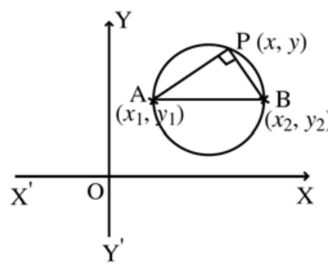


Fig. 5.9.

$$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$\therefore$ $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$ is the required equation of the circle.

5.5.3 The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Consider the equation $x^2 + y^2 + 2gx + 2fy + c = 0$

This can be written as $x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$

$$(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

$$[x-(-g)]^2 + [y-(-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

This is of the form $(x-h)^2 + (y-k)^2 = r^2$

$\therefore$ The considered equation represents a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$

$\therefore$ The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Note : The general second degree equation $ax^2+by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if (1) $a = b$ i.e. coefficient of x^2 = coefficient of y^2

(2) $h = 0$ i.e. no xy term

5.5.4 Parametric form

Consider a circle with radius r and centre at the origin. Let $P(x, y)$ be any point on the circle. Assume that OP makes an angle θ with the positive direction of x-axis. Draw the perpendicular PM to the x-axis.

From the figure (5.10), $\frac{x}{r} = \cos\theta$, $\frac{y}{r} = \sin\theta$.

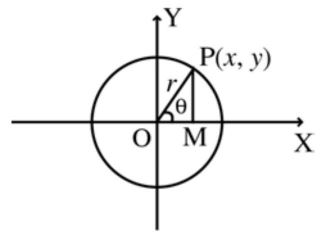


Fig. 5.10

Here x and y are the co-ordinates of any point on the circle. Note that these two co-ordinates depend on θ.

The value of r is fixed. The equations $x = r\cos\theta$, $y = r\sin\theta$ are called the parametric equations of the circle $x^2 + y^2 = r^2$. Here 'θ' is called the parameter and $0 \le \theta \le 2\pi$

Another parametric form:

We know that $\quad \sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \quad ; \quad \cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Let $t = \tan\frac{\theta}{2}$

If $0 \le \theta \le 2\pi$ then $-\infty < t < \infty$

$x = r\cos\theta \Rightarrow x = \dfrac{r(1-t^2)}{1+t^2} \quad ; \quad y = r\sin\theta \Rightarrow y = \dfrac{2rt}{1+t^2}$

Thus $x = \dfrac{r(1-t^2)}{1+t^2}$, $y = \dfrac{2rt}{1+t^2}$, $-\infty < t < \infty$ is another parametric equation of the circle $x^2 + y^2 = r^2$

Clearly $x = \dfrac{r(1-t^2)}{1+t^2}$, $y = \dfrac{2rt}{1+t^2}$ satisfy the equation $x^2 + y^2 = r^2$

Example 5.34: Find the equation of the circle if the centre and radius are $(2, -3)$ and 4 respectively.

Solution:

The equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$

Here $(h, k) = (2, -3)$ and $r = 4$ $\therefore$ $(x-2)^2 + (y+3)^2 = 4^2$

i.e. $x^2 + y^2 - 4x + 6y - 3 = 0$ is the required equation of the circle.

Example 5.35: Find the equation of the circle if (2, – 3) and (3, 1) are the extremities of a diameter.

Solution:

The equation of the circle is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

Here $(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (3, 1)$

$\therefore$ $(x-2)(x-3) + (y+3)(y-1) = 0$

$x^2 - 5x + 6 + y^2 + 2y - 3 = 0$

$\therefore$ The required equation is $x^2 + y^2 - 5x + 2y + 3 = 0$

Example 5.36: Find the centre and radius of the circle $x^2 + y^2 + 2x - 4y + 3 = 0$

Solution:

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Here $2g = 2$, $2f = -4$, $c = 3$

$\therefore$ centre is $(-g, -f) = (-1, 2)$

radius is $\sqrt{g^2 + f^2 - c} = \sqrt{1 + 4 - 3} = \sqrt{2}$ units.

Example 5.37: Find the centre and radius of the circle $3x^2+3y^2-2x+6y-6 = 0$

Solution:

The given equation is $3x^2 + 3y^2 - 2x + 6y - 6 = 0$

Rewriting the above, $x^2 + y^2 - \frac{2}{3}x + 2y - 2 = 0$

Comparing this with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$

We get $2g = -\frac{2}{3}$, $2f = 2$, $c = -2$

$\therefore$ centre is $(-g, -f) = \left(\frac{1}{3}, -1\right)$

radius is $\sqrt{g^2 + f^2 - c} = \sqrt{\frac{1}{9} + 1 + 2} = \frac{2\sqrt{7}}{3}$ units.

Example 5.38: If (4, 1) is one extremity of a diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$, find the other extremity.

Solution:

Comparing $x^2 + y^2 - 2x + 6y - 15 = 0$ with the general equation of the circle,

we get $2g = -2 \quad 2f = 6$

$\therefore$ centre is C $(-g, -f) = (1, -3)$

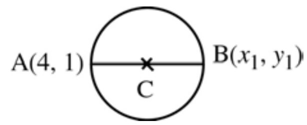


Fig. 5.11

Let B(x_1, y_1) be the other extremity and

A be (4, 1)

C is the mid point of AB

$$\therefore \quad \frac{x_1 + 4}{2} = 1, \quad \frac{y_1 + 1}{2} = -3 \Rightarrow x_1 = -2, \quad y_1 = -7$$

$\therefore$ The other extremity is $(-2, -7)$

Example 5.39: Find the equation the circle passing through the points (0,1), (2,3) and (–2, 5).

Solution:

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

The points (0, 1), (2, 3) and (– 2, 5) lie on the circle

$$\therefore \; 2f + c = -1 \quad \ldots (1)$$

$$4g + 6f + c = -13 \quad \ldots (2)$$

$$-4g + 10f + c = -29 \quad \ldots (3)$$

$$(1) - (2) \Rightarrow -4g - 4f = 12$$

$$g + f = -3 \quad \ldots (4)$$

$$(2) - (3) \Rightarrow 8g - 4f = 16$$

$$2g - f = 4 \quad \ldots (5)$$

$$(4) + (5) \Rightarrow 3g = 1 \Rightarrow g = \frac{1}{3}$$

$$(4) \Rightarrow f = -3 - \frac{1}{3} = -\frac{10}{3}$$

$$(1) \Rightarrow c = \frac{17}{3}$$

$$\therefore x^2 + y^2 + 2\left(\frac{1}{3}\right)x + 2\left(-\frac{10}{3}\right)y + \frac{17}{3} = 0$$

$\therefore$ $3x^2 + 3y^2 + 2x - 20y + 17 = 0$ is the required equation.

Example 5.40: Find the equation of the circle passing through the points (0, 1), (2, 3) and having the centre on the line $x - 2y + 3 = 0$

Solution:

The general equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

(0, 1) lies on the circle $\therefore\ 2f + c = -1$ … (1)

(2, 3) lies on the circle $\therefore\ 4g + 6f + c = -13$ … (2)

The centre $(-g, -f)$ lies on $x - 2y + 3 = 0$;

$\therefore\ -g + 2f = -3$ … (3)

(1) – (2) $\Rightarrow$ $-4g - 4f = 12$

i.e. $g + f = -3$ … (4)

(3) + (4) $\Rightarrow$ $3f = -6$ $\therefore f = -2$

(3) $\Rightarrow$ $g = -1$

(1) $\Rightarrow$ $c = 3$

$\therefore$ The required equation is $x^2 + y^2 - 2x - 4y + 3 = 0$

Example 5.41:

Find the values of a and b if the equation

$(a - 4)x^2 + by^2 + (b - 3)xy + 4x + 4y - 1 = 0$ represents a circle.

Solution:

The given equation is $(a - 4)x^2 + by^2 + (b - 3)xy + 4x + 4y - 1 = 0$

(i) coefficient of $xy = 0 \Rightarrow b - 3 = 0$ $\therefore\ b = 3$

(ii) coefficient of x^2 = co-efficient of y^2 $\Rightarrow\ a - 4 = b$

$\therefore\ a = 7$

Thus $a = 7$, $b = 3$

Example 5.42: Find the equation of the circle with centre (2, – 3) and radius 3. Show that it passes through the point (2, 0).

Solution:

If the centre is (h, k) and radius is r, then the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$.

Here $(h, k) = (2, -3)$ and $r = 3$.

$$(x - 2)^2 + (y + 3)^2 = 3^2$$

$(x - 2)^2 + (y + 3)^2 = 9$ is the required equation of the circle.

Putting (2, 0) in the equation of the circle, we get

L.H.S. = $(2 - 2)^2 + (0 + 3)^2 = 0 + 9 = 9$ = R.H.S.

Hence the circle passes through (2, 0)

Example 5.43:

Find the equation of the circle with centre (1, – 2) and passing through the point (4, 1)

Solution:

Let C be (1, – 2) and P be (4, 1)

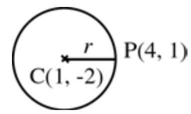


Fig. 5.12

Radius $r = CP = \sqrt{(1-4)^2 + (-2-1)^2} = \sqrt{9+9} = \sqrt{18}$

Thus the equation of the circle is $(x - h)^2 + (y - k)^2 = r^2 = r^2$

$\Rightarrow \quad (x-1)^2 + (y+2)^2 = \sqrt{18}^{\,2}$

i.e. $x^2 + y^2 - 2x + 4y - 13 = 0$ is the required equation.

Example 5.44: Find the parametric equations of the circle $x^2 + y^2 = 16$

Solution:

Here $r^2 = 16 \quad \Rightarrow \quad r = 4$. The parametric equations of the circle $x^2 + y^2 = r^2$ in parameter θ are $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ The parametric equations of the given circle $x^2 + y^2 = 16$ are

$x = 4\cos\theta$, $y = 4\sin\theta$, $0 \le \theta \le 2\pi$

Example 5.45: Find the cartesian equation of the circle whose parametric equations are

$x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

Solution:

To find the caretsian equation of the circle, eliminate the parameter ‘θ’ from the given equations, $\cos\theta = \frac{x}{2}$; $\sin\theta = \frac{y}{2}$

$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$\therefore x^2 + y^2 = 4$ is the required cartesian equation of the circle.

EXERCISE 5.5

(1) Find the centre and radius of the following circles:

(i) $x^2 + y^2 = 1$ (ii) $x^2 + y^2 - 4x - 6y - 9 = 0$

(iii) $x^2 + y^2 - 8x - 6y - 24 = 0$ (iv) $3x^2 + 3y^2 + 4x - 4y - 4 = 0$

(v) $(x-3)(x-5) + (y-7)(y-1) = 0$

(2) For what values of a and b does the equation $(a-2)x^2 + by^2 + (b-2)xy + 4x + 4y - 1 = 0$ represents a circle? Write down the resulting equation of the circle.

(3) Find the equation of the circle passing through the point (1, 2) and having its centre at (2, 3).

(4) $x + 2y = 7$, $2x + y = 8$ are two diameters of a circle with radius 5 units. Find the equation of the circle.

(5) The area of a circle is 16π square units. If the centre of the circle is $(7, -3)$, find the equation of the circle.

(6) Find the equation of the circle whose centre is $(-4, 5)$ and circumference is 8π units.

(7) Find the circumference and area of the circle $x^2 + y^2 - 6x - 8y + 15 = 0$

(8) Find the equation of the circle which passes through (2, 3) and whose centre is on x-axis and radius is 5 units.

(9) Find the equation of the circle described on the line joining the points (1, 2) and (2, 4) as its diameter.

(10) Find the equation of the circle passing through the points $(1, 0)$, $(0, -1)$ and $(0, 1)$.

(11) Find the equation of the circle passing through the points $(1, 1)$, $(2, -1)$ and $(3, 2)$.

(12) Find the equation of the circle that passes through the points (4, 1) and (6, 5) and has its centre on the line $4x + y = 16$.

(13) Find the equation of the circle whose centre is on the line $x = 2y$ and which passes through the points $(-1, 2)$ and $(3, -2)$.

(14) Find the cartesian equation of the circle whose parametric equations are $x = \frac{1}{4}\cos\theta$, $y = \frac{1}{4}\sin\theta$ and $0 \le \theta \le 2\pi$

(15) Find the parametric equation of the circle $4x^2 + 4y^2 = 9$

5.6. Tangent

5.6.1 Introduction

Let us consider a circle with centre at C and a straight line AB. This straight line can be related to the circle in 3 different positions as shown in the following figures.

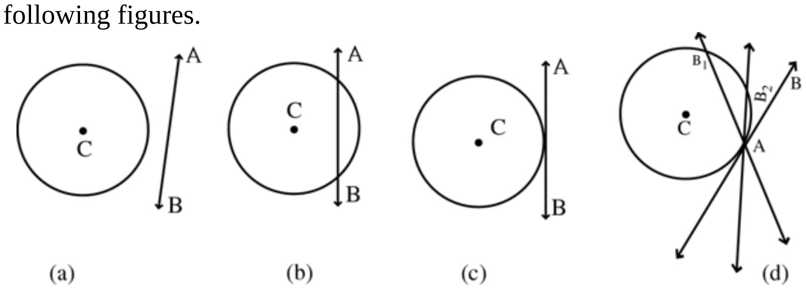


Fig. 5.13

In figure (5.13 a), the straight line AB does not touch or intersect the circle.

In figure (5.13 b), the straight line AB intersects the circle in two points and it is called a secant.

In figure (5.13 c), the straight line AB touches the circle at exactly one point, and it is called a tangent. In otherwords, the limiting form of a secant is called a tangent *(Fig. 5.13d)*

Definition : A tangent to a circle is a straight line which intersects (touches) the circle in exactly one point.

5.6.2 Equation of the tangent to a circle at a point (x_1, y_1)

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \dots (1)$

Let $P(x_1, y_1)$ be a given point on it.

$\therefore\ x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \dots (2)$

Let PT be the tangent at P.

The centre of the circle is $C(-g, -f)$.

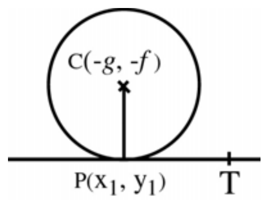


Fig. 5.14

$$\text{Slope of the CP} = \frac{y_1 + f}{x_1 + g}$$

$$\text{Since CP is perpendicular to PT, slope of PT} = -\left(\frac{x_1 + g}{y_1 + f}\right)$$

$\therefore$ Equation of the tangent PT is

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$(y - y_1)(y_1 + f) = -(x - x_1)(x_1 + g)$$

$$(y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$$

$$\Rightarrow \quad yy_1 - y_1^2 + fy - fy_1 + [xx_1 - x_1^2 + gx - gx_1] = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + fy + gx = x_1^2 + y_1^2 + gx_1 + fy_1$$

Add $gx_1 + fy_1 + c$ on both sides

$$xx_1 + yy_1 + gx + gx_1 + fy + fy_1 + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ is the required equation of the tangent at (x_1, y_1)

Corollary:

The equation of the tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

Note: To get the equation of the tangent at (x_1, y_1), replace x^2 as xx_1, y^2 as yy_1, x as $\frac{x + x_1}{2}$ and y as $\frac{y + y_1}{2}$ in the equation of the circle.

5.6.3 Length of the tangent to the circle from a point (x_1, y_1)

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

Let PT be the tangent to the circle from $P(x_1, y_1)$ outside it. We know that the co-ordinate of the centre C is $(-g, -f)$ and

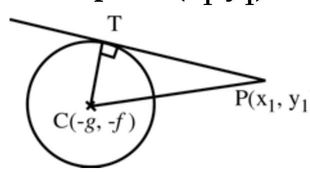


Fig. 5.15

$$\text{radius } r = CT = \sqrt{g^2 + f^2 - c}$$

From the right angled triangle PCT,

$$\begin{aligned} PT^2 &= PC^2 - CT^2 \\ &= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\ &= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \end{aligned}$$

$\therefore$ PT $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$, which is the length of the tangent from the point (x_1, y_1)

to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Note : (1) If the point P is on the circle then $PT^2 = 0$ (PT is zero).

(2) If the point P is outside the circle then $PT^2 > 0$ (PT is real)

(3) If the point P is inside the circle then $PT^2 < 0$ (PT is imaginary)

Corollary:

The constant c will be positive if the origin is outside the circle, zero if it is on the circle and negative if it is inside the circle.

5.6.4 The condition for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$

Let the line $y = mx + c$ be a tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1)

But the equation of the tangent at (x_1, y_1) to the circle

$x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

Thus the equations $y = mx + c$ and $xx_1 + yy_1 = a^2$ are representing the same straight line and hence their coefficients are proportional.

$$\therefore \frac{1}{y_1} = -\frac{m}{x_1} = \frac{c}{a^2}$$

$$\therefore \; x_1 = \frac{-a^2m}{c} \; , y_1 = \frac{a^2}{c}$$

But (x_1, y_1) is a point on the circle $x^2 + y^2 = a^2$

$$\therefore \; x_1^2 + y_1^2 = a^2 \quad \Rightarrow \frac{a^4 m^2}{c^2} + \frac{a^4}{c^2} = a^2$$

$$\Rightarrow \qquad a^2m^2 + a^2 = c^2 \;\; \Rightarrow \; a^2 (m^2 + 1) = c^2$$

i.e. $c^2 = a^2 (1 + m^2)$ is the required condition.

Note:(1)The point of contact of the tangent $y = mx + c$ to the circle $x^2+y^2= a^2$ is

$$\left[\frac{-am}{\sqrt{1+m^2}} \; , \; \frac{a}{\sqrt{1+m^2}}\right]$$

(2) The equation of any tangent to a circle is of the form

$$y = mx \pm a\sqrt{1+m^2}$$

5.6.5 Two tangents can be drawn from a point to a circle

Let (x_1, y_1) be the given point. We know that $y = mx \pm a\sqrt{1+m^2}$ is the equation of any tangent. It passes through (x_1, y_1).

$$\therefore y_1 = mx_1 \pm a\sqrt{1+m^2}$$

$$\Rightarrow \qquad y_1 - mx_1 = \pm a\sqrt{1+m^2}$$

$$\Rightarrow \qquad (y_1 - mx_1)^2 = a^2 (1 + m^2)$$

$$\Rightarrow \qquad y_1^2 + m^2 x_1^2 - 2mx_1y_1 - a^2 - a^2m^2 = 0$$

$$\Rightarrow \qquad m^2 (x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - a^2) = 0$$

This is a quadratic equation in 'm'. Thus 'm' has two values. But 'm' is the slope of the tangent. Thus two tangents can be drawn from a point to a circle.

Note : (1) If (x_1, y_1) is an exterior point (lies outside) then both the tangents are real and visible

(2) If (x_1, y_1) is an interior point (lies inside) the circle then both the tangents are imaginary and hence not visible.

(3) If (x_1, y_1) is a boundary point (lies on) then both the tangents coincide and appears to be one.

5.6.6. Equation of the chord of contact of tangents from a point to the circle

The general equation of the circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$ … (1)

Let $P(x_1, y_1)$ be a point outside the circle. Let the tangents from $P(x_1, y_1)$ touch the circle at $Q(x_2, y_2)$ and $R(x_3, y_3)$

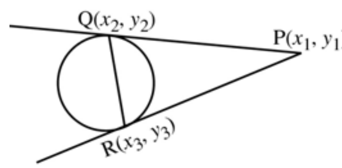


Fig. 5.16

The equation of the tangent PQ at Q (x_2, y_2) is

$$xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0 \quad \dots (2)$$

The equation of the tangent PR at $R(x_3, y_3)$ is

$$xx_3 + yy_3 + g(x + x_3) + f(y + y_3) + c = 0 \quad \dots (3)$$

But (x_1, y_1) satisfy the equations (2) and (3)

$\therefore\ x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$ and … (4)

$x_1x_3 + y_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0$ … (5)

But equations (4) and (5) show that (x_2, y_2) and (x_3, y_3) lie on the line

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Hence the straight line $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ represents the equation of QR, chord of contact of tangents from (x_1, y_1).

Example 5.46: Find the length of the tangent from (2, 3) to the circle $x^2 + y^2 - 4x - 3y + 12 = 0$.

Solution:

The length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$\therefore$ Length of the tangent to the given circle is $\sqrt{x_1^2 + y_1^2 - 4x_1 - 3y_1 + 12}$

$$= \sqrt{2^2 + 3^2 - 4.2 - 3.3 + 12}$$
$$= \sqrt{4 + 9 - 8 - 9 + 12}$$
$$= \sqrt{8} = 2\sqrt{2} \text{ units}$$

Example 5.47: Show that the point (2, 3) lies inside the circle

$$x^2 + y^2 - 6x - 8y + 12 = 0.$$

Solution:

The length of the tangent PT from P(x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

$$PT^2 = 2^2 + 3^2 - 6.2 - 8.3 + 12 = 4 + 9 - 12 - 24 + 12$$

$$= -11 < 0$$

The point (2, 3) lies inside the circle

Example 5.48: Find the equation of the tangent to the circle $x^2+y^2=25$ at (4, 3).

Solution:

The equation of the circle is $x^2 + y^2 = 25$.

The equation of the tangent at (x_1, y_1) is $xx_1 + yy_1 = 25$. Here $(x_1, y_1) = (4, 3)$.

$\therefore$ The equation of the tangent at (4, 3) is $4x + 3y = 25$

Example 5.49: If $y=3x+c$ is a tangent to the circle $x^2+y^2=9$, find the value of c.

Solution:

The condition for the line $y = mx + c$ to be a tangent to

$$x^2 + y^2 = a^2 \text{ is } c = \pm a\sqrt{1 + m^2}$$

Here $a = 3, m = 3$

$$\therefore c = \pm 3\sqrt{10}$$

Example 5.50: Find the equation of the tangent to

$$x^2 + y^2 - 4x + 4y - 8 = 0 \text{ at } (-2, -2)$$

Solution:

The equation of the tangent at (x_1, y_1) to the given circle is

$$xx_1 + yy_1 - 4\left(\frac{x + x_1}{2}\right) + 4\left(\frac{y + y_1}{2}\right) - 8 = 0$$

$$xx_1 + yy_1 - 2(x + x_1) + 2(y + y_1) - 8 = 0$$

At (– 2, – 2), the equation of the tangent is

$$-2x - 2y - 2(x - 2) + 2(y - 2) - 8 = 0$$

$\Rightarrow$ $-4x - 8 = 0$

$\Rightarrow$ $x + 2 = 0$ is the required equation of the tangent.

Example 5.51: Find the length of the chord intercepted by the circle

$$x^2 + y^2 - 2x - y + 1 = 0 \text{ and the line } x - 2y = 1.$$

Solution:

To find the end points of the chord, solve the equations of the circle and the line. Substitute $x = 2y + 1$ in the equation of the circle.

$$(2y + 1)^2 + y^2 - 2(2y + 1) - y + 1 = 0$$

$$4y^2 + 4y + 1 + y^2 - 4y - 2 - y + 1 = 0$$

$$5y^2 - y = 0 \qquad \therefore\ y(5y - 1) = 0$$

$$y = 0 \quad \Big| \quad y = \frac{1}{5}$$

$$\Rightarrow \qquad x = 1 \quad \Big| \quad x = \frac{7}{5}$$

$\therefore$ The two end points are (1, 0) and $\left(\frac{7}{5}, \frac{1}{5}\right)$

$\therefore$ Length of the chord $= \sqrt{\left(1 - \frac{7}{5}\right)^2 + \left(0 - \frac{1}{5}\right)^2} = \sqrt{\frac{4}{25} + \frac{1}{25}} = \frac{1}{\sqrt{5}}$ units

Example 5.52: Find the value of p if the line $3x + 4y - p = 0$ is a tangent to the circle $x^2 + y^2 = 16$.

Solution:

The condition for the tangency is $c^2 = a^2(1 + m^2)$.

Here $a^2 = 16$, $m = -\frac{3}{4}$, $c = \frac{p}{4}$

$$c^2 = a^2(1 + m^2) \quad \Rightarrow \quad \frac{p^2}{16} = 16\left(1 + \frac{9}{16}\right) = 25$$

$$p^2 = 16 \times 25$$

$$\therefore\ p = \pm 20$$

Example 5.53: Find the equation of the circle which has its centre at (2, 3) and touches the x-axis.

Solution:

Let P be a point on x-axis where it touches the circle.

Given that the centre C is (2, 3) and P is (2, 0)

$r = \text{CP} = \sqrt{(2-2)^2 + (3-0)^2} = 3$

The equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$

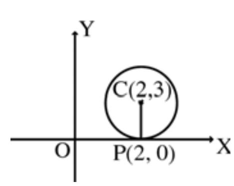


Fig. 5.17

$$(x - 2)^2 + (y - 3)^2 = 3^2$$

$$x^2 + y^2 - 4x - 6y + 4 = 0$$

EXERCISE 5.6

(1) Find the length of the tangent from (1, 2) to the circle
$x^2 + y^2 - 2x + 4y + 9 = 0$

(2) Prove that the tangents from (0, 5) to the circles
$x^2 + y^2 + 2x - 4 = 0$ and
$x^2 + y^2 - y + 1 = 0$ are equal.

(3) Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 8y - 5 = 0$ at (2, 1).

(4) Is the point (7, – 11) lie inside or outside the circle $x^2 + y^2 - 10x = 0$?

(5) Determine whether the points (– 2, 1), (0, 0) and (4, – 3) lie outside, on or inside the circle $x^2 + y^2 - 5x + 2y - 5 = 0$

(6) Find the co-ordinates of the point of intersection of the line $x + y = 2$ with the circle $x^2 + y^2 = 4$

(7) Find the equation of the tangent lines to the circle $x^2 + y^2 = 9$ which are parallel to $2x + y - 3 = 0$

(8) Find the length of the chord intercepted by the circle
$x^2 + y^2 - 14x + 4y + 28 = 0$ and the line $x - 7y + 4 = 0$

(9) Find the equation of the circle which has its centre at (5, 6) and touches (i) x-axis (ii) y-axis

(10) Find the equation of the tangent to $x^2 + y^2 - 2x - 10y + 1 = 0$ at (– 3, 2)

(11) Find the equation of the tangent to the circle $x^2 + y^2 = 16$ which are (i) perpendicular and (ii) parallel to the line $x + y = 8$

(12) Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 2y - 21 = 0$ at (1, 4).

(13) Find the value of p so that the line $3x + 4y - p = 0$ is a tangent to
$x^2 + y^2 - 64 = 0$

(14) Find the co-ordinates of the middle point of the chord which the circle
$x^2 + y^2 + 2x + y - 3 = 0$ cuts off by the line $y = x - 1$.

5.7. Family of circles

Concentric circles:

Two or more circles having the same centre are called concentric circles.

Circles touching each other:

Two circles may touch each other either internally or externally. Let C_1, C_2 be the centers of the circle and r_1, r_2 be their radii and P the point of contact.

Case (1):

The two circles touch externally.

The distance between their centres is equal to the sum of their radii.

(i.e.) $C_1C_2 = r_1 + r_2$

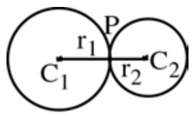


Fig. 5.18

Case (2) The two circles touch internally:

The distance between their centres is equal to the difference of their radii.

$$C_1 C_2 = C_1P - C_2P = r_1 - r_2$$

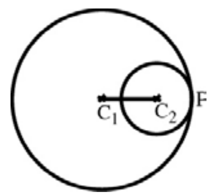


Fig. 5.19

Orthogonal circles:

Definition: Two circles are said to be orthogonal if the tangent at their point of intersection are at right angles.

Condition for two circles cut orthogonally

Let the two circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and cut each other orthogonally.

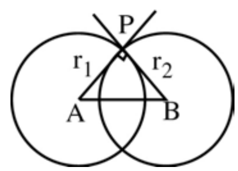


Fig. 5.20

Let A and B be the centres of the two circles

$\therefore$ A is $(-g_1, -f_1)$ and B is $(-g_2, -f_2)$ $r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$ and $r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$

In the right angled triangle APB, $AB^2 = AP^2 + PB^2$

i.e. $$(-g_1 + g_2)^2 + (-f_1 + f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$

$$\Rightarrow g_1^2 + g_2^2 - 2g_1g_2 + f_1^2 + f_2^2 - 2f_1 f_2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$

$$\Rightarrow -2g_1 g_2 - 2f_1f_2 = -c_1 - c_2$$

i.e. $$2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$$

is the required condition for orthogonality.

Example 5.54: Show that the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other.

Solution:

The given circles are

$$x^2 + y^2 - 4x + 6y + 8 = 0 \qquad \ldots (1)$$

and $$x^2 + y^2 - 10x - 6y + 14 = 0 \qquad \ldots (2)$$

(1) $\Rightarrow$ $g_1 = -2 \quad f_1 = 3, \quad c_1 = 8$. Centre is A(2, – 3)

$$\text{radius } r_1 = \sqrt{g_1^2 + f_1^2 - c_1} = \sqrt{4 + 9 - 8} = \sqrt{5}$$

(2) $\Rightarrow$ $g_2 = -5, f_2 = -3, \quad c_2 = 14$. Centre is B(5, 3)

$$\text{radius } r_2 = \sqrt{25 + 9 - 14} = \sqrt{20} = 2\sqrt{5}$$

$$\text{Distance between A and B} = \sqrt{(2-5)^2 + (-3-3)^2}$$

$$= \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}$$

$$= r_1 + r_2$$

$\therefore$ The circles touch each other.

Example 5.55: Find the equation of the circle, which is concentric with the circle

$x^2 + y^2 - 4x - 6y - 9 = 0$ and passing through the point (– 4, – 5).

Solution:

The given circle is $x^2 + y^2 - 4x - 6y - 9 = 0$

Centre $(-g, -f)$ is (2, 3)

The circle passes through the point (– 4, – 5).

$\therefore$ radius $= \sqrt{(2+4)^2 + (3+5)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

The equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$

Here $(h, k) = (2, 3), \quad r = 10$

$\therefore \; (x-2)^2 + (y-3)^2 = 10^2$

$x^2 + y^2 - 4x - 6y - 87 = 0$ is the required equation of the circle.

Example 5.56: Prove that the circles $x^2 + y^2 - 8x + 6y - 23 = 0$ and

$x^2 + y^2 - 2x - 5y + 16 = 0$ are orthogonal.

Solution:

The equations of the circle are

$$x^2 + y^2 - 8x + 6y - 23 = 0 \qquad \ldots (1)$$

$$x^2 + y^2 - 2x - 5y + 16 = 0 \qquad \ldots (2)$$

(1) $\Rightarrow$ $g_1 = -4$, $f_1 = 3$, $c_1 = -23$

(2) $\Rightarrow$ $g_2 = -1$, $f_2 = -\frac{5}{2}$, $c_2 = 16$

Condition for orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$$2g_1g_2 + 2f_1 f_2 = 2(-4)(-1) + 2(3)\left(-\frac{5}{2}\right) = 8 - 15 = -7$$

$$c_1 + c_2 = -23 + 16 = -7$$

$$\therefore \quad 2g_1g_2 + 2f_1 f_2 = c_1 + c_2$$

$\therefore$ The two circles cut orthogonally and hence they are orthogonal circles.

Example 5.57:

Find the equation of the circle which passes through the point (1, 2) and cuts orthogonally each of the circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 2x + 8y - 7 = 0$

Solution:

Let the required equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ … (1)

The point (1, 2) lies on the circle

$$\therefore \quad 1 + 4 + 2g + 4f + c = 0$$

$$2g + 4f + c = -5 \quad \text{… (2)}$$

The circle (1) cuts the circle $x^2 + y^2 = 9$ orthogonally.

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

$$\Rightarrow \quad 2g(0) + 2f(0) = c - 9$$

$$\therefore \quad c = 9 \quad \text{… (3)}$$

Again the circle (1) cuts $x^2 + y^2 - 2x + 8y - 7 = 0$ orthogonally.

$$\therefore \quad 2g(-1) + 2f(4) = c - 7$$

$$\Rightarrow \quad -2g + 8f = 9 - 7 = 2$$

$$\Rightarrow \quad -g + 4f = 1 \quad \text{… (4)}$$

(2) becomes $2g + 4f = -14$

$$\therefore \quad g + 2f = -7 \quad \text{… (5)}$$

(4) + (5) $\Rightarrow$ $6f = -6 \Rightarrow f = -1$

(5) $\Rightarrow$ $g - 2 = -7 \Rightarrow g = -5$

$\therefore$ The required equation of the circle is $x^2 + y^2 - 10x - 2y + 9 = 0$

EXERCISE 5.7

(1) Show that the circles $x^2 + y^2 - 2x + 6y + 6 = 0$ and $x^2 + y^2 - 5x + 6y + 15 = 0$ touch each other.

(2) Show that each of the circles $x^2 + y^2 + 4y - 1 = 0$, $x^2 + y^2 + 6x + y + 8 = 0$ and $x^2 + y^2 - 4x - 4y - 37 = 0$ touches the other two.

(3) Find the equation of the circle concentric with the circle $x^2 + y^2 - 2x - 6y + 4 = 0$ and having radius 7.

(4) Find the equation of the circle which is concentric with the circle $x^2 + y^2 - 8x + 12y + 15 = 0$ and passes through the point (5, 4)

(5) Show that the circle $x^2 + y^2 - 8x - 6y + 21 = 0$ is orthogonal to the circle $x^2 + y^2 - 2y - 15 = 0$

(6) Find the circles which cuts orthogonally each of the following circles

(i) $x^2 + y^2 + 2x + 4y + 1 = 0$, $x^2+y^2 - 4x+3 = 0$ and $x^2 + y^2 + 6y + 5 = 0$

(ii) $x^2 + y^2 + 2x + 17y + 4 = 0$, $x^2 + y^2 + 7x + 6y + 11 = 0$ and $x^2 + y^2-x + 22y + 3 = 0$

(7) Find the equation of the circle which passes through (1, – 1) and cuts orthogonally each of the circles $x^2 + y^2 + 5x - 5y + 9 = 0$ and $x^2 + y^2 - 2x + 3y - 7 = 0$

(8) Find the equation of the circle which passes through (1, 1) and cuts orthogonally each of the circles $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$

6. TRIGONOMETRY

6.1 Introduction:

Trigonometry is one of the oldest branch of Mathematics. The word trigonometry means “triangle measurement”. In olden days trigonometry was mainly used as a tool for use in astronomy. The early Babylonians divided the circle into 360 equal parts, giving us degrees, perhaps because they thought that there were 360 days in a year.

The sine function was invented in India, perhaps around 300 to 400 A.D. By the end of ninth century, all six trigometric functions and identities relating them were known to the Arabs.

In its earlier stages trigonometry was mainly concerned with establishing relations between the sides and angles of a triangle, but now it finds its application in various branches of science such as surveying, engineering, navigation etc. For every branch of higher Mathematics a knowledge of trigonometry is essential.

6.1.1 Angles:

An angle is defined as the amount of rotation of a revolving line from the initial position to the terminal position. Counter–clockwise rotations will be called positive and the clockwise will be called negative.

Consider a rotating ray OA with its end point at the origin O.

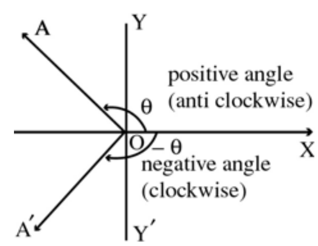


Fig. 6. 1

The rotating ray OA is often called the **terminal** side of the angle and the positive half of the *x*-axis (OX) is called the **initial side.**

The positive angle θ is $\angle XOA$ (counter-clockwise rotation)

The negative angle θ is $\angle XOA'$ (clockwise rotation)

Note : 1. one complete rotation (counter –clockwise) = 360° = 360 degree

2. If there is no rotation the measure of the angle is 0°.

6.1.2 Measurement of angles:

If a rotation from the initial position to the terminal position is $\left(\frac{1}{360}\right)^{\text{th}}$ of the revolution, the angle is said to have a measure of one degree and written as 1°. A degree is divided into minutes, and minute is divided into seconds.

i.e. 1 degree (1°) = 60 minutes ($60'$)

1 minute ($1'$) = 60 seconds ($60''$)

In theoretical work another system of measurement of angles is used which is known as circular measure. A radian is taken as the unit of measurement.

6.1.3 Radian measure:

Definition:

One radian, written as 1^c is the measure of an angle subtended at the centre O of a circle of radius r by an arc of length r.

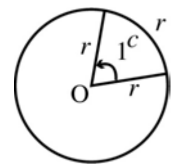


Fig. 6. 2

Note : 1. To express the measure of an angle as a real number, we use radian measure.

2. The word "radians" is optional and often omitted. Thus if no unit is given for a rotation, it is understood to be in radians.

3. 'c' in 1^c indicates the circular measure.

6.1.4 Relation between Degrees and Radians

Since a circle of radius r has a circumference of $2\pi r$, a circle of radius 1 unit (which is referred to as an unit circle) has circumference 2π. When θ is a complete rotation, P travels the circumference of an unit circle completely.

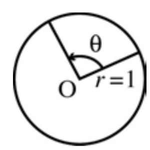


Fig. 6. 3

If θ is a complete rotation (counter-clockwise) then $\theta = 2\pi$ radian. On the other hand we already know that one complete rotation (counter-clockwise) is 360°, consequently, $360^\circ = 2\pi$ radians or $180^\circ = \pi$ radian. It follows that $1^\circ = \dfrac{\pi}{180}$ radian and $\dfrac{180^\circ}{\pi} = 1$ radian. Therefore $1^\circ = 0.01746$ radian (app.) and 1 radian $= 180^\circ \times \dfrac{7}{22} = 57^\circ\ 16'$ (app.).

Conversion for some special angles:

degrees	30°	45°	60°	90°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

(Table 6.1)

Example 6.1: Convert

(i) 150° into radians (ii) $\dfrac{3\pi}{4}$ into degrees (iii) $\dfrac{1}{4}$ radians into degrees.

Solution:

(i) $150° = 150 \times \frac{\pi}{180}$ radians $= \frac{5}{6}\pi$

(ii) $\frac{3\pi}{4}$ radians $= \frac{3\pi}{4} \times \frac{180°}{\pi} = 135°$

(iii) $\frac{1}{4}$ radians $= \frac{1}{4} \times \frac{180}{\pi} = \frac{1}{4} \times 180 \times \frac{7}{22} = 14°\ 19'\ 5''$

6.1.5 Quadrants

Let X′OX and YOY′ be two lines at right angles to each other as in the fig. (6.4) we call X′OX and YOY′ as *x*-axis and *y*-axis respectively.

Clearly these axes divide the entire plane into four equal parts, called quadrants.

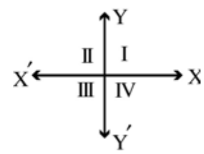


Fig. 6. 4

The parts XOY, YOX′, X′OY′ and Y′OX are known as the first, the second, the third and the fourth quadrant respectively.

Angle in standard position:

If the vertex of an angle is at O and its initial side lies along *x*-axis, then the angle is said to be in standard position.

Angle in a Quadrant:

An angle is said to be in a particular quadrant, if the terminal side of the angle in standard position lies in that quadrant.

Example 6.2: Find the quadrants in which the terminal sides of the following angles lie.

(i) – 60°

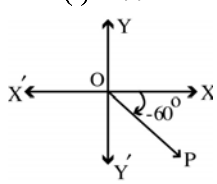


Fig. 6. 5 a

From Fig (6.5a) the terminal side of – 60° lies in IV quadrant.

(ii) – 300°

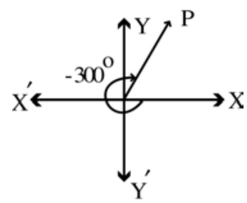


Fig. 6.5 b

From Fig (6.5b) the terminal side of – 300° lies in I quadrant.

(iii) 1295°

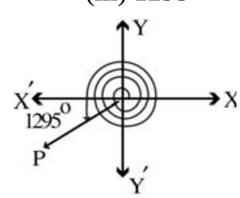


Fig. 6.5 c

From Fig (6.5c)

1295° = 3×360°+180° + 35°

The terminal side lies in III quadrant.

EXERCISE 6.1

(1) Convert the following degree measure into radian measure.

(i) 30° (ii) 100° (iii) 200° (iv) – 320°

(v) – 85° (vi) 7° 30′

(2) Find the degree measure corresponding to the following radian measure

(i) $\left(\frac{\pi}{8}\right)$ (ii) $\left(\frac{18\pi}{5}\right)$ (iii) – 3 (iv) $\left(\frac{7\pi}{12}\right)$

(3) Determine the quadrants in which the following degrees lie.

(i) 380° (ii) – 140° (iii) 1100°

6.2. Trigonometrical ratios and Identities

6.2.1 Trigonometrical ratios:

In the co-ordinate plane, consider a point A on the positive side of x-axis. Let this point revolve about the origin in the anti clockwise direction through an angle θ and reach the point P. Now $\underline{|XOP} = \theta$. Let the point P be (x, y). Draw PL perpendicular to OX.

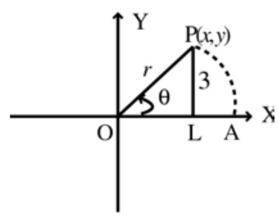


Fig. 6. 6

The triangle OLP is a right angled triangle, in which θ is in standard position. Also, from the ΔOLP, we have

OL = x = Adjacent side ; PL = y = opposite side ;

OP = $\sqrt{x^2 + y^2}$ = Hypotenuse (= $r > 0$)

The trigonometrical ratios (circular functions) are defined as follows :

The sine of the angle θ is defined as the ratio $\frac{y}{r}$ it is denoted by sinθ.

i.e. $\sin\theta = \frac{y}{r}$; cosecant value at $\theta = \frac{r}{y}$ = cosec θ ; $y \neq 0$

and $\cos\theta = \frac{x}{r}$; secant value at $\theta = \frac{r}{x}$ = sec θ ; $x \neq 0$

$\tan\theta = \frac{y}{x}$; cotangent value at $\theta = \frac{x}{y}$ = cot θ ; $y \neq 0$

Note : 1. From the definition, observe that tan θ and sec θ are not defined if $x = 0$, while cot θ and cosec θ are not defined if $y = 0$.

2. cosec θ, sec θ and cotθ are the reciprocals of sinθ, cos θ and tanθ respectively.

Example 6.3: If (2, 3) is a point on the terminal side of θ, find all the six trigonometrical ratios.

Solution:

P(x, y) is (2, 3) and it lies in the first quadrant.

$\therefore\ x = 2, \quad y = 3\ ;\ r = \sqrt{x^2 + y^2} = \sqrt{4+9} = \sqrt{13}$

$\therefore \sin\theta = \dfrac{y}{r} = \dfrac{3}{\sqrt{13}}\ ;\ \cos\theta = \dfrac{x}{r} = \dfrac{2}{\sqrt{13}}\ ;\ \tan\theta = \dfrac{y}{x} = \dfrac{3}{2}$

$\operatorname{cosec}\theta = \dfrac{\sqrt{13}}{3}\ ;\ \sec\theta = \dfrac{\sqrt{13}}{2}\ ;\ \cot\theta = \dfrac{2}{3}$

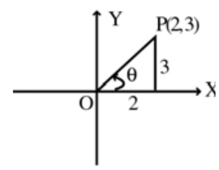


Fig. 6. 7

Note : 1. From example (6.3), we see that all the trigonometrical ratios are positive when the terminal side of angle θ lies in first quadrant.

Now, let us observe the sign of trigonometrical ratios if the point on the terminal side of θ lies in the other quadrants. (other than the first quadrant).

Example 6.4: If (– 2, – 3) is a point on the terminal side of θ. Find all the six trigonometrical ratios.

Solution:

P(x, y) is (– 2, – 3) and it lies in the third quadrant

$\therefore\ x = -2\ ,\ y = -3\ ;$

$r = \sqrt{x^2 + y^2} = \sqrt{4+9} = \sqrt{13}$

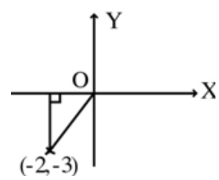


Fig. 6. 8

$\sin\theta = \dfrac{y}{r} = \dfrac{-3}{\sqrt{13}} = -\,ve\ ;\ \cos\theta = \dfrac{x}{r} = \dfrac{-2}{\sqrt{13}} = -\,ve\ ;\ \tan\theta = \dfrac{y}{x} = \dfrac{-3}{-2} = \dfrac{3}{2} = +\,ve$

$\operatorname{cosec}\theta = \dfrac{r}{y} = \dfrac{\sqrt{13}}{-3} = -\,ve\ ;\ \sec\theta = \dfrac{r}{x} = \dfrac{\sqrt{13}}{-2} = -\,ve\ ;\ \cot\theta = \dfrac{-2}{-3} = \dfrac{2}{3} = +\,ve$

As example illustrates, trigonometric functions may be negative. For instance, since r is always positive, $\sin\theta = \dfrac{y}{r}$ and $\operatorname{cosec}\theta = \dfrac{r}{y}$ have the same sign as y. Thus sin θ and cosec θ are positive when θ is in the first or second quadrants, and negative when θ is in the third or fourth quadrants. The signs of the other trigonometric functions can be found similarly. The following table indicates the signs depending on where θ lies.

Quadrants / Functions	I	II	III	IV
Sin	+	+	–	–
Cos	+	–	–	+
Tan	+	–	+	–
Cosec	+	+	–	–
Sec	+	–	–	+
Cot	+	–	+	–

II sin cosec	I All
III tan cot	IV cos sec

Table (6.2)

6.2.2 Trigonometrical ratios of particular angles:

Let X′OX and YOY′ be the co-ordinate axes. With O as centre and unit radius draw a circle cutting the co-ordinate axes at A, B, A′ and B′ as shown in the figure.

Suppose that a moving point starts from A and move along the circumference of the circle. Let it cover an arc length. θ and take the final position P. Let the co-ordinates of this point be P(x, y). Then by definition, $x = \cos\theta$ and $y = \sin\theta$.

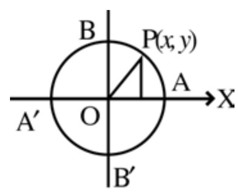


Fig. 6. 9

We consider the arc length θ to be positive or negative according as the variable point moves in the anti clockwise or clockwise direction respectively.

Range of cos θ and sinθ :

Since for every point (x, y) on the unit circle, we have

$-1 \le x \le 1$ and $-1 \le y \le 1$, therefore $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

Values of cosθ and sinθ for θ = 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π.

We know that the circumference of a circle of unit radius is 2π. If the moving point starts from A and moves in the anti clockwise direction then at the points A, B, A′, B′ and A the arc lengths covered are $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π respectively.

Also, the co-ordinates of these points are:A(1, 0), B(0, 1), A′(–1, 0), B′(0,– 1) and A(1, 0)

At the point:

A(1, 0), $\theta = 0 \Rightarrow \cos 0 = 1$ and $\sin 0 = 0$

B (0, 1),	$\theta = \frac{\pi}{2}$	$\Rightarrow$	$\cos\frac{\pi}{2}$	$= 0$	and	$\sin\frac{\pi}{2} = 1$
A′ (–1, 0),	$\theta = \pi$	$\Rightarrow$	$\cos\pi$	$= -1$	and	$\sin\pi = 0$
B′ (0, – 1),	$\theta = 3\frac{\pi}{2}$	$\Rightarrow$	$\cos 3\frac{\pi}{2}$	$= 0$	and	$\sin 3\frac{\pi}{2} = -1$
A (1, 0),	$\theta = 2\pi$	$\Rightarrow$	$\cos 2\pi$	$= 1$	and	$\sin 2\pi = 0$

6.2.3 Trigonometrical ratios of 30°, 45° and 60° :

Consider an isosceles right-angled triangle whose equal sides are 1. Its hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Its base angle is 45°.

$\therefore \sin 45° = \frac{1}{\sqrt{2}}$; $\cos 45° = \frac{1}{\sqrt{2}}$; $\tan 45° = 1$

$\operatorname{cosec} 45° = \sqrt{2}$; $\sec 45° = \sqrt{2}$; $\cot 45° = 1$

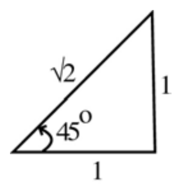


Fig. 6. 10

Opposite side = 1
adjacent side = 1
hypotenuse = $\sqrt{2}$

Consider an equilateral triangle ABC of side 2 units. Each of its angle is 60°. Let CD be the bisector of angle C. Then angle ACD is 30°. Also AD = 1 and CD = $\sqrt{2^2 - 1^2} = \sqrt{3}$. Now in the right angled triangle ACD

For 30°

opposite side = 1
adjacent side = $\sqrt{3}$
hypotenuse = 2

$\sin 30° = \frac{1}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$

$\therefore \operatorname{cosec} 30° = 2$

$\therefore \sec 30° = \frac{2}{\sqrt{3}}$

$\therefore \cot 30° = \sqrt{3}$

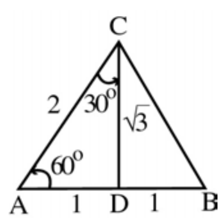


Fig. 6. 11

For 60°

opposite side = $\sqrt{3}$
adjacent side = 1
hypotenuse = 2

$\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{1}{2}$

$\tan 60° = \sqrt{3}$

$\therefore \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}$

$\sec 60° = 2$

$\therefore \cot 60° = \frac{1}{\sqrt{3}}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

Table 6.3

Important results:

For all values of θ, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Proof:

Let X′OX and YOY′ be the co-ordinate axes. With O as centre and unit radius draw a circle meeting OX at A. Now let a moving point start from A and move in anti clockwise direction and take the final position P(x, y) so that arc AP = θ.

On the other hand, if the point starts from A and moves in the clockwise direction through the arc length AP′ equal to arc length AP. Then arc AP′= $-\theta$.

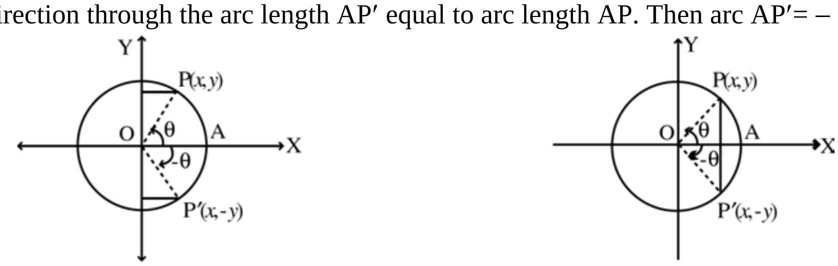


Fig. 6. 12

Thus $\angle AOP = \theta$ and $\angle AOP' = -\theta$

From the co-ordinate geometry, we know that the co-ordinates of P′ are (x, $-y$).

Clearly, $\cos\theta$ and $\cos(-\theta)$ are respectively the distances of points P and P′ from y axis and clearly each one of them is equal to x.

$\therefore \cos(-\theta) = \cos\theta$

Clearly, $\sin\theta$ and $\sin(-\theta)$ are respectively the distances of points P and P′ from x-axis. As $\sin\theta = y$ and $\sin(-\theta) = -y$, we have $\sin(-\theta) = -\sin\theta$

Deductions $\quad \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$; $\sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta$; $\cot(-\theta) = -\cot\theta$

6.2.4 T-ratios of (90° ± θ), (180° ± θ), (270° ± θ) and (360° ± θ):

It is evident that, when θ is a small angle (0 < θ < 90°), then 90° – θ, 90° + θ etc., are in the quadrants as given below:

Angle	Quadrant
90° – θ	Q_1 (first quadrant)
90° + θ	Q_2
180° – θ	Q_2
180° + θ	Q_3
270° – θ	Q_3
270° + θ	Q_4
360° – θ ; also equal to "– θ"	Q_4
360° + θ	Q_1

Table 6.4:

Let P(α, β) be a point in the first quadrant. Let $\underline{|XOP}$ = θ°.

$$\therefore \sin\theta = \frac{\beta}{OP} \;;\; \cos\theta = \frac{\alpha}{OP} \;;\; \tan\theta = \frac{\beta}{\alpha}$$

$$\operatorname{cosec}\theta = \frac{OP}{\beta} \;;\; \sec\theta = \frac{OP}{\alpha} \;;\; \cot\theta = \frac{\alpha}{\beta}$$

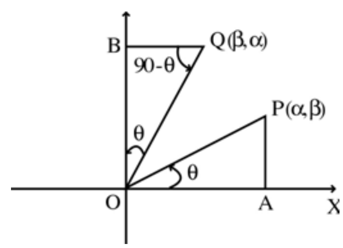


Fig. 6. 13

T-ratios of (90° – θ)

Let Q be a point in the first quadrant such that

$\underline{|XOQ}$ = 90° – θ and OQ = OP.

Let PA and QB be perpendicular to OX and OY respectively.

Then ΔOAP ≡ ΔOBQ and Q is (β, α). Hence

$$\sin(90° - \theta) = \frac{y \text{ co-ordinate of Q}}{OQ} = \frac{\alpha}{OP} = \cos\theta$$

$$\cos(90° - \theta) = \frac{x \text{ co-ordinate of Q}}{OQ} = \frac{\beta}{OP} = \sin\theta$$

$$\tan(90° - \theta) = \frac{y \text{ co-ordinate of Q}}{x \text{ coordinate of Q}} = \frac{\alpha}{\beta} = \cot\theta$$

Similarly, $\operatorname{cosec}(90° - \theta) = \sec\theta$

$\sec(90° - \theta) = \operatorname{cosec}\theta$

$\cot(90° - \theta) = \tan\theta$

T-ratios of (90° + θ)

Let R be a point in the second quadrant such that $\underline{|XOR}$ = 90° + θ and OR = OP

Let RC be perpendicular to x axis.

Then ΔOAP ≡ ΔRCO and R is (– β, α), Hence

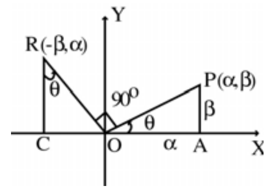


Fig. 6. 14

$$\sin(90° + \theta) = \frac{y \text{ co-ordinate of R}}{\text{OR}} = \frac{\alpha}{\text{OP}} = \cos\theta$$

$$\cos(90° + \theta) = \frac{x \text{ co-ordinate of R}}{\text{OR}} = \frac{-\beta}{\text{OP}} = -\sin\theta$$

$$\tan(90° + \theta) = \frac{y \text{ co-ordinate of R}}{x \text{ coordinate of R}} = \frac{\alpha}{-\beta} = -\cot\theta$$

Similarly, cosec (90° + θ) = sec θ

sec (90° + θ) = – cosec θ

cot (90° + θ) = – tan θ

T-ratios of (180° – θ)

Let S be a point in the second quadrant such that $\underline{|XOS}$ =180° –θ and OS =OP

Draw SD perpendicular to x-axis

Thus Δ OAP ≡ Δ ODS and S is (– α, β). Hence

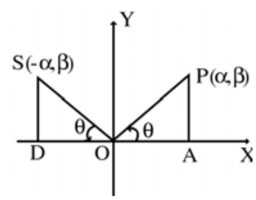


Fig. 6. 15

$$\sin(180° - \theta) = \frac{y \text{ co-ordinate of S}}{\text{OS}} = \frac{\beta}{\text{OP}} = \sin\theta$$

$$\cos(180° - \theta) = \frac{x \text{ co-ordinate of S}}{\text{OS}} = \frac{-\alpha}{\text{OP}} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{y \text{ co-ordinate of S}}{x \text{ coordinate of S}} = \frac{\beta}{-\alpha} = -\tan\theta$$

Similarly, cosec (180° – θ) = cosec θ

sec (180° – θ) = – sec θ

cot (180° – θ) = – cot θ

T-ratios of (180° + θ)

Let T be a point in the third quadrant such that $\underline{|XOT}$ = 180° + θ and OT = OP

Draw TE perpendicular to x-axis

Then ΔOAP ≡ ΔOET and T is (– α, – β). Hence

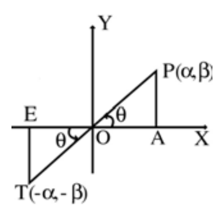


Fig. 6. 16

$$\sin(180° + \theta) = \frac{y \text{ co-ordinate of T}}{\text{OT}} = \frac{-\beta}{\text{OP}} = -\sin\theta$$

$$\cos(180° + \theta) = \frac{x \text{ co-ordinate of T}}{\text{OT}} = \frac{-\alpha}{\text{OP}} = -\cos\theta$$

$$\tan(180° + \theta) = \frac{y \text{ co-ordinate of T}}{x \text{ coordinate of T}} = \frac{-\beta}{-\alpha} = \tan\theta$$

Similarly, $\text{cosec}\ (180° + \theta) = -\text{cosec}\ \theta$

$\sec\ (180° + \theta) = -\sec\theta$

$\cot\ (180° + \theta) = \cot\theta$

Remark: To determine the trigonometric ratios of any angle, follow the procedure given below

(i) Write the angle in the form $k\frac{\pi}{2} \pm \theta$; $k \in$ Z.

(ii) Determine the quadrant in which the terminal side of the angle lies.

(iii) Detrmine the sign of the given trigonometric function in that particular quadrant, using $\frac{\text{S} \quad \text{A}}{\text{T} \quad \text{C}}$ rule.

(iv) If k is even, trigonometric function of allied angle equals the same function of θ.

(v) If k is odd, then adopt the following changes:

sine ↔ cos ; tan ↔ cot ; sec ↔ cosec

Trigonometrical ratios for related angles

Angle / function	− θ	90 − θ	90 + θ	180 −θ	180+θ	270−θ	270+θ	360−θ or − θ
sin	−sinθ	**cos** θ	**cos** θ	**sin** θ	− **sin** θ	−**cos** θ	−**cos** θ	− **sin** θ
cos	**cos** θ	**sin** θ	− **sin** θ	−**cos** θ	−**cos** θ	−**sin** θ	**sin** θ	**cos** θ
tan	−**tan** θ	**cot** θ	−**cot** θ	−**tan** θ	**tan** θ	**cot** θ	−**cot** θ	−**tan** θ
cosec	−**cosec**θ	**sec** θ	**sec** θ	**cosec** θ	−**cosec**θ	− **sec** θ	−**sec** θ	−**cosec**θ
sec	**sec** θ	**cosec** θ	−**cosec**θ	− **sec** θ	− **sec** θ	−**cosec**θ	**cosec** θ	**sec** θ
cot	− **cot** θ	**tan** θ	− **tan** θ	− **cot** θ	**cot** θ	**tan** θ	−**tan** θ	− **cot** θ

Table 6.5

Note : Since 360° corresponds to one full revolution, sine of the angles 360°+45°;720°+45°;1080°+45° are equal to sine of 45°. This is so for the other trigonemetrical ratios. That is, when an angle exceeds 360°, it can be reduced to an angle between 0° and 360° by wiping out integral multiples of 360°.

Example 6.5:

Simplify : (i) tan 735° (ii) cos 980° (iii) sin 2460° (iv) cos (–870°)
(v) sin (–780°) (vi) cot (–855°) (vii) cosec 2040° (viii) sec (– 1305°)

Solution:

(i) $\tan(735°) = \tan(2 \times 360° + 15°) = \tan 15°$

(ii) $\cos 980° = \cos(2 \times 360° + 260°) = \cos 260°$

$= \cos(270° - 10°) = -\sin 10°$

(iii) $\sin(2460°) = \sin(6 \times 360° + 300°) = \sin(300°)$

$= \sin(360° - 60°)$

$= -\sin 60°$

$= -\dfrac{\sqrt{3}}{2}$

(iv) $\cos(-870°) = \cos(870°) = \cos(2 \times 360° + 150°)$

$= \cos 150 = \cos(180° - 30°)$

$= -\cos 30° = -\dfrac{\sqrt{3}}{2}$

(v) $\sin(-780°) = -\sin 780°$

$= -\sin(2 \times 360° + 60°)$

$= -\sin 60° = -\dfrac{\sqrt{3}}{2}$

(vi) $\cot(-855°) = -\cot(855°) = -\cot(2 \times 360° + 135°)$

$= -\cot(135°) = -\cot(180° - 45°)$

$= \cot 45° = 1$

(vii) $\mathrm{cosec}(2040°) = \mathrm{cosec}(5 \times 360° + 240°) = \mathrm{cosec}(240°)$

$= \mathrm{cosec}(180° + 60°) = -\mathrm{cosec}(60°)$

$= -\dfrac{2}{\sqrt{3}}$

(viii) $\sec(-1305°) = \sec(1305°) = \sec(3 \times 360° + 225°)$

$= \sec(225°) = \sec(270° - 45°)$

$= -\mathrm{cosec}\, 45° = -\sqrt{2}$

Example 6.6: Simplify : $\dfrac{\cot(90° - \theta)\ \sin(180° + \theta)\ \sec(360° - \theta)}{\tan(180° + \theta)\ \sec(-\theta)\ \cos(90° + \theta)}$

Solution: The given expression $= \dfrac{\tan\theta\,(-\sin\theta)\,(\sec\theta)}{\tan\theta\,(\sec\theta)\,(-\sin\theta)}$

$= 1$

Example 6.7:

Without using the tables, prove that $\sin 780^\circ\ \sin 480^\circ + \cos 120^\circ \cos 60^\circ = \frac{1}{2}$

Solution:

$$\sin 780^\circ = \sin(2 \times 360^\circ + 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 480^\circ = \sin(360^\circ + 120^\circ)$$

$$= \sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}\ ;\ \cos 60^\circ = \frac{1}{2}$$

$$\text{L.H.S.} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}\ \text{ R.H.S.}$$

6.2.5 Special properties of Trigonometrical functions:

Periodic function:

A function $f(x)$ is said to be a periodic function with period α if $f(x + \alpha)= f(x)$. The least positive value of α is called the fundamental period of the function.

All the circular functions (trigonometrical functions) are periodic functions.

For example,

$$\sin(x + 2\pi) = \sin x\ ;\ \sin(x + 4\pi) = \sin x\ ;\ \sin(x + 6\pi) = \sin x$$

$$\sin(x + 2n\pi) = \sin x,\ n \in Z$$

Here $\alpha = \ldots\ldots -6\pi, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$. But the fundamental period must be the least positive quantity. Therefore $\alpha = 2\pi$ is the fundamental period.

Thus sine function is a periodic function with fundamental period 2π. Similarly one can prove that the functions $\cos x$, $\operatorname{cosec} x$ and $\sec x$ are also periodic functions with fundamental period 2π while $\tan x$ and $\cot x$ are periodic with fundamental period π.

6.2.6 Odd and even functions:

We know that, if $f(x) = f(-x)$, then the function is an even function and if $f(-x) = -f(x)$ then the function is an odd function.

Consider $f(x) = \sin x$; $f(-x) = \sin(-x) = -\sin x = -f(x)$ i.e. $f(x) = -f(-x)$

$\therefore$ $\sin x$ is an odd function. Similarly we can prove that cosec x, tan x and cot x are odd functions.

Consider $f(x) = \cos x$; $f(-x) = \cos(-x) = \cos x = f(x)$. $\therefore$ $\cos x$ is an even function. Similarly we can prove sec x is an even function.

Note : We can read more about odd and even function in Chapter 7.

EXERCISE 6.2

(1) If $\sin\theta = \frac{11}{12}$, find the value of

$\sec(360° - \theta) \cdot \tan(180° - \theta) + \cot(90° + \theta)\sin(270° + \theta)$

(2) Express the following as functions of positive acute angles:-

(i) $\sin(-840°)$ (ii) $\cos(1220°)$ (iii) $\cot(-640°)$ (iv) $\tan(300°)$

(v) $\operatorname{cosec}(420°)$ (vi) $\sin(-1110°)$ (vii) $\cos(-1050°)$

(3) Prove that $\dfrac{\sin 300° \cdot \tan 330° \cdot \sec 420°}{\cot 135° \cdot \cos 210° \cdot \operatorname{cosec} 315°} = -\sqrt{\dfrac{2}{3}}$

(4) Prove that $\left\{1+\cot\alpha - \sec\left(\alpha + \frac{\pi}{2}\right)\right\}\left\{1 + \cot\alpha + \sec\left(\alpha + \frac{\pi}{2}\right)\right\} = 2\cot\alpha$

(5) Express the following as functions of A :

(i) $\sec\left(A - \frac{3\pi}{2}\right)$ (ii) $\operatorname{cosec}\left(A - \frac{\pi}{2}\right)$ (iii) $\tan\left(A - \frac{3\pi}{2}\right)$

(iv) $\cos(720° + A)$ (v) $\tan(A + \pi)$

(6) Prove that $\dfrac{\sin(180°+A) \cdot \cos(90°-A) \cdot \tan(270°-A)}{\sec(540° - A)\cos(360°+A)\operatorname{cosec}(270°+A)} = -\sin A\cos^2 A$

(7) Prove that $\sin\theta \cdot \cos\theta\left\{\sin\left(\frac{\pi}{2} - \theta\right) \cdot \operatorname{cosec}\theta + \cos\left(\frac{\pi}{2} - \theta\right)\sec\theta\right\} = 1$

(8) Find the values of :-

(i) $\cos(135°)$ (ii) $\sin(240°)$ (iii) $\sec(225°)$ (iv) $\cos(-150°)$

(v) $\cot(315°)$ (vi) $\operatorname{cosec}(-300°)$ (vii) $\cot\frac{5\pi}{4}$ (viii) $\tan\left(-\frac{5\pi}{6}\right)$

(9) If A, B, C, D are angles of a cyclic quadrilateral prove that
$\cos A + \cos B + \cos C + \cos D = 0$.

(10) Find the values of the following expressions :

(i) $\tan^2 30° + \tan^2 45° + \tan^2 60°$ (ii) $\sin\frac{\pi}{6} \cdot \cos\frac{\pi}{3} + \cos\frac{\pi}{6} \cdot \sin\frac{\pi}{3}$

(iii) $\cos\frac{\pi}{6} \cdot \cos\frac{\pi}{3} - \sin\frac{\pi}{6} \cdot \sin\frac{\pi}{3}$ (iv) $\cos 45°.\cos 60° - \sin 45°.\sin 60°$

(v) $\tan^2 60° + 2\tan^2 45°$ (vi) $\tan^2 45° + 4\cos^2 60°$

(vii) $\cot 60° \cdot \tan 30° + \sec^2 45° \cdot \sin 90°$

(viii) $\tan^2 60° + 4\cot^2 45° + 3\sec^2 30° + \cos^2 90°$

(ix) $\tan^2 30° + 2\sin 60° + \tan 45° - \tan 60° + \cos^2 30°$

(x) $\frac{1}{2}\sin^2 60° - \frac{1}{2}\sec 60° \tan^2 30° + \frac{4}{5}\sin^2 45° \cdot \tan^2 60°$

(11) If $\cos\theta = -\frac{1}{2}$ and $\tan\theta > 0$ show that $\dfrac{5\tan\theta + 4\sin\theta}{\sqrt{3}\cos\theta - 3\sin\theta} = 3$.

6.2.7 Trigonometrical identities:

As in variables, $\sin\theta \cdot \sin\theta = (\sin\theta)^2$. This will be written as $\sin^2\theta$. Similarly

$\tan\theta \quad \tan^2\theta = \tan^3\theta$ etc. We can derive some fundamental trigonometric identities as follows:

Consider the unit circle with centre at the origin O. Let P(x, y) be any point on the circle with $\angle XOP = \theta$.

Draw PL perpendicular to OX. Now, triangle OLP is a right angled triangle in which (hypotenuse) OP = r = 1 unit, and x and y are adjacent and opposite sides respectively.

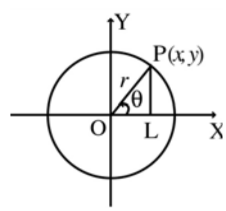


Fig. 6. 17

Now we have $\cos\theta = \frac{x}{1} = x$ and $\sin\theta = \frac{y}{1} = y$; $\tan\theta = \frac{y}{x}$

and $r^2 = x^2 + y^2 = 1$

From ΔOLP, we have $x^2 + y^2 = r^2 = 1$

i.e. $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$

$$1 + \tan^2\theta = 1 + \frac{y^2}{x^2} = \frac{x^2 + y^2}{x^2} = \left(\frac{1}{x}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2 = \sec^2\theta$$

$$1 + \cot^2\theta = 1 + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2} = \left(\frac{1}{y}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2 = \text{cosec}^2\theta$$

Thus we have the identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \text{cosec}^2\theta$$

$\therefore$ From these we also have

$$\sec^2\theta - \tan^2\theta = 1$$
$$\text{cosec}^2\theta - \cot^2\theta = 1$$

Example 6.8: Show that $\cos^4 A - \sin^4 A = 1 - 2\sin^2 A$

Solution:
$$\begin{aligned}\cos^4 A - \sin^4 A &= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)\\ &= \cos^2 A - \sin^2 A = 1 - \sin^2 A - \sin^2 A\\ &= 1 - 2\sin^2 A\end{aligned}$$

Example 6.9: Prove that $\sec^2 A + \text{cosec}^2 A = \sec^2 A \,.\, \text{cosec}^2 A$

Solution:
$$\begin{aligned}\sec^2 A + \text{cosec}^2 A &= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}\\ &= \frac{\sin^2 A + \cos^2 A}{\cos^2 A \,.\, \sin^2 A} = \frac{1}{\cos^2 A \,.\, \sin^2 A}\\ &= \sec^2 A \,.\, \text{cosec}^2 A\end{aligned}$$

Example 6.10: Show that $\cos A\sqrt{1 + \cot^2 A} = \sqrt{\text{cosec}^2 A - 1}$

Solution:
$$\begin{aligned}\cos A\sqrt{1 + \cot^2 A} &= \cos A\sqrt{\text{cosec}^2 A} = \cos A \,.\, \text{cosec} A\\ &= \frac{\cos A}{\sin A} = \cot A = \sqrt{\text{cosec}^2 A - 1}\end{aligned}$$

Example 6.11: If $a\sin^2\theta + b\cos^2\theta = c$, show that $\tan^2\theta = \dfrac{c - b}{a - c}$

Solution:
$$a\sin^2\theta + b\cos^2\theta = c.$$

Dividing both sides by $\cos^2\theta$, we get $a\tan^2\theta + b = c\sec^2\theta$

$$\begin{aligned}a\tan^2\theta + b &= c(1 + \tan^2\theta)\\ \tan^2\theta(a - c) &= c - b\\ \therefore \tan^2\theta &= \frac{c - b}{a - c}\end{aligned}$$

Example 6.12: that $\sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \text{cosec}\, A - \cot A$

Solution: consider, $\dfrac{1 - \cos A}{1 + \cos A} = \dfrac{1 - \cos A}{1 + \cos A} \times \dfrac{1 - \cos A}{1 - \cos A}$

$$= \frac{(1-\cos A)^2}{1-\cos^2 A} = \left(\frac{1-\cos A}{\sin A}\right)^2$$

$$\therefore \sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{1-\cos A}{\sin A} = \frac{1}{\sin A} - \frac{\cos A}{\sin A}$$

$$= \operatorname{cosec} A - \cot A$$

Example 6.13:

If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, show that $x^2 + y^2 = a^2 + b^2$

Solution:

$$\begin{aligned} x^2 + y^2 &= (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2 \\ &= a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta \\ &\quad + a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta \\ &= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) \\ &= a^2 + b^2 \end{aligned}$$

Example 6.14: Show that $\sin^2 A.\tan A + \cos^2 A \,.\, \cot A + 2\sin A \,.\, \cos A = \tan A + \cot A$

Solution:

$$\begin{aligned} \text{L.H.S.} &= \sin^2 A \,.\, \frac{\sin A}{\cos A} + \cos^2 A \,.\, \frac{\cos A}{\sin A} + 2\sin A\cos A \\ &= \frac{\sin^3 A}{\cos A} + \frac{\cos^3 A}{\sin A} + 2\sin A \,.\, \cos A \\ &= \frac{\sin^4 A + \cos^4 A + 2\sin^2 A \,.\, \cos^2 A}{\sin A \,.\, \cos A} \\ &= \frac{(\sin^2 A + \cos^2 A)^2}{\sin A \,.\, \cos A} = \frac{1}{\sin A \,.\, \cos A} \\ &= \frac{\sin^2 A + \cos^2 A}{\sin A \,.\, \cos A} \qquad [\because \sin^2 A + \cos^2 A = 1] \\ &= \frac{\sin^2 A}{\sin A\cos A} + \frac{\cos^2 A}{\sin A\cos A} \end{aligned}$$

Hence the result $\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \tan A + \cot A = \text{R.H.S.}$

Example 6.15: Show that $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$

Solution:

$$\begin{aligned} (\sin x - \cos x)^4 &= \left[(\sin x - \cos x)^2\right]^2 = [\sin^2 x + \cos^2 x - 2\sin x\cos x]^2 \\ &= [1 - 2\sin x\cos x]^2 \\ &= 1 - 4\sin x\cos x + 4\sin^2 x\cos^2 x \qquad \text{...... (i)} \\ (\sin x + \cos x)^2 &= \sin^2 x + \cos^2 x + 2\sin x \,.\, \cos x \\ &= 1 + 2\sin x\cos x \qquad \text{...... (ii)} \\ \sin^6 x + \cos^6 x &= (\sin^2 x)^3 + (\cos^2 x)^3 \end{aligned}$$

$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \,.\, \cos^2 x\, (\sin^2 x + \cos^2 x)$

$= 1 - 3\sin^2 x\ \cos^2 x$ …… (iii)

Using (i), (ii) and (iii)L.H.S.

$= 3(1 - 4\sin x \cos x + 4\sin^2 x \,.\, \cos^2 x)$

$+ 6(1 + 2\sin x\ \cos x) + 4(1 - 3\sin^2 x\ \cos^2 x)$

$= 3 + 6 + 4$

$= 13 =$ R.H.S.

Example 6.16: Prove that $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1 + \sin\theta}{\cos\theta}$

Solution: L.H.S. $= \dfrac{\tan\theta + \sec\theta - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$

$= \dfrac{\tan\theta + \sec\theta - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\tan\theta - \sec\theta + 1}$

$= \dfrac{(\tan\theta + \sec\theta)(1 - \sec\theta + \tan\theta)}{(\tan\theta - \sec\theta + 1)} = \tan\theta + \sec\theta$

$= \dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta} = \dfrac{\sin\theta + 1}{\cos\theta} =$ R.H.S.

EXERCISE 6.3

(1) Prove the following:

(i) $\sin^4 A - \cos^4 A = 1 - 2\cos^2 A$

(ii) $\sin^3 A - \cos^3 A = (\sin A - \cos A)(1 + \sin A\ \cos A)$

(iii) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

(iv) $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \mathrm{cosec}^2\theta$

(v) $\dfrac{1}{1 + \sin\theta} + \dfrac{1}{1 - \sin\theta} = 2\sec^2\theta$ (vi) $\dfrac{\sec x + \tan x}{\sec x - \tan x} = (\sec x + \tan x)^2$

(vii) $\dfrac{\mathrm{cosec}\ \theta}{\cot\theta + \tan\theta} = \cos\theta$ (viii) $\dfrac{1}{\tan\theta + \sec\theta} = \sec\theta - \tan\theta$

(ix) $\dfrac{1}{\mathrm{cosec}\theta - \cot\theta} = \dfrac{1 + \cos\theta}{\sin\theta}$

(x) $(\sec\theta + \cos\theta)(\sec\theta - \cos\theta) = \tan^2\theta + \sin^2\theta$

(2) If $\tan\theta + \sec\theta = x$, show that $2\tan\theta = x - \dfrac{1}{x}$, $2\sec\theta = x + \dfrac{1}{x}$

Hence show that $\sin\theta = \dfrac{x^2 - 1}{x^2 + 1}$

(3) If $\tan\theta + \sin\theta = p$, $\tan\theta - \sin\theta = q$ and $p > q$ then show that $p^2 - q^2 = 4\sqrt{pq}$

(4) Prove that $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\mathrm{cosec}^2 A} - \dfrac{\mathrm{cosec} A}{\sec^2 A}$

(5) Prove that $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \sin A + \cos A$

(6) Prove the following :

(i) $\sqrt{\dfrac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$ $(\sin A \neq 1)$

(ii) $\sqrt{\dfrac{1 + \cos A}{1 - \cos A}} = \mathrm{cosec} A + \cot A$ $(\cos A \neq 1)$

(iii) $\sqrt{\dfrac{1 - \sin\theta}{1 + \sin\theta}} = \sec\theta - \tan\theta$

(7) If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, show that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

(8) Prove that $(1 + \tan A + \sec A)(1 + \cot A - \mathrm{cosec} A) = 2$

6.3 Compound Angles

6.3.1 Compound Angles A + B and A – B

In the previous chapter we have found the trigonometrical ratios of angles such as $90° \pm \theta$, $180° \pm \theta$, … which involves single angle only. In this chapter we shall express the trigonometrical ratios of compound angles such as A + B, A – B, … interms of trigonometrical ratios of A, B, ….

It is important to note that the relation $f(x + y) = f(x) + f(y)$ is not true for all functions of a real variable. As an example all the six trigonometrical ratios do not satisfy the above relation.

cos (A + B) is not equal to cosA + cosB.

Let us develop the identity

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

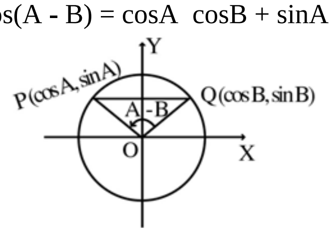


Fig. 6. 18

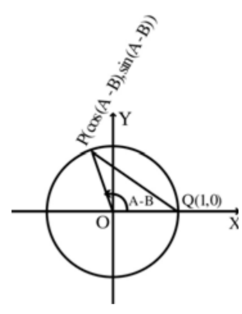


Fig. 6. 19

Let P and Q be any two points on the unit circle such that $\angle XOP = A$ and $\angle XOQ = B$. Then the coordinates of P and Q are (cos A, sinA) and (cosB, sinB) respectively.

$$\begin{aligned}
PQ^2 &= (\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\
&= (\cos^2 A - 2\cos A \cos B + \cos^2 B) + (\sin^2 A - 2\sin A \sin B + \sin^2 B) \\
&= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2\cos A \cos B - 2\sin A \cos B \\
&= 1 + 1 - 2\cos A \cos B - 2\sin A \sin B = 2 - 2(\cos A \cos B + \sin A \sin B) \quad \ldots (1)
\end{aligned}$$

Now imagine that the unit circle above is rotated so that the point Q is at (1, 0). The length PQ has not changed.

$$\begin{aligned}
PQ^2 &= [\cos(A - B) - 1]^2 + [\sin(A - B) - 0]^2 \\
&= [\cos^2(A - B) - 2\cos(A - B) + 1] + \sin^2(A - B) \\
&= [\cos^2(A - B) + \sin^2(A - B)] + 1 - 2\cos(A - B) = 1 + 1 - 2\cos(A - B) \\
&= 2 - 2\cos(A - B) \qquad \ldots (2)
\end{aligned}$$

From (1) and (2), $2 - 2\cos(A - B) = 2 - 2(\cos A \cos B + \sin A \sin B)$

$\Rightarrow \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$

Next let us consider cos(A + B). This is equal to $\cos[A - (-B)]$ and by cosine of a difference identity, we have the following:

$$\cos(A + B) = \cos A \cos(-B) + \sin A \cdot \sin(-B)$$

But $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B.$$

To develop an identity for sin(A + B), we recall the following:

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

In this identity we shall substitute A + B for θ

$$\sin(A + B) = \cos\left[\frac{\pi}{2} - (A + B)\right] = \cos\left[\left(\frac{\pi}{2} - A\right) - B\right]$$

We can now use the identity for the cosine of a difference.

$$\begin{aligned}
&= \cos\left(\frac{\pi}{2} - A\right) \cdot \cos B + \sin\left(\frac{\pi}{2} - A\right) \cdot \sin B \\
&= \sin A \cdot \cos B + \cos A \cdot \sin B
\end{aligned}$$

Thus, $\sin(A + B) = \sin A \cdot \cos B + \cos A \sin B$

To find an identity for the sine of a difference, we can use the identity just derived, substituting – B for B

$$\sin(A - B) = \sin[A + (-B)]$$
$$= \sin A\cos(-B) + \cos A \,.\, \sin(-B)$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$

An identity for the tangent of a sum can be derived using identities already established.

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$
$$= \frac{\sin A \,.\, \cos B + \cos A \sin B}{\cos A\cos B - \sin A\sin B}$$

Divide both Numerator and Denominator by cosA cosB

$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}$$

Similarly, an identity for the tangent of a difference can be established.

It is given by $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \,.\, \tan B}$

(1) $\sin(A + B) = \sin A\cos B + \cos A\sin B$

(2) $\sin(A - B) = \sin A\cos B - \cos A\sin B$

(3) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

(4) $\cos(A - B) = \cos A\cos B + \sin A\sin B$

(5) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(6) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \,.\, \tan B}$

Example 6.17: Find the values of (i)cos 15° (ii)cos 105° (iii)sin 75° (iv)tan 15°

Solution:

(i) $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}\,\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\,\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(ii) $\cos 105° = \cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$

$$= \frac{1}{2}\ \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\ \frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

(iii) $\sin 75° = \sin(45° + 30°) = \sin 45°\ \cos 30° + \cos 45°\ \sin 30°$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(iv) $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45°\ \tan 30°} = \dfrac{1 - \dfrac{1}{\sqrt{3}}}{1 + 1 \cdot \dfrac{1}{\sqrt{3}}}$

$$= \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2 - \sqrt{3}$$

Example 6.18: If A, B are acute angles, $\sin A = \frac{3}{5}$; $\cos B = \frac{12}{13}$, find $\cos(A + B)$

Solution: $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$$

$$\therefore \cos(A + B) = \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{33}{65}$$

Example 6.19: Show that (i) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$

$$\begin{aligned}
\sin(A+B)\sin(A-B) &= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \\
&= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\
&= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B \\
&= \sin^2 A - \sin^2 B
\end{aligned}$$

$$\begin{aligned}
\cos(A+B)\cos(A-B) &= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\
&= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\
&= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B \\
&= \cos^2 A - \sin^2 B
\end{aligned}$$

Example 6.20: If $A + B = 45°$, show that $(1 + \tan A)(1 + \tan B) = 2$ and hence deduce the value of $\tan 22\frac{1}{2}°$

Solution: Given $A + B = 45° \Rightarrow \tan(A + B) = \tan 45°$

$$\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = 1$$

i.e. $\tan A + \tan B = 1 - \tan A \cdot \tan B$

i.e. $1 + \tan A + \tan B = 2 - \tan A \tan B$ (add 1 on both sides)

$1 + \tan A + \tan B + \tan A \tan B = 2$

i.e. $(1 + \tan A)(1 + \tan B) = 2$

Take A = B then $2A = 45° \Rightarrow A = 22\frac{1}{2} = B$

$$\therefore \left(1 + \tan 22\frac{1}{2}\right)^2 = 2 \quad \Rightarrow \quad 1 + \tan 22\frac{1}{2} = \pm\sqrt{2}$$

$$\therefore \tan 22\frac{1}{2} = \pm\sqrt{2} - 1$$

Since $22\frac{1}{2}$ is acute, $\tan 22\frac{1}{2}$ is positive and therefore $\tan 22\frac{1}{2} = \sqrt{2} - 1$

Example 6.21:

(i) Prove that $\dfrac{\tan 69° + \tan 66°}{1 - \tan 69° \ \tan 66} = -1$ (ii) $\dfrac{\tan(A - B) + \tan B}{1 - \tan(A - B)\tan B} = \tan A$

(iii) $\dfrac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°} = \tan 62°$

Solution:

(i) $\dfrac{\tan 69° + \tan 66°}{1 - \tan 69° \ \tan 66°} = \tan(69° + 66°)$

$= \tan(135°) = \tan(90° + 45°) = -\cot 45° = -1$

(ii) $\dfrac{\tan(A - B) + \tan B}{1 - \tan(A - B)\tan B} = \tan\left[(A - B) + B\right] = \tan A$

(iii) L.H.S. $= \dfrac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°}$

Divide both Numerator and Denominator by cos17°

L.H.S. $= \dfrac{1 + \tan 17°}{1 - \tan 17°} = \dfrac{\tan 45° + \tan 17°}{1 - \tan 45° \ \tan 17°}$ ($\because \tan 45° = 1$)

$= \tan(45° + 17°) = \tan 62° =$ R.H.S.

Example 6.22: Prove that (i) $\tan\left(\dfrac{\pi}{4} + \theta\right) \ \tan\left(\dfrac{\pi}{4} - \theta\right) = 1$

(ii) If $\tan A = 3$ and $\tan B = \dfrac{1}{2}$, prove that $A - B = \dfrac{\pi}{4}$

Solution:

(i) $$\text{L.H.S.} = \tan\left(\frac{\pi}{4} + \theta\right) \tan\left(\frac{\pi}{4} - \theta\right)$$

$$= \left(\frac{1 + \tan\theta}{1 - \tan\theta}\right)\left(\frac{1 - \tan\theta}{1 + \tan\theta}\right) = 1 \qquad \left(\because \tan\frac{\pi}{4} = 1\right)$$

(ii) $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{3 - \frac{1}{2}}{1 + 3 \cdot \frac{1}{2}} = \frac{\frac{5}{2}}{\frac{5}{2}} = 1 = \tan\frac{\pi}{4}$$

$$\tan(A - B) = \tan\frac{\pi}{4} \Rightarrow A - B = \frac{\pi}{4}$$

Example 6.23: If $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$ where $(\alpha + \beta)$ and $(\alpha - \beta)$ are acute, find $\tan 2\alpha$.

Solution:

$$\cos(\alpha + \beta) = \frac{4}{5} \Rightarrow \tan(\alpha + \beta) = \frac{3}{4}$$

$$\sin(\alpha - \beta) = \frac{5}{13} \Rightarrow \tan(\alpha - \beta) = \frac{5}{12}$$

$$2\alpha = (\alpha + \beta) + (\alpha - \beta)$$

$$\therefore \tan 2\alpha = \tan[(\alpha + \beta) + (\alpha - \beta)]$$

$$= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta) \cdot \tan(\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}} = \frac{\frac{14}{12}}{\frac{11}{16}} = \frac{56}{33}$$

Example 6.24: Prove that $\tan 3A - \tan 2A - \tan A = \tan A \tan 2A \tan 3A$

Solution:

$$\tan 3A = \tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

i.e. $$\tan 3A(1 - \tan A \tan 2A) = \tan A + \tan 2A$$

i.e. $$\tan 3A - \tan A \tan 2A \tan 3A = \tan A + \tan 2A$$

$$\therefore \tan 3A - \tan 2A - \tan A = \tan A \tan 2A \tan 3A$$

EXERCISE 6.4

(1) Find the values of (i) $\sin 15°$ (ii) $\cos 75°$ (iii) $\tan 75°$ (iv) $\sin 105°$

(2) Prove that

(i) $\sin(45°+A) = \frac{1}{\sqrt{2}}(\sin A + \cos A)$ (ii) $\cos(A+45°) = \frac{1}{\sqrt{2}}(\cos A - \sin A)$

(3) Prove that

(i) $\sin(45° + A) - \cos(45° + A) = \sqrt{2}\ \sin A$

(ii) $\sin(30° + A) + \sin(30° - A) = \cos A$

(4) Prove that (i) $\cos(A + B)\ \cos(A - B) = \cos^2 B - \sin^2 A$

(ii) $\sin(A + B)\sin(A - B) = \cos^2 B - \cos^2 A$

(5) Prove that $\cos^2 15° + \cos^2 45° + \cos^2 75° = \dfrac{3}{2}$

(6) Prove that (i) $\sin A + \sin(120° + A) + \sin(240° + A) = 0$

(ii) $\cos A + \cos(120° + A) + \cos(120° - A) = 0$

(7) Show that

(i) $\cos 15° - \sin 15° = \dfrac{1}{\sqrt{2}}$ (ii) $\tan 15° + \cot 15° = 4$ (iii) $\cot 75° + \tan 75° = 4$

(8) (i) Find $\sin 45° + \sin 30°$ and compare with $\sin 75°$

(ii) Find $\cos 45° - \cos 30°$ and compare with $\cos 15°$.

(9) Show that

(i) $\tan 70° = 2\tan 50° + \tan 20°$

(ii) $\tan 72° = \tan 18° + 2\tan 54°$ (Hint : $\tan A \tan B = 1$ if $A + B = 90°$)

(iii) $\dfrac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°} = \tan 56°$ (iv) $\dfrac{\cos 29° + \sin 29°}{\cos 29° - \sin 29°} = \tan 74°$

(10) Prove that $\dfrac{\sin(A - B)}{\sin A\ \sin B} + \dfrac{\sin(B - C)}{\sin B \sin C} + \dfrac{\sin(C - A)}{\sin C \sin A} = 0$

(11) (i) If $\tan A = \dfrac{5}{6}$, $\tan B = \dfrac{1}{11}$ show that $A + B = 45°$

(ii) If $\tan\alpha = \dfrac{1}{2}$ and $\tan\beta = \dfrac{1}{3}$, show that $\alpha + \beta = \dfrac{\pi}{4}$

(12) If $A + B = 45°$, show that $(\cot A - 1)(\cot B - 1) = 2$ and deduce the value of $\cot 22\dfrac{1}{2}^{\circ}$

(13) If $A + B + C = \pi$, prove that

(i) $\tan A + \tan B + \tan C = \tan A\ \tan B \tan C$

(ii) $\tan 2A + \tan 2B + \tan 2C = \tan 2A\ \tan 2B\ \tan 2C$

(14) If $\sin A = \dfrac{1}{3}$, $\sin B = \dfrac{1}{4}$ find $\sin(A + B)$, where A and B are acute.

(15) Prove that (i) $\sin(A + 60°) + \sin(A - 60°) = \sin A$

(ii) $\tan 4A\ \tan 3A\ \tan A + \tan 3A + \tan A - \tan 4A = 0$

6.3.2 Multiple angle identities:

Identities involving sin2A, cos2A, tan3A etc. are called multiple angle identities. To develop these identities we shall use sum identities from the preceding lesson.

We first develop an identity for sin2A.

Consider $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and put $B = A$

$$\sin 2A = \sin(A + A) = \sin A \cos A + \cos A \sin A$$
$$= 2\sin A \cos A$$

Thus we have the identity $\boxed{\sin 2A = 2\sin A \,.\, \cos A}$

Identities involving cos2A and tan2A can be derived in much the same way as the identity above

$$\cos 2A = \cos(A + A) = \cos A \cos A - \sin A \sin A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$

Thus we have the identity $\boxed{\cos 2A = \cos^2 A - \sin^2 A}$

Similarly we can derive $\boxed{\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}}$

The other useful identities for cos2A can easily be derived as follows:

$$\cos 2A = \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A$$
$$= 1 - 2\sin^2 A$$
$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A)$$
$$= 2\cos^2 A - 1$$

From $\cos 2A = 1 - 2\sin^2 A$, also we have

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

Also, $\cos 2A = 2\cos^2 A - 1$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

Hence $\tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}$

$$\sin 2A = 2\sin A \cos A$$
$$= \frac{2\sin A}{\cos A} \cos^2 A = \frac{2\tan A}{\sec^2 A} = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A \left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$$

$$= \cos^2 A\,(1 - \tan^2 A)$$

$$= \frac{1 - \tan^2 A}{\sec^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Thus we have

$$\sin 2A = 2 \sin A \,.\, \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

6.3.3: Trigonometrical ratios of A in terms of trigonometrical ratios of $\frac{A}{2}$

$$\sin A = \sin\left(2 \times \frac{A}{2}\right)$$

$$= 2 \sin \frac{A}{2} \,.\, \cos \frac{A}{2}$$

$$\cos A = \cos\left(2 \times \frac{A}{2}\right) = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$= 2\cos^2 \frac{A}{2} - 1$$

$$= 1 - 2\sin^2 \frac{A}{2}$$

$$\tan A = \tan\left(2 \times \frac{A}{2}\right)$$

$$= \frac{2\tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

Similarly, we can prove the following identities

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$$

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$$

$$\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A}$$

Also note that $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$ and $\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$

Example 6.25: If $\sin\theta = \frac{3}{8}$ and θ is acute, find $\sin 2\theta$?

Solution: $\sin\theta = \frac{3}{8}$; $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{64}} = \frac{\sqrt{55}}{8}$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{3}{8}\cdot\frac{\sqrt{55}}{8} = \frac{3\sqrt{55}}{32}$$

Example 6.26: Find (i) $\sin 15°$ (ii) $\tan 15°$

Solution : (i) $\sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$

(ii) $\tan 15° = \tan\frac{30°}{2} = \frac{1-\cos 30°}{\sin 30°} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3}$

6.3.4 Trigonometrical ratios involving 3A

$$\begin{aligned}\sin 3A &= \sin(2A+A) = \sin 2A\cdot\cos A + \cos 2A\cdot\sin A\\ &= 2\sin A\cos^2 A + (1-2\sin^2 A)\sin A\\ &= 2\sin A(1-\sin^2 A) + (1-2\sin^2 A)\sin A\end{aligned}$$

$$= 3\sin A - 4\sin^3 A$$

Similarly, $\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \cdot \tan A}$$

$$= \frac{\left(\frac{2\tan A}{1 - \tan^2 A}\right) + \tan A}{1 - \tan A \cdot \frac{2\tan A}{1 - \tan^2 A}}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Example 6.27: Prove that $\cos^4 A - \sin^4 A = \cos 2A$

Solution:

$$\text{L.H.S.} = (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$
$$= 1 \cdot \cos 2A = \cos 2A = \text{R.H.S.}$$

Example 6.28:

Show that $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

Solution:

$$\text{R.H.S.} = \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1} = \frac{\frac{1}{\tan^3 A} - \frac{3}{\tan A}}{\frac{3}{\tan^2 A} - 1} = \frac{1 - 3\tan^2 A}{3\tan A - \tan^3 A}$$

$$= \frac{1}{\tan 3A} = \cot 3A = \text{L.H.S.}$$

Example 6.29:

If $\tan A = \dfrac{1 - \cos B}{\sin B}$, prove that $\tan 2A = \tan B$, where A and B are acute angles.

Solution : $\text{R.H.S} = \dfrac{1 - \cos B}{\sin B} = \dfrac{2\sin^2 \frac{B}{2}}{2\sin \frac{B}{2} \cdot \cos \frac{B}{2}} = \dfrac{\sin \frac{B}{2}}{\cos \frac{B}{2}} = \tan \dfrac{B}{2}$

$$\therefore \tan \frac{B}{2} = \tan A$$

$$\Rightarrow\; A = \frac{B}{2} \quad \Rightarrow\; B = 2A$$

Therefore $\qquad \tan 2A = \tan B$

Example 6.30: Show that $4 \sin A \sin (60° + A) . \sin (60° – A) = \sin 3A$

Solution: $\quad$ L.H.S. $= 4 \sin A \sin (60° + A) . \sin (60° – A)$

$$= 4\sin A \left\{ \sin (60° + A) . \sin (60° – A)\right\}$$

$$= 4\sin A \{\sin^2 60 – \sin^2 A\}$$

$$= 4\sin A \left\{\frac{3}{4} – \sin^2 A\right\} = 3\sin A – 4\sin^3 A = \sin 3A$$

$$= \text{R.H.S.}$$

Example 6.31: Prove that $\cos 20° \;\cos 40° \;\cos 80° = \frac{1}{8}$

$$\text{L.H.S.} = \cos 20° \;\cos 40° \;\cos 80°$$

$$= \cos 20° \;\left\{\cos (60° – 20°) \;\cos(60° + 20°)\right\}$$

$$= \cos 20° \;\left[\cos^2 60° – \sin^2 20°\right]$$

$$= \cos 20° \left[\frac{1}{4} – \sin^2 20°\right]$$

$$= \frac{1}{4} \;\cos 20° \;\left\{1 – 4(1 – \cos^2 20°)\right\}$$

$$= \frac{1}{4} \;\{4\cos^3 20° – 3 \cos 20°\} \;=\; \frac{1}{4} \;[\cos 3 \times 20°]$$

$$= \frac{1}{4} \times \cos 60° = \frac{1}{8} = \text{R.H.S.}$$

Example 6.32: Find the values of:

(i) $\sin 18°$ (ii) $\cos 18°$ (iii) $\cos 36°$ (iv) $\sin 36°$ (v) $\sin 54°$ (vi) $\cos 54°$

Solution:

(i) Let $\theta = 18°$ then $\quad 5\theta = 90° \Rightarrow\; 2\theta = 90° – 3\theta$

$$\Rightarrow\; \sin 2\theta = \sin(90° – 3\theta) = \cos 3\theta$$

$$\Rightarrow\; 2\sin\theta \cos\theta = 4\cos^3\theta – 3\cos\theta$$

$$\Rightarrow\; 2\sin\theta = 4\cos^2\theta – 3 \qquad (\because \cos \theta \neq 0)$$

$$\Rightarrow\; 2\sin\theta = 1 – 4\sin^2\theta$$

$$\Rightarrow\; 4\sin^2\theta + 2\sin\theta – 1 = 0$$

$$\Rightarrow \quad \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

since $\sin 18°$ is positive, $\sin 18° = \dfrac{-1+\sqrt{5}}{4}$

(ii) $\cos 18° = \sqrt{1-\sin^2 18} = \sqrt{1-\left(\dfrac{\sqrt{5}-1}{4}\right)^2} = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

(iii) $\cos 36° = 1 - 2\sin^2 18° = \dfrac{\sqrt{5}+1}{4}$

(iv) $\sin 36° = \sqrt{1-\cos^2 36°} = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

(v) $\sin 54° = \sin(90° - 36°) = \cos 36° = \dfrac{\sqrt{5}+1}{4}$

(vi) $\cos 54° = \cos(90° - 36°) = \sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

EXERCISE 6.5

(1) Prove the following:

(i) $2\sin 15° \cos 15° = \dfrac{1}{2}$

(ii) $\sin\dfrac{\pi}{8} \cos\dfrac{\pi}{8} = \dfrac{1}{2\sqrt{2}}$

(iii) $\sin 72° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

(iv) $\cos 72° = \dfrac{\sqrt{5}-1}{4}$

(v) $1 - 2\sin^2 22\dfrac{1}{2}° = \dfrac{1}{\sqrt{2}}$

(vi) $\dfrac{2\tan 22\frac{1}{2}°}{1-\tan^2 22\frac{1}{2}°} = 1$

(2) Show that $8\cos^3\dfrac{\pi}{9} - 6\cos\dfrac{\pi}{9} = 1$

(3) If $\tan\dfrac{\theta}{2} = \left(2-\sqrt{3}\right)$ find the value of $\sin\theta$

(4) Prove that $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$

(5) Prove that

(i) $\cos^2\left(\dfrac{\pi}{4}-\theta\right) - \sin^2\left(\dfrac{\pi}{4}-\theta\right) = \sin 2\theta$

(ii) $\sec 2\theta + \tan 2\theta = \tan\left(\dfrac{\pi}{4}+\theta\right)$

(6) (i) If $\tan\theta = 3$ find $\tan 3\theta$

(ii) If $\sin A = \dfrac{3}{5}$ find $\sin 3A$

(7) If $\tan\alpha = \frac{1}{3}$ and $\tan\beta = \frac{1}{7}$ show that $2\alpha + \beta = \frac{\pi}{4}$

(8) If $2\cos\theta = x + \frac{1}{x}$ then prove that $\cos 2\theta = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$

6.3.5 Transformation of a product into a sum or difference

We know that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad \ldots (1)$$

and

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \quad \ldots (2)$$

Adding (1) and (2), we get

$$\sin(A + B) + \sin(A - B) = 2\sin A \cos B \quad \ldots (I)$$

Subtracting (2) from (1)

$$\sin(A + B) - \sin(A - B) = 2\cos A \sin B \quad \ldots (II)$$

Again

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad \ldots (3)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad \ldots (4)$$

(3) + (4) $\Rightarrow$ $\cos(A + B) + \cos(A - B) = 2\cos A \cos B \quad \ldots (III)$

(4) – (3) $\cos(A + B) - \cos(A - B) = -2\sin A \sin B \quad \ldots (IV)$

Now, let $A + B = C$ and $A - B = D$ then

$2A = C + D$ (OR) $A = \frac{C + D}{2}$ and $2B = C - D$ (OR) $B = \frac{C - D}{2}$

Putting these values of A and B in the above four formulae I, II, III and IV, we get

1) $\sin C + \sin D = 2\sin\frac{C + D}{2} \cdot \cos\frac{C - D}{2}$

2) $\sin C - \sin D = 2\cos\frac{C + D}{2} \cdot \sin\frac{C - D}{2}$

3) $\cos C + \cos D = 2\cos\frac{C + D}{2} \cdot \cos\frac{C - D}{2}$

4) $\cos D - \cos C = 2\sin\frac{C + D}{2} \cdot \sin\frac{C - D}{2}$

Example 6.33: Express as sum or difference of following expressions.

(i) $2\sin 2\theta \cdot \cos\theta$ (ii) $2\cos 2\theta \cos\theta$ (iii) $2\sin 3A \cdot \sin A$

(iv) $\cos 7\theta . \cos 5\theta$ (v) $\cos\frac{3A}{2} . \cos\frac{5A}{2}$ (vi) $\cos 3\theta . \sin 2\theta$ (vii) $2\cos 3A \cdot \sin 5A$

Solution:

(i) $2\sin 2\theta \cdot \cos\theta = \sin(2\theta + \theta) + \sin(2\theta - \theta) = \sin 3\theta + \sin\theta$

(ii) $2\cos 2\theta \cdot \cos\theta = \cos(2\theta + \theta) + \cos(2\theta - \theta) = \cos 3\theta + \cos\theta$

(iii) $2\sin 3A \cdot \sin A = \cos(3A - A) - \cos(3A + A) = \cos 2A - \cos 4A$

(iv) $\cos 7\theta \cdot \cos 5\theta = \frac{1}{2}[\cos(7\theta + 5\theta) + \cos(7\theta - 5\theta)] = \frac{1}{2}[\cos 12\theta + \cos 2\theta]$

(v) $\cos\frac{3A}{2} \cdot \cos\frac{5A}{2} = \frac{1}{2}\left[\cos\left(\frac{3A}{2} + \frac{5A}{2}\right) + \cos\left(\frac{3A}{2} - \frac{5A}{2}\right)\right]$

$= \frac{1}{2}[\cos 4A + \cos(-A)] = \frac{1}{2}[\cos 4A + \cos A]$

(vi) $\cos 3\theta \cdot \sin 2\theta = \frac{1}{2}[\sin(3\theta + 2\theta) - \sin(3\theta - 2\theta)] = \frac{1}{2}[\sin 5\theta - \sin\theta]$

(vii) $2\cos 3A \cdot \sin 5A = \sin(3A + 5A) - \sin(3A - 5A) = \sin 8A - \sin(-2A)$

$= \sin 8A + \sin 2A$

Example 6.34: Express the following in the form of a product:

(i) $\sin 4A + \sin 2A$ (ii) $\sin 5A - \sin 3A$ (iii) $\cos 3A + \cos 7A$

(iv) $\cos 2A - \cos 4A$ (v) $\cos 60° - \cos 20°$ (vi) $\cos 55° + \sin 55°$

Solution:

(i) $\sin 4A + \sin 2A = 2\sin\left(\frac{4A + 2A}{2}\right)\cos\left(\frac{4A - 2A}{2}\right) = 2\sin 3A \cos A$

(ii) $\sin 5A - \sin 3A = 2\cos\left(\frac{5A + 3A}{2}\right)\sin\left(\frac{5A - 3A}{2}\right) = 2\cos 4A \sin A$

(iii) $\cos 3A + \cos 7A = 2\cos\left(\frac{3A + 7A}{2}\right)\cos\left(\frac{3A - 7A}{2}\right)$

$= 2\cos 5A \cos(-2A) = 2\cos 5A \cos 2A$

(iv) $\cos 2A - \cos 4A = -2\sin\left(\frac{2A + 4A}{2}\right)\sin\left(\frac{2A - 4A}{2}\right)$

$= -2\sin 3A \sin(-A) = 2\sin 3A \sin A$

(v) $\cos 60° - \cos 20° = -2\sin\frac{(60° + 20°)}{2}\sin\left(\frac{60° - 20°}{2}\right) = -2\sin 40° \sin 20°$

(vi) $\cos 55° + \sin 55° = \cos 55° + \cos(90° - 55°) = \cos 55° + \cos 35°$

$= 2\cos\frac{55° + 35°}{2}\cos\frac{55° - 35°}{2} = 2\cos 45° \cos 10°$

$= 2\frac{1}{\sqrt{2}}\cos 10° = \sqrt{2}\cos 10°$

Example 6.35: Show that $\sin 20° \sin 40° \ \sin 80° = \frac{\sqrt{3}}{8}$

Solution:

$$\text{L.H.S.} = \sin 20 \ \sin 40° \ \sin 80° = \sin 20° \ \frac{1}{2} \ \{\cos 40° - \cos 120°\}$$

$$= \frac{1}{2} \ \sin 20° \left\{\cos 40° + \frac{1}{2}\right\}$$

$$= \frac{1}{2} \ \sin 20° \cos 40° + \frac{1}{4} \ \sin 20°$$

$$= \frac{1}{4} \ (\sin 60° - \sin 20°) + \frac{1}{4} \ \sin 20° = \frac{1}{4} \ \sin 60°$$

$$= \frac{\sqrt{3}}{8} = \text{R.H.S.}$$

Example 6.36: Prove that $4(\cos 6° + \sin 24°) = \sqrt{3} + \sqrt{15}$

Solution:

$$4(\cos 6° + \sin 24°) = 4 \ (\sin 84° + \sin 24°) \quad [\because \cos 6° = \cos(90°-84) = \sin 84°]$$

$$= 4 \ . \ 2\sin\left(\frac{84° + 24°}{2}\right) \ \cos\left(\frac{84° - 24°}{2}\right)$$

$$= 8 \ \sin 54° \ . \ \cos 30° = 8\left(\frac{\sqrt{5} + 1}{4}\right) . \left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{15} + \sqrt{3}$$

Example 6.37:

Prove that (i) $\cos 20° + \cos 100° + \cos 140° = 0$ (ii) $\sin 50° - \sin 70° + \sin 10° = 0$

Solution:

(i) $$\text{L.H.S.} = \cos 20° + (\cos 100° + \cos 140°)$$

$$= \cos 20° + 2\cos\left(\frac{100 + 140°}{2}\right) \ . \cos\left(\frac{100 - 140°}{2}\right)$$

$$= \cos 20° + 2\cos 120° \cos(-20°) = \cos 20° + 2\left(-\frac{1}{2}\right) \ \cos 20°$$

$$= \cos 20° - \cos 20° = 0 = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = \sin 50° - \sin 70° + \sin 10°$$

$$= 2 \cos\left(\frac{50 + 70°}{2}\right) \ . \sin\left(\frac{50 - 70°}{2}\right) + \sin 10°$$

$= 2\cos 60° \,.\, \sin(-10°) + \sin 10° = 2 \times \frac{1}{2}\,(-\sin 10°) + \sin 10°$

$= -\sin 10° + \sin 10° = 0 = \text{R.H.S.}$

6.3.6 Conditional Identities

Example 6.38:

If $A + B + C = \pi$, prove that $\sin 2A + \sin 2B + \sin 2C = 4\sin A\ \sin B\ \sin C$

Solution:

$$
\begin{aligned}
\text{L.H.S.} &= \sin 2A + \sin 2B + \sin 2C = (\sin 2A + \sin 2B) + \sin 2C \\
&= 2\sin(A + B)\cos(A - B) + \sin 2C \\
&= 2\sin(\pi - C)\ \cos(A - B) + \sin 2C \\
&= 2\sin C\ \cos(A - B) + 2\sin C\ \cos C \\
&= 2\sin C\,\{\cos(A - B) + \cos C\} \\
&= 2\sin C\,\{\cos(A - B) + \cos(180 - \overline{A + B})\} \\
&= 2\sin C\,\{\cos(A - B) - \cos(A + B)\} = 2\sin C\ \{2\sin A\ \sin B\} \\
&= 4\sin A \sin B \sin C = \text{R.H.S.}
\end{aligned}
$$

Example 6.39:

If $A + B + C = 180°$ Prove that $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C$

Solution:

$$
\begin{aligned}
\text{L.H.S.} &= \cos 2A + (\cos 2B - \cos 2C) \\
&= 1 - 2\sin^2 A + \{-2\sin(B + C)\sin(B - C)\} \\
&= 1 - 2\sin^2 A - 2\sin(180° - A)\ \sin(B - C) \\
&= 1 - 2\sin^2 A - 2\sin A \sin(B - C) \\
&= 1 - 2\sin A\,[\sin A + \sin(B - C)] \\
&= 1 - 2\sin A\,[\sin(B + C) + \sin(B - C)],\ [\because A = 180° - (B + C)] \\
&= 1 - 2\sin A\,[2\sin B\ \cos C] \\
&= 1 - 4\sin A \sin B \cos C = \text{R.H.S.}
\end{aligned}
$$

Example 6.40:

If $A+B+C = \pi$, prove that $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$

Solution:

$$
\begin{aligned}
\text{L.H.S.} &= \cos^2 A + \cos^2 B - \cos^2 C\ = (1 - \sin^2 A) + \cos^2 B - \cos^2 C \\
&= 1 + (\cos^2 B - \sin^2 A) - \cos^2 C \\
&= 1 + \cos(A + B)\,.\,\cos(A - B) - \cos^2 C \\
&= 1 + \cos(\pi - C)\cos(A - B) - \cos^2 C \\
&= 1 - \cos C.\ \cos(A - B) - \cos^2 C
\end{aligned}
$$

$= 1 - \cos C\ [\cos(A - B) + \cos C]$

$= 1 - \cos C\ [\cos(A - B) - \cos(A + B)] = 1 - \cos C\ [2\sin A \sin B]$

$= 1 - 2\sin A \sin B \cos C = \text{R.H.S.}$

EXERCISE 6.6

(1) Express in the form of a sum or difference:

(i) $2\sin 4\theta \cos 2\theta$ (ii) $2\cos 8\theta \cos 6\theta$ (iii) $2\cos 7\theta \sin 3\theta$

(iv) $2\sin 3A \sin A$ (v) $2\cos 6A \sin 3A$ (vi) $\cos 4\theta \sin 9\theta$

(vii) $\cos\dfrac{3A}{2} \sin\dfrac{A}{2}$ (viii) $\sin\dfrac{7A}{2} \cos\dfrac{5A}{2}$ (ix) $\cos\dfrac{5\theta}{3} \cos\dfrac{4\theta}{3}$

(2) Express in the form of a product:

(i) $\sin 13A + \sin 5A$ (ii) $\sin 13A - \sin 5A$ (iii) $\cos 13A + \cos 5A$

(iv) $\cos 13A - \cos 5A$ (v) $\sin 52° - \sin 32°$ (vi) $\cos 51° + \cos 23°$

(vii) $\sin 80° - \cos 70°$ (viii) $\sin 50° + \cos 80°$ (ix) $\sin 20° + \cos 50°$

(x) $\cos 35° + \sin 72°$

(3) Prove that $\sin 20° \sin 40° \sin 60° \sin 80° = \dfrac{3}{16}$

(4) Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = \dfrac{1}{16}$

(5) Prove that $\sin 50° - \sin 70° + \cos 80° = 0$

(6) Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\left(\dfrac{\alpha + \beta}{2}\right)$

(7) Prove that (i) $\dfrac{\sin 3A - \sin A}{\cos A - \cos 3A} = \cot 2A$ (ii) $\dfrac{\cos 2A - \cos 3A}{\sin 2A + \sin 3A} = \tan\dfrac{A}{2}$

(8) $A + B + C = \pi$, prove that $\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C$

(9) If $A + B + C = 180°$,

prove that $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$

(10) If $A+B+C = \pi$, prove that $\tan\dfrac{A}{2} \tan\dfrac{B}{2} + \tan\dfrac{B}{2} \tan\dfrac{C}{2} + \tan\dfrac{C}{2} \tan\dfrac{A}{2} = 1$

(11) If $A + B + C = 90°$, show that $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C} = \cot A \cot B$

(12) Prove that $A + B + C = \pi$, prove that $\sin^2\dfrac{A}{2} + \sin^2\dfrac{B}{2} + \sin^2\dfrac{C}{2}$

$= 1 - 2\sin\dfrac{A}{2} \sin\dfrac{B}{2} \sin\dfrac{C}{2}$

6.4 Trigonometrical Equations

An equation involving trigonometrical function is called a trigonometrical equation.

$\cos\theta = \frac{1}{2}$, $\tan\theta = 0$, $\cos^2\theta - 2\sin\theta = \frac{1}{2}$ are some examples for trigonometrical equations. To solve these equations we find all replacements for the variable θ that make the equations true.

A solution of a trigonometrical equation is the value of the unknown angle that satisfies the equation. A trigonometrical equation may have infinite number of solutions. The solution in which the absolute value of the angle is the least is called **principal solution.** Note that trigonometrical equations are different from trigonometrical identities. It is possible that some equations may not have solution. For example $\cos\theta = 4$ has no solution. The expression involving integer '*n*' which gives all solutions of a trigonometrical equation is called the general solution.

6.4.1 General solutions of sin θ = 0 ; cosθ = 0 ; tan θ = 0

Consider the unit circle with centre at O(0, 0)

Let a revolving line OP, starting from OX, trace $\angle XOP = \theta$ Draw PM perpendicular to OX

(1) $\sin\theta = 0$

In the right angled triangle OMP we have OP = 1 unit,

$\sin\theta = \frac{MP}{OP} \Rightarrow \sin\theta = MP$

If $\sin\theta = 0$, then MP = 0, i.e. OP coincides with OX or OX′

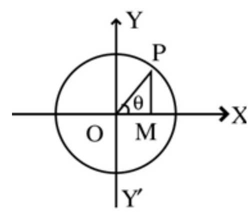


Fig. 6.20

$\therefore \angle XOP = \theta = 0, \pi, 2\pi, 3\pi, \ldots$[in the anti clockwise direction]

or $\theta = -\pi, -2\pi, -3\pi, \ldots\ldots$ [in the clockwise direction]

i.e. θ = 0 or any + *ve* or – *ve* integral multiple of π.

Hence the general solution of $\sin\theta = 0$ is given by $\theta = n\pi$, $n \in Z$,

where Z is the set of all integers.

(2) $\cos\theta = 0$

In the right angled triangle OMP we have $\cos\theta = \frac{OM}{OP} = OM$ ($\because$ OP = 1 unit)

If $\cos\theta = 0$, then OM = 0

i.e. OP coincides with OY or OY′

i.e. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, 5\frac{\pi}{2}$, …. …. [in anticlockwise direction)

or $\theta = -\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}$ ……[in clockwise direction]

i.e. $\theta = \pm\left(\text{odd multiple of } \frac{\pi}{2}\right)$

Hence the general value of θ is given by $\theta = (2n + 1)\frac{\pi}{2}$, $n \in Z$

(3) $\tan\theta = 0$

In the right angled triangle OMP, if $\tan\theta = 0$ then $\frac{MP}{OM} = 0$ or MP = 0

Proceeding as in (1), we get $\theta = n\pi$, $n \in Z$

Thus, (1) If $\sin\theta = 0$, $\theta = n\pi$, $n \in Z$

(2) If $\cos\theta = 0$, $\theta = (2n + 1)\frac{\pi}{2}$, $n \in Z$

(3) If $\tan\theta = 0$, $\theta = n\pi$, $n \in Z$

When a trigonometrical equation is solved, among all solutions the solution which is in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ for sine, in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for tangent and in $[0, \pi]$ for cosine, are the principal values of those functions.

Example 6.41:

Find the principal value of the following :

(i) $\cos x = \frac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\frac{\sqrt{3}}{2}$ (iii) $\text{cosec}\theta = -\frac{2}{\sqrt{3}}$

(iv) $\cot\theta = -1$ (v) $\tan\theta = \sqrt{3}$

Solution: (i) $\cos x = \frac{\sqrt{3}}{2} > 0$

∴ x lies in the first or fourth quadrant. Principal value of x must be in $[0, \pi]$. Since $\cos x$ is positive the principal value is in the first quadrant $\cos x = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ and $\frac{\pi}{6} \in [0, \pi]$

∴ The principal value of x is $\frac{\pi}{6}$.

(ii) $\cos\theta = -\frac{\sqrt{3}}{2} < 0$

Since cos θ is negative, θ lies in the second or third quadrant. But the principal value must be in [0, π] i.e. in first or second quadrant. The principal value is in the second quadrant.

$\therefore \cos\theta = -\frac{\sqrt{3}}{2} = \cos(180° - 30°) = \cos 150°.$

The principal value is $\theta = 150° = \frac{5\pi}{6}$.

(iii) $\operatorname{cosec}\theta = -\frac{2}{\sqrt{3}} \Rightarrow \sin\theta = -\frac{\sqrt{3}}{2} < 0$

$\therefore$ θ lies in the third or fourth quadrant. But principal value must be in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

i.e. in first or fourth quadrant. $\therefore \theta = -\frac{\pi}{3}$

(iv) $\cot\theta = -1 \quad \therefore \tan\theta = -1 < 0$

$\therefore$ θ is in the second or fourth quadrant. Principal value of θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ the solution is in the fourth quadrant.

$\cot\left(-\frac{\pi}{4}\right) = -1 \Rightarrow \theta = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

6.4.2 General solutions of sin θ = sin α ; cos θ = cos α ; tan θ = tan α

(1) $\sin\theta = \sin\alpha \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ i.e. $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \sin\theta - \sin\alpha = 0$$

$$\Rightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right) . \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \cos\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2}, \text{ or } \frac{\theta-\alpha}{2} = n\pi, \quad n \in Z$$

$\Rightarrow$ θ + α = odd multiple of π or θ – α = even multiple of π

$\Rightarrow$ θ = (odd multiple of π) – α … (1)

or θ = (even multiple of π) + α …(2)

Combining (1) and (2), we have

$$\theta = n\pi + (-1)^n.\alpha, \text{ where } n \in Z$$

(2) $\cos\theta = \cos\alpha$ $\quad 0 \le \alpha \le \pi \quad$ i.e. $\alpha \in [0, \pi]$

$$\Rightarrow \cos\theta - \cos\alpha = 0$$

$$\Rightarrow -2\sin\left(\frac{\theta+\alpha}{2}\right) . \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \sin\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = n\pi ; n\in Z \text{ or } \frac{\theta-\alpha}{2} = n\pi ; n \in Z$$

$$\Rightarrow \theta = 2n\pi - \alpha \text{ or } \theta = 2n\pi + \alpha$$

Hence $\theta = 2n\,\pi \pm \alpha$ ‘ $n \in Z$.

(3) $\tan\theta = \tan\alpha$ $\quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \quad$ i.e. $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$

$$\Rightarrow \sin(\theta - \alpha) = 0$$

$$\Rightarrow \theta - \alpha = n\pi, n \in Z$$

$$\Rightarrow \theta = n\pi + \alpha, n \in Z$$

Thus, we have $\sin\theta = \sin\alpha \quad \Rightarrow \theta = n\pi + (-1)^n\,\alpha ; n \in Z$

$\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha ; n \in Z$

$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha ; n \in Z$

Example 6.42: Find the general solution of the following :

(i) $\sin\theta = \frac{1}{2}$ (ii) $\sec\theta = -\sqrt{2}$ (iii) $\cos^2\theta = \frac{1}{4}$ (iv) $\cot^2\theta = 3$ (v) $\sec^2\theta = \frac{4}{3}$

Solution: (i) $\sin\theta = \frac{1}{2}$

$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$ which is of the form $\sin\theta = \sin\alpha$ where $\alpha = \frac{\pi}{6}$

$\therefore$ The general solution is $\theta = n\pi + (-1)^n . \frac{\pi}{6} ; n \in Z$

(ii) $\sec\theta = -\sqrt{2} \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} < 0$

Principal value of θ lies in $[0, \pi]$

As $\cos\theta$ is negative, the principal value of θ lies in second quadrant.

$$\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\therefore\ \theta = 2n\pi \pm \frac{3\pi}{4}\ ;\ n \in Z$$

(iii) We know that $\cos 2\theta = 2\cos^2\theta - 1$

$$= 2\left(\frac{1}{4}\right) - 1 = \frac{1}{2} - 1 = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\frac{2\pi}{3}$$

$$\therefore\ 2\theta = 2n\pi \pm \frac{2\pi}{3}\ ;\ n \in Z$$

$$\theta = n\pi \pm \frac{\pi}{3}\ ;\ n \in Z$$

(iv) We know that $1 + \cot^2\theta = \text{cosec}^2\theta \Rightarrow 1 + 3 = \text{cosec}^2\theta$

$$\therefore\ \text{cosec}^2\theta = 4 \quad \text{or } \sin^2\theta = \frac{1}{4}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{1}{4}\right) = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\therefore\ 2\theta = 2n\pi \pm \frac{\pi}{3}\ ;\ n \in Z$$

$$\theta = n\pi \pm \frac{\pi}{6}\ ;\ n \in Z$$

(v) We know that $\tan^2\theta = \sec^2\theta - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{1}{2}$$

$$\cos 2\theta = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$2\theta = 2n\pi \pm \frac{\pi}{3}\ ;\quad n \in Z$$

$$\theta = n\pi \pm \frac{\pi}{6}\ ;\quad n \in Z$$

Note : Solve : $\sin\theta = \frac{-\sqrt{3}}{2}$

There are two solutions in $0 \le \theta < 2\pi$ i.e. $\theta = -\frac{\pi}{3}$ and $\frac{4\pi}{3}$

The general solution is

$$\theta = n\pi + (-1)^n \left(-\frac{\pi}{3}\right) \;;\quad n \in Z \qquad \ldots (1)$$

Even if we take $$\theta = n\,\pi + (-1)^n \left(\frac{4\pi}{3}\right) \;;\; n \in Z \qquad \ldots (2)$$

The solution will be the same although these two structures are different.

Here the solution sets of (1) and (2) are same. But the order in which they occur are different.

For example Put $n = 1$ in (1), we get, $\theta = \frac{4\pi}{3}$

Put $n = 0$ in (2), we get, $\theta = \frac{4\pi}{3}$

It is a convention to take that value of θ whose absolute value is least as α (principal value) to define the general solution.

Example 6.43: Solve : $2\cos^2\theta + 3\sin\theta = 0$

Solution:

$$\begin{aligned}
2\cos^2\theta + 3\sin\theta = 0 \quad &\Rightarrow \quad 2\,(1 - \sin^2\theta) + 3\,\sin\theta = 0 \\
&\Rightarrow \quad 2\sin^2\theta - 3\sin\theta - 2 = 0 \\
&\Rightarrow \quad (2\,\sin\theta + 1)\,(\sin\theta - 2) = 0 \\
&\Rightarrow \quad \sin\theta = \frac{-1}{2} \qquad (\because \sin\theta = 2 \text{ is not possible}) \\
&\Rightarrow \quad \sin\theta = -\sin\frac{\pi}{6} \\
&\Rightarrow \quad \sin\theta = \sin\left(-\frac{\pi}{6}\right) \\
&\Rightarrow \quad \theta = -\frac{\pi}{6} \\
&\Rightarrow \quad \theta = n\pi + (-1)^n \,.\left(-\frac{\pi}{6}\right) \;;\; n \in Z
\end{aligned}$$

Example 6.44: Solve : $2\tan\theta - \cot\theta = -1$

Solution:

$$2\tan\theta - \cot\theta = -1$$

$$2\tan\theta - \frac{1}{\tan\theta} = -1$$

$$\Rightarrow 2\tan^2\theta + \tan\theta - 1 = 0$$

$$(2\tan\theta - 1)(\tan\theta + 1) = 0$$

$$2\tan\theta - 1 = 0 \quad \text{or } \tan\theta + 1 = 0$$

$$\tan\theta = \frac{1}{2} \quad \text{or } \tan\theta = -1$$

When $\quad \tan\theta = -1 = -\tan\frac{\pi}{4}$

$$\tan\theta = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \theta = n\pi + \left(-\frac{\pi}{4}\right)$$

$$= n\pi - \frac{\pi}{4} \;\; ; \; n \in Z$$

When $\quad \tan\theta = \frac{1}{2} = \tan\beta$ (say)

$$\therefore \; \theta = n\pi + \beta$$

$$= n\pi + \tan^{-1}\left(\frac{1}{2}\right)$$

Hence $\quad \theta = n\pi - \frac{\pi}{4} \quad \text{or } \theta = n\pi + \tan^{-1}\left(\frac{1}{2}\right); \; n \in Z$

Example 6.45: Solve : $\sin 2x + \sin 6x + \sin 4x = 0$

Solution:

$\sin 2x + \sin 6x + \sin 4x = 0$ or $(\sin 6x + \sin 2x) + \sin 4x = 0$

or $2\sin 4x \,.\, \cos 2x + \sin 4x = 0$

$$\sin 4x\,(2\cos 2x + 1) = 0$$

when $\sin 4x = 0 \Rightarrow 4x = n\pi$ or $x = \frac{n\pi}{4}$; $n \in Z$

When $2\cos 2x + 1 = 0 \Rightarrow \cos 2x = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$$\therefore \; 2x = 2n\pi \pm \frac{2\pi}{3} \quad \text{or} \quad x = n\,\pi \pm \frac{\pi}{3}$$

Hence $x = \frac{n\pi}{4}$ or $x = n\pi \pm \frac{\pi}{3}$; $n \in Z$

Example 6.46: Solve : $2\sin^2 x + \sin^2 2x = 2$

Solution: $\quad 2\sin^2 x + \sin^2 2x = 2$

$$\therefore \; \sin^2 2x = 2 - 2\sin^2 x$$

$$= 2(1 - \sin^2 x)$$

$$\sin^2 2x = 2\cos^2 x$$

$\Rightarrow\quad 4\sin^2 x\cos^2 x - 2\cos^2 x = 0$

$\Rightarrow\quad 2(1-\cos^2 x)\cos^2 x - \cos^2 x = 0$

$\Rightarrow\quad 2\cos^4 x - \cos^2 x = 0$

$\Rightarrow\quad \cos^2 x\,(2\cos^2 x - 1) = 0$

$\Rightarrow\quad \cos^2 x = 0$	$\cos^2 x = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$
$\Rightarrow\quad \cos^2 x = \cos^2\frac{\pi}{2}$	$\cos^2 x = \cos^2\frac{\pi}{4}$
$\Rightarrow\quad x = n\pi \pm \frac{\pi}{2},\ n \in Z$	$x = m\pi \pm \frac{\pi}{4},\ m \in Z$

Example 6.47: Solve : $\tan^2\theta + \left(1-\sqrt{3}\right)\tan\theta - \sqrt{3} = 0$

$\Rightarrow\quad \tan^2\theta + \tan\theta - \sqrt{3}\tan\theta - \sqrt{3} = 0$

$\Rightarrow\quad \tan\theta\,(\tan\theta + 1) - \sqrt{3}\,(\tan\theta + 1) = 0$

$\Rightarrow\quad (\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$

$\Rightarrow\quad \tan\theta = -1$	$\tan\theta = \sqrt{3}$
$\Rightarrow\quad \tan\theta = \tan\left(-\frac{\pi}{4}\right)$	$\tan\theta = \tan\frac{\pi}{3}$
$\Rightarrow \theta = n\pi - \frac{\pi}{4},\quad n \in Z$	$m\pi + \frac{\pi}{3},\quad m \in Z$

6.4.3 Solving equation of the form $a\cos\theta + b\sin\theta = c$. where $c^2 \le a^2 + b^2$

$$a\cos\theta + b\sin\theta = c \qquad \ldots (1)$$

Divide each term by $\sqrt{a^2+b^2}$,

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Choose $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$; $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$ and $\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$

$\therefore$ (1) becomes $\cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos\beta$

$\Rightarrow\quad \cos(\theta-\alpha) = \cos\beta$

$\Rightarrow\quad \theta - \alpha = 2n\pi \pm \beta$

$\Rightarrow\quad \theta = 2n\pi + \alpha \pm \beta,\ n \in Z$

Example 6.48: Solve : $\sqrt{3}\sin x + \cos x = 2$

Solution: This is of the form a cos$x$ + b sin$x$ = c, where $c^2 \leq a^2 + b^2$

So dividing the equation by $\sqrt{(\sqrt{3})^2 + 1^2}$ or 2

We get $\frac{\sqrt{3}}{2}$ sin x + $\frac{1}{2}$ cos$x$ = 1 $\Rightarrow$ sin $\frac{\pi}{3}$. sin$x$ + cos $\frac{\pi}{3}$. cos x = 1

$$\text{i.e.} \quad \cos\left(x - \frac{\pi}{3}\right) = 1$$

$$\cos\left(x - \frac{\pi}{3}\right) = \cos 0$$

$$x - \frac{\pi}{3} = 2n\pi \pm 0$$

$$\text{i.e.} \quad x = 2n\pi + \frac{\pi}{3}, \quad n \in Z$$

EXERCISE 6.7

(1) Find the principal value of the following equations:

(i) $\sin\theta = \frac{1}{\sqrt{2}}$ (ii) $2\cos\theta - 1 = 0$ (iii) $\sqrt{3}\cot\theta = 1$

(iv) $\sqrt{3}\sec\theta = 2$ (v) $\sin x = -\frac{\sqrt{3}}{2}$ (vi) $\tan\theta = -\frac{1}{\sqrt{3}}$

(vii) $\sec x = 2$

(2) Find the general solution of the following equation:

(i) $\sin 2\theta = \frac{1}{2}$ (ii) $\tan\theta = -\sqrt{3}$ (iii) $\cos 3\theta = \frac{-1}{\sqrt{2}}$

(3) Solve the following :

(i) $\sin 3x = \sin x$ (ii) $\sin 4x + \sin 2x = 0$ (iii) $\tan 2x = \tan x$

(4) Solve the following:

(i) $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$ (ii) $\cos^2 x + \sin^2 x + \cos x = 0$

(iii) $\cos x + \cos 2x + \cos 3x = 0$ (iv) $\sin 2x + \sin 4x = 2\sin 3x$

(5) Solve the following:

(i) $\sin\theta + \cos\theta = \sqrt{2}$ (ii) $\sin\theta - \cos\theta = -\sqrt{2}$

(iii) $\sqrt{2}\sec\theta + \tan\theta = 1$ (iv) $\mathrm{cosec}\,\theta - \cot\theta = \sqrt{3}$

6.5 Properties of Triangles

Consider a triangle ABC.

It has three angles A, B and C.

The sides opposite to the angles A, B, C are denoted by the corresponding small letters a, b, c respectively.

Thus a = BC, b = CA, c = AB.

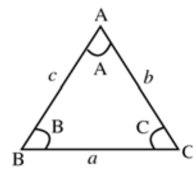


Fig. 6.21

We can establish number of formulae connecting these three angles and sides.

I. Sine formula:

In any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$. Where R is the radius of the circum circle of the triangle ABC.

In fig(6.22) O is the circumcentre of the triangle ABC. R is the radius of the circumcircle. Draw OD perpendicular to BC. Now BC = a, BD = $\dfrac{a}{2}$

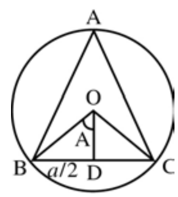


Fig. 6.22

Clearly ΔBOC is an isosceles triangle.

We know that $\angle BOC = 2 \angle BAC = 2A$

$\therefore \angle BOD = A$

From the right angled triangle BOD,

$$\sin A = \frac{BD}{R} = \frac{a/2}{R} = \frac{a}{2R}$$

$$\therefore 2R \sin A = a \quad \text{or} \quad \frac{a}{\sin A} = 2R$$

Similarly, we can prove $\dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

II. Napier's formulae

In any triangle ABC

(1) $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$

(2) $\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$

(3) $\tan \dfrac{C-A}{2} = \dfrac{c-a}{c+a} \cot \dfrac{B}{2}$ are true

These are called Napier's formulae

Result (1): $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

Proof: From sine formulae

$$\frac{a-b}{a+b}\cot\frac{C}{2} = \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}\cot\frac{C}{2}$$

$$= \frac{\sin A - \sin B}{\sin A + \sin B}\cot\frac{C}{2}$$

$$= \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}\cot\frac{C}{2}$$

$$= \cot\left(\frac{A+B}{2}\right)\tan\frac{A-B}{2}\cot\frac{C}{2}$$

$$= \cot\left(90-\frac{C}{2}\right)\tan\frac{A-B}{2}\cot\frac{C}{2}$$

$$= \tan\frac{C}{2}\tan\frac{A-B}{2}\cot\frac{C}{2} = \tan\frac{A-B}{2}$$

$$\therefore \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

Similarly, we can prove other two results (2) and (3)

III. Cosine formulae

In any triangle ABC, the following results are true with usual notation.

Results:

(1) $a^2 = b^2 + c^2 - 2bc\cos A$ (2) $b^2 = c^2 + a^2 - 2ca\cos B$

(3) $c^2 = a^2 + b^2 - 2ab\cos C$

These are called cosine formulae

Result (1): $a^2 = b^2 + c^2 - 2bc\cos A$

Proof:

Draw CD perpendicular to AB.

Now $a^2 = BC^2 = CD^2 + BD^2$

$= (AC^2 - AD^2) + (AB - AD)^2$

$= AC^2 - AD^2 + AB^2 + AD^2 - 2AB \times AD$

$= AC^2 + AB^2 - 2AB \times (AC\cos A)$

$a^2 = b^2 + c^2 - 2bc\cos A$

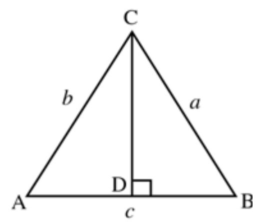


Fig. 6.23

Similarly we can prove the other results (2) and (3)

We can rewrite the formulae in different formats.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \; ; \cos B = \frac{c^2 + a^2 - b^2}{2ca} \; ; \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

IV. Projection formulae

In any triangle ABC

(1) $a = b \cos C + c \cos B$ (2) $b = c \cos A + a \cos C$ (3) $c = a \cos B + b \cos A$

are true with usual notations and these are called projection formulae.

Result (1): $a = b \cos C + c \cos B$

Proof:

In triangle ABC, draw AD perpendicular to BC.

From the right angled triangles ABD and ADC,

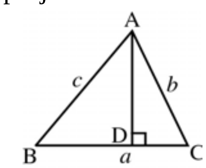


Fig. 6.24

$$\cos B = \frac{BD}{AB} \Rightarrow BD = AB \times \cos B$$

$$\cos C = \frac{DC}{AC} \Rightarrow DC = AC \times \cos C$$

$$\begin{aligned} \text{But} \quad BC &= BD + DC = AB \cos B + AC \cos C \\ a &= c \cos B + b \cos C \\ \text{or} \quad a &= b \cos C + c \cos B \end{aligned}$$

Similarly, we can prove the other formulae (2) and (3)

V. Sub-multiple (half) angle formulae

In any triangle ABC, the following results are true.

(1) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (2) $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(3) $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ (4) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(5) $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (6) $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(7) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (8) $\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

(9) $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

where $s = \frac{a+b+c}{2}$

The above results are called sub-multiple angles (or half angle) formulae.

Result (1): $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Proof : We know that $\cos 2A = 1 - 2\sin^2 A$

$$2\sin^2 A = 1 - \cos 2A$$

Replacing A by $\frac{A}{2}$, $2\sin^2\frac{A}{2} = 1 - \cos A$

$$= 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$$

$$= \frac{(a+b+c-2c)(a+b+c-2b)}{2bc}$$

$$= \frac{(2s-2c)(2s-2b)}{2bc} \qquad \because a+b+c = 2s$$

$$2\sin^2\frac{A}{2} = \frac{2(s-c)\,2(s-b)}{2bc}$$

$$\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{(s-b)(s-c)}{bc}}$$

Since $\frac{A}{2}$ is acute, $\sin\frac{A}{2}$ is always positive.

Thus $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Similarly we can prove the other two sine related formulae (2) and (3)

Result (4): $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Proof : We know that $\cos 2A = 2\cos^2 A - 1$

$$2\cos^2 A = 1 + \cos 2A$$

Replacing A by $\frac{A}{2}$, $2\cos^2\frac{A}{2} = 1 + \cos A$

$$= 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{(b + c + a)\,(b + c + a - 2a)}{2bc} = \frac{2s(2s - 2a)}{2bc}$$

$$2\cos^2\frac{A}{2} = \frac{2s \times 2(s - a)}{2bc}$$

$$\cos^2\frac{A}{2} = \frac{s(s - a)}{bc}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}$$

Similarly we can prove other two cosine related formulae (5) and (6)

Result (7): $$\tan\frac{A}{2} = \sqrt{\frac{(s - b)\,(s - c)}{s(s - a)}}$$

Proof: $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{\frac{(s - b)\,(s - c)}{bc}}}{\sqrt{\frac{s(s - a)}{bc}}}$$

$$= \sqrt{\frac{(s - b)\,(s - c)}{s(s - a)}}$$

Similarly we can prove other two tangent related formulae (8) and (9)

VI. Area formulae (Δ denotes area of a triangle)

In any triangle ABC

(1) $\Delta = \frac{1}{2}ab\sin C$ (2) $\Delta = \frac{1}{2}bc\sin A$ (3) $\Delta = \frac{1}{2}ca\sin B$

(4) $\Delta = \frac{abc}{4R}$ (5) $\Delta = 2R^2\sin A\sin B\sin C$ (6) $\Delta = \sqrt{s(s - a)\,(s - b)\,(s - c)}$

are true with the usual notations and these are called Area formulae.

Result (1): $\Delta = \frac{1}{2}ab\sin C$

Proof :

Draw AD perpendicular to BC

Δ = Area of triangle ABC

$= \frac{1}{2} \times BC \times AD = \frac{1}{2} \times BC \times AC \times \sin C$

$= \frac{1}{2}ab\sin C$ $\left[\because \sin C = \frac{AD}{AC} \Rightarrow AD = AC \times \sin C\right]$

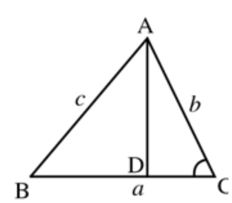


Fig. 6.25

Similarly we can prove the results (2) and (3)

Result (4): $\Delta = \dfrac{abc}{4R}$

Proof:

We know that $\Delta = \dfrac{1}{2}\, ab \sin C$

$= \dfrac{1}{2}\, ab \dfrac{c}{2R}$ $\qquad \because \dfrac{c}{\sin C} = 2R$

$= \dfrac{abc}{4R}$

Result (5): $\Delta = 2R^2 \sin A \sin B \sin C$

Proof:

We know that $\Delta = \dfrac{1}{2}\, ab \sin C$

$= \dfrac{1}{2}\, 2R \sin A\; 2R \sin B \sin C$ $\qquad \because a = 2R \sin A$

$= 2R^2 \sin A \sin B \sin C$ $\qquad b = 2R \sin B$

Result (6) Prove that $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Proof:

We know that $\Delta = \dfrac{1}{2}\, ab \sin C$

$= \dfrac{1}{2}\, ab\, 2\sin\dfrac{C}{2} \cos\dfrac{C}{2}$

$= ab \sqrt{\dfrac{(s-a)(s-b)}{ab}} \sqrt{\dfrac{s(s-c)}{ab}}$

$= \sqrt{s(s-a)(s-b)(s-c)}$

Example 6.49: In a triangle ABC prove that $a \sin A - b \sin B = c \sin(A - B)$

Solution:

By sine formulae we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$\therefore\ a = 2R \sin A,\quad b = 2R \sin B,\quad c = 2R \sin C$

$$\begin{aligned} a \sin A - b \sin B &= 2R\sin A \sin A - 2R \sin B \sin B \\ &= 2R(\sin^2 A - \sin^2 B) \\ &= 2R \sin(A + B) \sin(A - B) \\ &= 2R \sin(180 - C) \sin(A - B) \end{aligned}$$

$$= 2R \sin C \sin(A - B)$$
$$= c \sin(A - B)$$

Example 6.50: Prove that $\dfrac{\sin(A - B)}{\sin(A + B)} = \dfrac{a^2 - b^2}{c^2}$

Solution:

By sine formula $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\frac{a^2 - b^2}{c^2} = \frac{(2R \sin A)^2 - (2R \sin B)^2}{(2R \sin C)^2}$$
$$= \frac{4R^2 \sin^2 A - 4R^2 \sin^2 B}{4R^2 \sin^2 C} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$
$$= \frac{\sin(A + B)\sin(A - B)}{\sin^2 C} \qquad [\sin C = \sin(A + B)]$$
$$= \frac{\sin(A + B)\sin(A - B)}{\sin^2(A + B)} = \frac{\sin(A - B)}{\sin(A + B)}$$

Example 6.51: Prove that $\sum a \sin(B - C) = 0$

Solution:

$$\begin{aligned}
\sum a \sin(B - C) &= a \sin(B - C) + b \sin(C - A) + c \sin(A - B) \\
&= 2R \sin A \sin(B - C) + 2R \sin B \sin(C - A) + 2R \sin C \sin(A - B) \\
\sin A &= \sin(B + C),\ \sin B = \sin(C + A)\ ;\ \sin C = \sin(A + B) \\
&= 2R \sin(B + C)\sin(B - C) + 2R \sin(C + A)\sin(C - A) \\
&\quad + 2R \sin(A + B)\sin(A - B) \\
&= 2R[\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B] \\
&= 0
\end{aligned}$$

Example 6.52: Prove that $\cos\dfrac{B - C}{2} = \dfrac{b + c}{a} \sin\dfrac{A}{2}$

Solution: $\dfrac{b + c}{a} \sin\dfrac{A}{2} = \dfrac{2R \sin B + 2R \sin C}{2R \sin A} \sin\dfrac{A}{2}$

$$= \frac{\sin B + \sin C}{\sin A} \sin\frac{A}{2}$$
$$= \frac{2 \sin\frac{B + C}{2} \cos\frac{B - C}{2}}{2 \sin\frac{A}{2} \cos\frac{A}{2}} \sin\frac{A}{2}$$

$$= \frac{\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\sin\left(\frac{180-A}{2}\right)\cos\frac{B-C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\sin\left(90-\frac{A}{2}\right)\cos\frac{B-C}{2}}{\cos\frac{A}{2}}$$

$$= \cos\frac{B-C}{2} \qquad \because \sin\left(90-\frac{A}{2}\right) = \cos\frac{A}{2}$$

Example 6.53: In any triangle ABC prove that

$$\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0$$

Solution:

$$\frac{a^2 \sin(B-C)}{\sin A} = \frac{(2R \sin A)^2 \sin(B-C)}{\sin A} = \frac{4R^2 \sin^2 A \sin(B-C)}{\sin A}$$

$$= 4R^2 \sin A \sin(B-C) = 4R^2 \sin(B+C)\sin(B-C)$$

$$= 4R^2 (\sin^2 B - \sin^2 C) = 4R^2 \sin^2 B - 4R^2 \sin^2 C$$

$$= b^2 - c^2$$

Similarly $\frac{b^2 \sin(C-A)}{\sin B} = c^2 - a^2$

$$\frac{c^2 \sin(A-B)}{\sin C} = a^2 - b^2$$

$$\therefore \frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C}$$

$$= b^2 - c^2 + c^2 - a^2 + a^2 - b^2$$

$$= 0$$

EXERCISE 6.8

In any triangle ABC prove that

(1) $a^2 = (b+c)^2 \sin^2\frac{A}{2} + (b-c)^2 \cos^2\frac{A}{2}$

(2) $\sum a(b^2 + c^2) \cos A = 3abc$

(3) $\sum a(\sin B - \sin C) = 0$

(4) $\sum (b + c) \cos A = a + b + c$

(5) $a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0$

(6) $a(b \cos C - c \cos B) = b^2 - c^2$

(7) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$

(8) $\dfrac{\tan A}{\tan B} = \dfrac{c^2 + a^2 - b^2}{b^2 + c^2 - a^2}$

(9) If $a \cos A = b \cos B$ then show that the triangle is either an isosceles triangle or right angled triangle?

6.6. Solution of triangles

We know that a triangle has six parts (or elements). Consider a triangle ABC. With usual symbols, the sides *a*, *b*, *c* and the angles A, B, C are parts of the triangle ABC.

The process of finding the unknown parts of a triangle is called the solution of triangle. If three parts of a triangle (atleast one of which is a side) are given then the other parts can be found. Here, we shall discuss the following three types.

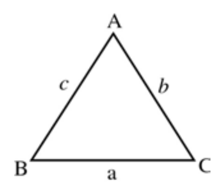


Fig. 6.26

1) Any three sides (SSS) are given.

2) Any one side and two angles (SAA) are given.

3) Any two sides and the included angle (SAS) are given.

Type I: Given three sides (SSS)

To solve this type, we can use any one of the following formulae.

(a) Cosine formula (b) Sine formula (c) Half angle formula .

It is better to use cosine formula if the sides are small, while use half angle formula if the sides are large.

Example 6.54: Given $a = 8$, $b = 9$, $c = 10$, find all the angles.

Solution: To find A, use the formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{81 + 100 - 64}{180} = \frac{117}{180}$$

$$A = 49° 28'$$

Similarly

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{100 + 64 - 81}{160} = \frac{83}{160}$$

$$B = 58° \; 51'$$

But $\quad A + B + C = 180°$

$$\therefore C = 180° - (49° \; 28' + 58° \; 51')$$

$$= 71° \; 41'$$

Thus $\quad A = 49° \; 28', B = 58° \; 51', C = 71° \; 41'$

Note: In the above example the numbers are smaller and hence we used cosine formula.

Example 6.55: Given $a = 31$, $b = 42$, $c = 57$, find all the angles.

Solution: Since the sides are larger quantities, use half angle formulae

$$s = \frac{a + b + c}{2} = 65$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \left(\frac{23 \times 8}{65 \times 34}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \log\left[\tan\frac{A}{2}\right] = \frac{1}{2}\,[\log 23 + \log 8 - \log 65 - \log 34]$$

$$= \frac{1}{2}\,[1.3617 + 0.9031 - 1.8129 - 1.5315]$$

$$= \frac{1}{2}\,[-1.0796] = \frac{1}{2}\,[-2 + 0.9204]$$

$$= \frac{1}{2}\left[\bar{2} + 0.9204\right] = \bar{1}.4602$$

$$\Rightarrow \quad \frac{A}{2} = 16° \; 6' \Rightarrow A = 32° \; 12'$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \left(\frac{8 \times 34}{65 \times 23}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \log\left[\tan\frac{B}{2}\right] = \frac{1}{2}\,[\log 8 + \log 34 - \log 65 - \log 23]$$

$$= \frac{1}{2}\,[-0.7400] = \frac{1}{2}\,[-2 + 1.2600]$$

$$= \frac{1}{2}\left[\bar{2} + 1.2600\right] = \bar{1}.6300$$

$$\Rightarrow \qquad \frac{B}{2} = 23° \, 6' \Rightarrow B = 46° \, 12'$$

$$C = 180 - (A + B) = 101° \, 36'$$

Thus $\quad A = 32° \, 12' \quad B = 46° \, 12' \quad C = 101° \, 36'$

Type II: Given one side and any two angles (SAA)

To solve this type, draw a sketch of the triangle roughly, for better understanding and use sine formula.

Example 6.56: In a triangle ABC, A = 35° 17′, C = 45° 13′, b = 42.1. Solve the triangle.

Solution:

The unknown parts are B, a, c

B = 180 – (A + C) = 180 – (35° 17′ + 45° 13′)

= 99° 30′

To find the sides, use sine formula

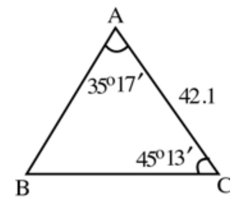


Fig. 5.27

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad a = \frac{b \sin A}{\sin B} = \frac{42.1 \times \sin 35° \, 17'}{\sin 99° \, 30'}$$

$$\log a = \log 42.1 + \log \sin 35° \, 17' - \log \sin 99° \, 30'$$

$$= 1.6243 + \bar{1}.7616 - \bar{1}.9940$$

$$= 1.3859 - \bar{1}.9940$$

$$= 1.3859 - [-1 + 0.9940] = 1.3919$$

$$\Rightarrow \quad a = 24.65$$

$$\text{Again} \quad c = \frac{b \sin C}{\sin B} = \frac{42.1 \times \sin 45° \, 13'}{\sin 99' \, 30'}$$

$$\log c = \log 42.1 + \log \sin 45° \, 13' - \log \sin 99° \, 30'$$

$$= 1.6243 + \bar{1}.8511 - \bar{1}.9940$$

$$= 1.4754 - \bar{1}.9940$$

$$= 1.4754 - [-1 + 0.9940] = 1.4814$$

$$\Rightarrow \quad c = 30.3$$

Thus $\quad B = 99° \, 30', \quad a = 24.65, \quad c = 30.3$

Type III: Given two sides and the included angle (SAS)

Since two sides and the included angle are given, the third side can be found by using the proper cosine formula. Then one can apply the sine formula to calculate the other elements.

Example 6.57: Solve the triangle ABC if $a = 5$, $b = 4$ and $C = 68°$.

Solution: To find c, use $c^2 = a^2 + b^2 - 2ab \cos C$

$$c^2 = 25 + 16 - 2 \times 5 \times 4 \cos 68°$$
$$= 41 - 40 \times 0.3746 = 26.016$$
$$c = 5.1$$

To find the other two angles, use sine formula.

$$\Rightarrow \quad \sin B = \frac{b \sin C}{c} = \frac{4 \times \sin 68°}{5.1}$$
$$\log \sin B = \log 4 + \log \sin 68° - \log 5.1$$
$$= 0.6021 + \bar{1}.9672 - .7075$$
$$= 0.5693 - 0.7075 = -0.1382$$
$$= \bar{1}.8618$$
$$B = 46° 40'$$
$$\Rightarrow \quad A = 180 - (B + C) = 180 - (114° 40')$$
$$= 65° 20'$$

Thus $\quad B = 46° 40'$, $A = 65° 20'$, $c = 5.1$

Note: To find the angles A and B one can also use the tangent formula

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}$$

6.7 Inverse Trigonometrical functions (Inverse circular functions)

The quantities $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, … are called inverse circular functions. **$\sin^{-1}x$ is an angle θ, whose sine is x.** Similarly $\cos^{-1}x$ denotes an angle whose cosine is x and so on. The principal value of an inverse function is that value of the general value which is numerically the least. It may be positive or negative. When there are two values, one is positive and the other is negative such that they are numerically equal, then the principal value is the positive one.

For example the principal values of $\cos^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{3}$ and not $-\frac{\pi}{3}$ though $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

Note : $\sin^{-1}x$ is different from $(\sin x)^{-1}$. $\sin^{-1}$ in $\sin^{-1}x$ denotes the inverse of the circular function. But $(\sin x)^{-1}$ is the reciprocal of $\sin x$ i.e. $\frac{1}{\sin x}$.

The Domain and Range of Inverse Trigonometrical functions are given below:

	Function	Domain	Range (Principal Value)
1.	$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2.	$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
3.	$y = \tan^{-1}x$	R	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
4.	$y = \operatorname{cosec}^{-1}x$	$x \ge 1$ or $x \le -1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
5.	$y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$0 < y \le \pi$; $y \ne \frac{\pi}{2}$
6.	$y = \cot^{-1} x$	R	$0 < y < \pi$

Table 6.6

Example 6.58: Find the principal values of:

(i) $\operatorname{Sin}^{-1}\left(\frac{1}{2}\right)$ (ii) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (iv) $\sin^{-1}(-1)$

(v) $\cos^{-1}\left(-\frac{1}{2}\right)$ (vi) $\operatorname{cosec}^{-1}(-2)$

Solution:

(i) Let $\sin^{-1}\left(\frac{1}{2}\right) = y$, where $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$.

Then $\sin^{-1}\left(\frac{1}{2}\right) = y \Rightarrow \sin y = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow y = \frac{\pi}{6}$

$\therefore$ The principal value of $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$

(ii) Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$, where $0 < y < \frac{\pi}{2}$, then,

$\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y \Rightarrow \sec y = \frac{2}{\sqrt{3}} = \sec\frac{\pi}{6} \Rightarrow y = \frac{\pi}{6}$

$\therefore$ The principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Then $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = y \Rightarrow \tan y = -\frac{1}{\sqrt{3}} = \tan\left(-\frac{\pi}{6}\right) \Rightarrow y = \frac{-\pi}{6}$

$\therefore$ The principal values of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $\frac{-\pi}{6}$

(iv) Let $\sin^{-1}(-1) = y$, where $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$

Then, $\sin^{-1}(-1) = y \Rightarrow \sin y = -1$

$-1 = \sin\left(-\frac{\pi}{2}\right) \Rightarrow y = -\frac{\pi}{2}$

$\therefore$ The principal value of $\sin^{-1}(-1)$ is $-\frac{\pi}{2}$

(v) Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$, where $0 \leq y \leq \pi$, then

$\cos^{-1}\left(-\frac{1}{2}\right) = y \Rightarrow \cos y = -\frac{1}{2}$

$\cos y = -\cos\frac{\pi}{3} \Rightarrow \cos y = \cos\left(\pi - \frac{\pi}{3}\right) \Rightarrow y = \left(\frac{2\pi}{3}\right)$

$\therefore$ The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$

(vi) Let $\mathrm{cosec}^{-1}(-2) = y$, where $-\frac{\pi}{2} \leq y < 0$

$\mathrm{cosec}^{-1}(-2) = y \Rightarrow \mathrm{cosec}\, y = -2 = \mathrm{cosec}\left(\frac{-\pi}{6}\right) \Rightarrow y = \frac{-\pi}{6}$

$\therefore$ The principal value of $\mathrm{cosec}^{-1}(-2)$ is $\frac{-\pi}{6}$

Example 6.59:

(i) If $\cot^{-1}\left(\frac{1}{7}\right) = \theta$, find the value of $\cos\theta$ (ii) If $\sin^{-1}\left(\frac{1}{2}\right) = \tan^{-1}x$, find the value of x

Solution:

(i) $\cot^{-1}\left(\frac{1}{7}\right) = \theta \Rightarrow \cot\theta = \frac{1}{7} \therefore \tan\theta = 7$

$\Rightarrow \sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + 49}$

$$\sec\theta = 5\sqrt{2}$$

$$\Rightarrow \cos\theta = \frac{1}{5\sqrt{2}}$$

(ii) $\tan^{-1}x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \quad \therefore \tan^{-1}x = \frac{\pi}{6}$

$$\Rightarrow x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \quad \Rightarrow x = \frac{1}{\sqrt{3}}$$

Properties of principal inverse Trigonometric functions:

Property (1):

(i) $\sin^{-1}(\sin x) = x$ (ii) $\cos^{-1}(\cos x) = x$ (iii) $\tan^{-1}(\tan x) = x$

(iv) $\cot^{-1}(\cot x) = x$ (v) $\sec^{-1}(\sec x) = x$ (vi) $\mathrm{cosec}^{-1}(\mathrm{cosec}\, x) = x$

Proof:

(i) Let $\sin x = y$, then $x = \sin^{-1}(y)$ … (1)

$\therefore x = \sin^{-1}(\sin x)$ by (1)

Similarly, the other results may be proved.

Property (2):

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \mathrm{cosec}^{-1}x$ (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$ (iv) $\mathrm{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x$

(v) $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x$ (vi) $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}x$

Proof:

(i) Let $\sin^{-1}\left(\frac{1}{x}\right) = \theta \Rightarrow \sin\theta = \frac{1}{x}$

$$\Rightarrow \mathrm{cosec}\theta = x$$

$$\Rightarrow \theta = \mathrm{cosec}^{-1}(x)$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{x}\right) = \mathrm{cosec}^{-1}x$$

Similarly the other results can be proved.

Property (3):

(i) $\sin^{-1}(-x) = -\sin^{-1}x$ (ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$ (iv) $\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$ (vi) $\cot^{-1}(-x) = -\cot^{-1}x$

Proof:

(i) Let $\sin^{-1}(-x) = \theta \quad \therefore -x = \sin\theta$

$$\Rightarrow \quad x = -\sin\theta$$

$$x = \sin(-\theta)$$

$$\Rightarrow \quad -\theta = \sin^{-1} x$$

$$\Rightarrow \quad \theta = -\sin^{-1}x$$

$$\Rightarrow \quad \sin^{-1}(-x) = -\sin^{-1}x$$

(ii) Let $\cos^{-1}(-x) = \theta \quad \Rightarrow \quad -x = \cos\theta$

$$\Rightarrow \quad x = -\cos\theta = \cos(\pi - \theta)$$

$$\Rightarrow \quad \pi - \theta = \cos^{-1}x$$

$$\Rightarrow \quad \theta = \pi - \cos^{-1}x$$

$$\Rightarrow \quad \cos^{-1}(-x) = \pi - \cos^{-1}x$$

Similarly the other results may be proved.

Property (4):

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ (iii) $\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2}$

Proof:

(i) Let $\sin^{-1}x = \theta \Rightarrow x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \theta$$

$$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$

$$\Rightarrow \quad \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

Similarly (ii) and (iii) can be proved.

Property (5):

If $xy < 1$, then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof: Let $\tan^{-1}x = \theta_1$ and $\tan^{-1}y = \theta_2$ then $\tan\theta_1 = x$ and $\tan\theta_2 = y$

$$\Rightarrow \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \,.\, \tan\theta_2} = \frac{x+y}{1-xy}$$

$$\Rightarrow \theta_1 + \theta_2 = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Note: Similarly, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Property (6): $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

Proof: Let $\theta_1 = \sin^{-1}x$ and $\theta_2 = \sin^{-1}y$ then $\sin\theta_1 = x$ and $\sin\theta_2 = y$

$$\Rightarrow \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$

$$= \left(\sin\theta_1\sqrt{1-\sin^2\theta_2} + \sqrt{1-\sin^2\theta_1}\,\sin\theta_2\right)$$

$$= \left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\Rightarrow \theta_1 + \theta_2 = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

Example 6.60:

Prove that (i) $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\frac{2}{9}$ (ii) $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$

Solution:

(i) $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\cdot\frac{1}{13}}\right) = \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\left(\frac{2}{9}\right)$

(ii) Let $\cos^{-1}\frac{4}{5} = \theta$ then $\cos\theta = \frac{4}{5}$ $\therefore$ $\tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\frac{3}{4}$

$\cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$

$$\therefore \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{3}{5}}{1-\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}\right\} = \tan^{-1}\frac{27}{11}$$

Example 6.61: Show that $\tan^{-1} + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-yz-zx-xy}\right)$

Solution: $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y}{1-xy}\right] + \tan^{-1}z$

$$= \tan^{-1}\frac{\frac{x+y}{1-xy}+z}{1-\frac{(x+y)z}{1-xy}} = \tan^{-1}\left[\frac{\frac{x+y+z-xyz}{1-xy}}{\frac{1-xy-xz-yz}{1-xy}}\right]$$

$$= \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$$

Example 6.62: Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

$$\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4} \Rightarrow \tan^{-1}\left[\frac{2x+3x}{1-6x^2}\right] = \tan^{-1}(1)$$

$$\therefore \frac{5x}{1-6x^2} = 1 \Rightarrow 1-6x^2 = 5x \quad \therefore 6x^2+5x-1=0$$

i.e. $(x+1)(6x-1) = 0 \Rightarrow x = -1 \text{ or } \frac{1}{6}$

The negative value of x is rejected since it makes R.H.S. negative. $\therefore x = \frac{1}{6}$

Example 6.63:

Evaluate : (i) $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$ (ii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ (iii) $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$

Solution: (i) Let $\cos^{-1}\frac{3}{5} = \theta$. Then, $\cos\theta = \frac{3}{5}$

$$\therefore \sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

(ii) Let $\tan^{-1}\left(\frac{3}{4}\right) = \theta$ then, $\tan\theta = \frac{3}{4}$

$$\therefore \cos\left(\tan^{-1}\frac{3}{4}\right) = \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{4}{5}$$

(iii) Let $\cos^{-1}\frac{4}{5} = \theta$; then $\cos\theta = \frac{4}{5}$

$$\sin\left[\frac{1}{2}\cos^{-1}\frac{4}{5}\right] = \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \frac{1}{\sqrt{10}}$$

Example 6.64: Evaluate : $\cos\left[\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right]$

Solution: Let $\sin^{-1}\frac{3}{5} = A \quad \therefore \sin A = \frac{3}{5} \Rightarrow \cos A = \frac{4}{5}$

Let $\sin^{-1}\frac{5}{13} = B \quad \therefore \sin B = \frac{5}{13} \Rightarrow \cos B = \frac{12}{13}$

$$\therefore \cos\left[\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right] = \cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$= \left(\frac{4}{5}\cdot\frac{12}{13}-\frac{3}{5}\cdot\frac{5}{13}\right) = \frac{33}{65}$$

EXERCISE 6.9

(1) Find the principal value of

(i) $\sin^{-1}\frac{\sqrt{3}}{2}$ (ii) $\cos^{-1}\left(\frac{1}{2}\right)$ (iii) $\operatorname{cosec}^{-1}(-1)$

(iv) $\sec^{-1}\left(-\sqrt{2}\right)$ (v) $\tan^{-1}\left(\sqrt{3}\right)$ (vi) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(2) Prove that (i) $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\frac{3}{4}$ (ii) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$

(iii) $\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

(3) Evaluate:

(i) $\cos\left(\sin^{-1}\frac{5}{13}\right)$ (ii) $\cos\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]$ (iii) $\tan\left(\cos^{-1}\frac{8}{17}\right)$ (iv) $\sin\left[\cos^{-1}\frac{1}{2}\right]$

(4) Prove the following:

(i) $\tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right] = \frac{x}{2}$ (ii) $\cos^{-1}(2x^2-1) = 2\cos^{-1}x$

(iii) $\tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] = 3\tan^{-1}x$ (iv) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$

(5) Prove that $2\tan^{-1}\frac{2}{3} = \tan^{-1}\left(\frac{12}{5}\right)$

(6) Prove that $\tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}$

(7) Solve : $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

(8) Solve $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}$, where $x > 0$

(9) Solve : $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{4}{7}$

(10) Prove the following:

(i) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\ \sqrt{1-y^2}\right]$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\,.\sqrt{1-y^2}\right]$

OBJECTIVE TYPE QUESTIONS

Choose the correct or most suitable answer

(1) The order of matrix B = [1 2 5 7] is

(1) 1×4 (2) 4×1 (3) 2×1 (4) 1×1

(2) Number of elements in a matrix of order 2×3 is

(1) 5   (2) 2   (3) 3   (4) 6

(3) If A = $\begin{bmatrix} 2 & 1 & 4 \\ -3 & 2 & 1 \end{bmatrix}$ and X + A = 0 then matrix X is

(1) $\begin{bmatrix} 2 & 1 & 4 \\ -3 & 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} -2 & -1 & -4 \\ 3 & -2 & -1 \end{bmatrix}$

(3) $\begin{bmatrix} -2 & -1 & -4 \\ 3 & 2 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -2 & -1 \end{bmatrix}$

(4) The product of the matrices $[7 \;\; 5 \;\; 3]\begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix}$ is equal to

(1) [70]   (2) [49]   (3) [15]   (4) 70

(5) The type of the matrix $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ is

(1) a scalar matrix   (2) a diagonal matrix

(3) a unit matrix   (4) diagonal and scalar

(6) If $[2 \;\; x \;\; -1]\begin{bmatrix} 0 \\ x \\ 3 \end{bmatrix} = [13]$ then the value of x is

(1) 5   (2) 2   (3) ± 3 (4) ± 4

(7) Matrix A is of order 2×3 and B is of order 3×2 then order of matrix BA is

(1) 3×3 (2) 2×3 (3) 2×2 (4) 3×2

(8) If $[3 \;\; -1 \;\; 2]B = [5 \;\; 6]$ the order of matrix B is

(1) 3×1 (2) 1×3 (3) 3×2 (4) 1×1

(9) The true statements of the following are

(i) Every unit matrix is a scalar matrix but a scalar matrix need not be a unit matrix.

(ii) Every scalar matrix is a diagonal matrix but a diagonal matrix need not be a scalar matrix.

(iii) Every diagonal matrix is a square matrix but a square matrix need not be a diagonal matrix.

(1) (i), (ii), (iii) (2) (i) and (ii) (3) (ii) and (iii) (4) (iii) and (i)

(10) The matrix $\begin{bmatrix} 8 & 5 & 7 \\ 0 & 6 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ is

(1) the upper triangular (2) lower triangular

(3) square matrix (4) null matrix

(11) The minor of 2 in $\begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix}$ is

(1) 0 (2) 1 (3) 2 (4) – 3

(12) The cofactor of – 7 in $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ is

(1) – 18 (2) 18 (3) – 7 (4) 7

(13) If A = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and | A | = 2 then | 3 A | is

(1) 54 (2) 6 (3) 27 (4) – 54

(14) In a third order determinant the cofactor of a_{23} is equal to the minor of a_{23} then the value of the minor is

(1) 1 (2) Δ (3) $-\Delta$ (4) 0

(15) The solution of $\begin{vmatrix} 2x & 3 \\ 2 & 3 \end{vmatrix} = 0$ is

(1) $x = 1$ (2) $x = 2$ (3) $x = 3$ (4) $x = 0$

(16) The value of $\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x & 3y & 3z \end{vmatrix}$ is

(1) 1 (2) xyz (3) $x + y + z$ (4) 0

(17) If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$ then $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$ ie equal to

(1) Δ (2) $-\Delta$ (3) 3Δ (4) -3Δ

(18) The value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 7 & 6 & 5 \\ 1 & 2 & 3 \end{vmatrix}$ is

(1) 0 (2) 5 (3) 10 (4) – 10

(19) If A is a square matrix of order 3 then $|kA|$ is

(1) $k|A|$ (2) $-k|A|$ (3) $k^3|A|$ (4) $-k^3|A|$

(20) If $\Delta = \begin{vmatrix} 1 & 4 & 3 \\ -1 & 1 & 5 \\ 3 & 2 & -1 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} 2 & 8 & 6 \\ -2 & 2 & 10 \\ 6 & 4 & -2 \end{vmatrix}$ then

(1) $\Delta_1 = 2\Delta$ (2) $\Delta_1 = 4\Delta$ (3) $\Delta_1 = 8\Delta$ (4) $\Delta = 8\Delta_1$

(21) If $\Delta_1 = \begin{vmatrix} 7 & 6 & 1 \\ 5 & 3 & 8 \\ 8 & 2 & 4 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 7 & 6 & 1 \\ 8 & 2 & 4 \\ 10 & 6 & 16 \end{vmatrix}$ then

(1) $\Delta_1 = -2\Delta_2$ (2) $\Delta_2 = -2\Delta_1$ (3) $\Delta_1 = 2\Delta_2$ (4) $\Delta_1 = -2\Delta_2$

(22) Two rows of a determinant Δ are identical when $x = -a$ then the factor of Δ is

(1) $x + a$ (2) $x - a$ (3) $(x + a)^2$ (4) $(x - a)^2$

(23) The factor of the determinant $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$ is

(1) $x + 2$ (2) $x - 3$ (3) $2x + 1$ (4) $x + 3$

(24) If all the three rows are identical in a determinant Δ on putting $x = a$ then the factor of Δ is

(1) $x - a$ (2) $x + a$ (3) $(x - a)^2$ (4) $(x + a)^2$

(25) The factor of the determinant $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$ is

(1) x (2) $x + b$ (3) $x + c$ (4) $x - a + b + c$

(26) The value of the determinant $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2$ is

(1) abc (2) 0 (3) $a^2b^2c^2$ (4) $-abc$

(27) The value of the product $\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} \times \begin{vmatrix} 2 & 0 \\ 1 & -4 \end{vmatrix}$ is

(1) 56 (2) – 56 (3) – 1 (4) – 63

(28) If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_1, B_1, C_1 …… are the cofactors of a_1, b_1, c_1 …… then $a_1A_2 + b_1B_2 + c_1C_2$ is equal to

(1) Δ (2) 0 (3) $-\Delta$ (4) Δ^2

(29) Given that the value of a third order determinant is 11 then the value of the determinant formed by the respective co-factors as its elements will be

(1) 11 (2) 121 (3) 1331 (4) 0

(30) A factor of the determinant $\begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$ is

(1) $x + y$ (2) $a + b$ (3) $x - y$ (4) $a + b + c$

(31) The position vector of A is $2\vec{i} + 3\vec{j} + 4\vec{k}$, $\overrightarrow{AB} = 5\vec{i} + 7\vec{j} + 6\vec{k}$ then position vector of B is

(1) $7\vec{i} + 10\vec{j} + 10\vec{k}$ (2) $7\vec{i} - 10\vec{j} + 10\vec{k}$

(3) $7\vec{i} + 10\vec{j} - 10\vec{k}$ (4) $-7\vec{i} + 10\vec{j} - 10\vec{k}$

(32) If $\vec{a}$ is a non zero vector and k is a scalar such that $\left|k\vec{a}\right| = 1$ then k is equal to

(1) $\left|\vec{a}\right|$ (2) 1 (3) $\dfrac{1}{\left|\vec{a}\right|}$ (4) $\pm\dfrac{1}{\left|\vec{a}\right|}$

(33) Let $\vec{a}$, $\vec{b}$ be the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ determined by two adjacent sides of regular hexagon ABCDEF. The vector represented by $\overrightarrow{EF}$ is

(1) $\vec{a} - \vec{b}$ (2) $\vec{a} + \vec{b}$ (3) $2\vec{a}$ (4) $-\vec{b}$

(34) If $\overrightarrow{AB} = k\,\overrightarrow{AC}$ where k is a scalar then

(1) A, B, C are collinear (2) A, B, C are coplanar

(3) $\overrightarrow{AB}$, $\overrightarrow{AC}$ have the same magnitude (4) A, B, C are coincident

(35) The position vectors of A and B are $\vec{a}$ and $\vec{b}$. P divides AB in the ratio 3 : 1. Q is the mid point of AP. The position vector of Q is

(1) $\dfrac{5\vec{a} + 3\vec{b}}{8}$ (2) $\dfrac{3\vec{a} + 5\vec{b}}{2}$ (3) $\dfrac{5\vec{a} + 3\vec{b}}{4}$ (4) $\dfrac{3\vec{a} + \vec{b}}{4}$

(36) If G is the centriod of a triangle ABC and O is any other point then $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ is equal to

(1) $\vec{O}$ (2) $\overrightarrow{OG}$ (3) $3\,\overrightarrow{OG}$ (4) $4\,\overrightarrow{OG}$

(37) If G is the centriod of a triangle ABC then $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}$ is equal to

(1) $3\left(\vec{a} + \vec{b} + \vec{c}\right)$ (2) $\overrightarrow{OG}$ (3) $\vec{O}$ (4) $\dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

(38) If G is the centriod of a triangle ABC and G′ is the centroid of triangle A′ B′ C′ then $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} =$

(1) $\overrightarrow{GG'}$ (2) $3\overrightarrow{GG'}$ (3) $2\overrightarrow{GG'}$ (4) $4\overrightarrow{GG'}$

(39) If the initial point of vector $-2\vec{i} - 3\vec{j}$ is (– 1, 5, 8) then the terminal point is

(1) $3\vec{i} + 2\vec{j} + 8\vec{k}$ (2) $-3\vec{i} + 2\vec{j} + 8\vec{k}$

(3) $-3\vec{i} - 2\vec{j} - 8\vec{k}$ (4) $3\vec{i} + 2\vec{j} - 8\vec{k}$

(40) Which of the following vectors has the same direction as the vector $\vec{i} - 2\vec{j}$

(1) $-\vec{i} + 2\vec{j}$ (2) $2\vec{i} + 4\vec{j}$ (3) $-3\vec{i} + 6\vec{j}$ 4) $3\vec{i} - 6\vec{j}$

(41) If $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$, $\vec{c} = \vec{i} - 2\vec{j} + 2\vec{k}$, then a unit vector parallel to $\vec{a} + \vec{b} + \vec{c}$ is

(1) $\dfrac{\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{6}}$ (2) $\dfrac{\vec{i} - \vec{j} + \vec{k}}{\sqrt{3}}$ (3) $\dfrac{2\vec{i} + \vec{j} + \vec{k}}{\sqrt{6}}$ (4) $\dfrac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$

(42) If $\vec{a} = 2\vec{i} + \vec{j} - 8\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} - 4\vec{k}$ then the magnitude of $\vec{a} + \vec{b} =$

(1) 13 (2) 13/3 (3) 3/13 (4) 4/13

(43) If the position vectors of P and Q are

$2\vec{i} + 3\vec{j} - 7\vec{k}$, $4\vec{i} - 3\vec{j} + 4\vec{k}$, then the direction cosines of $\overrightarrow{PQ}$ are

(1) $\dfrac{2}{\sqrt{161}}, \dfrac{-6}{\sqrt{161}}, \dfrac{11}{\sqrt{161}}$ (2) $\dfrac{-2}{\sqrt{161}}, \dfrac{-6}{\sqrt{161}}, \dfrac{-11}{\sqrt{161}}$

(3) 2, – 6, 11 (4) 1, 2, 3

(44) If $\dfrac{ax}{(x+2)(2x-3)} = \dfrac{2}{x+2} + \dfrac{3}{2x-3}$ then $a =$

(1) 4 (2) 5 (3) 7 (4) 8

(45) If $nPr = 720\, nCr$, then the value of r is

(1) 6 (2) 5 (3) 4 (4) 7

(46) How many different arrangements can be made out of letters of words ENGINEERING

(1) 11! (2) $\dfrac{11!}{(3!)^2\,(2!)^2}$ (3) $\dfrac{11!}{3!\,.\,2!}$ (4) $\dfrac{11!}{3!}$

(47) The number of 4 digit numbers, that can be formed by the digits 3, 4, 5, 6, 7, 8, 0 and no digit is being repeated, is

(1) 720 (2) 840 (3) 280 (4) 560

(48) The number of diagonals that can be drawn by joining the vertices of an octagon is

(1) 28 (2) 48 (3) 20 (4) 24

(49) A polygon has 44 diagonals then the number of its sides is

(1) 11 (2) 7 (3) 8 (4) 12

(50) 20 persons are invited for a party. The number of ways in which they and the host can be seated at a circular table if two particular persons be seated on either side of the host is equal to

(1) 18! 2! (2) 18! 3! (3) 19! 2! (4) 20! 2!

(51) If n is a positive integer then the number of terms in the expansion of $(x + a)^n$ is

(1) n (2) $n - 1$ (3) $n + 1$ (4) $n + 2$

(52) The values of $nC0 - nC1 + nC2 - nC3 + \dots (-1)^n . nCn$ is

(1) 2^{n+1} (2) n (3) 2^n (4) 0

(53) The sum of the coefficients in the expansion of $(1 - x)^{10}$ is

(1) 0 (2) 1 (3) 10^2 (4) 1024

(54) The largest coefficient in the expansion of $(1 + x)^{24}$ is

(1) 24C24 (2) 24C13 (3) 24C12 (4) 24C11

(55) The total number of terms in the expansion of $\left[(a + b)^2\right]^{18}$ is

(1) 11 (2) 36 (3) 37 (4) 35

(56) Sum of the binomial coefficients is

(1) $2n$ (2) n^2 (3) 2^n (4) $n + 17$

(57) The last term in the expansion of $\left(2 + \sqrt{3}\right)^8$ is

(1) 81 (2) 27 (3) $\sqrt{3}$ (4) 3

(58) If a, b, c are in A.P., then 3^a, 3^b, 3^c are in

(1) A.P. (2) G.P. (3) H.P. (4) A.P. and G.P.

(59) If the n^{th} term of an A.P. is $(2n - 1)$, then the sum of n terms is

(1) $n^2 - 1$ (2) $(2n - 1)$ (3) n^2 (4) $n^2 + 1$

(60) The sum of n terms of an A.P. is n^2. Then its common difference is

(1) 2 (2) -2 (3) ± 2 (4) 1

(61) The sum to the first 25 terms of the series 1 + 2 + 3 … … … is

(1) 305 (2) 325 (3) 315 (4) 335

(62) The n^{th} term of the series 3 + 7 + 13 + 21 + 31 + … … … is

(1) $4n - 1$ (2) $n^2 + 2n$ (3) $(n^2 + n + 1)$ (4) $(n^3 + 2)$

(63) What number must be added to 5, 13 and 29 so that sum may form a G.P?

(1) 2 (2) 3 (3) 4 (4) 5

(64) The third term of a G.P. is 5, the product of its first five terms is

(1) 25 (2) 625 (3) 3125 (4) 625×25

(65) The first term of a G.P. is 1. The sum of third and fifth terms is 90. Find the common ratio of the G.P.

(1) ± 2 (2) $\sqrt{10}$ (3) ± 3 (4) -3

(66) When the terms of a G.P. are written in reverse order the progression formed is

(1) A.P. (2) G.P. (3) H.P. (4) A.P. and H.P.

(67) If A, G, H are respectively arithmetic mean, geometric mean and harmonic mean then

(1) A > G > H (2) A < G > H (3) A < G < H (4) A > G < H

(68) The A.M. between two numbers is 5 and the G.M. is 4. Then H.M. between them is

(1) $3\frac{1}{5}$ (2) 1 (3) 9 (4) $1\frac{1}{4}$

(69) If a, b, c are in A.P. as well as in G.P. then

(1) $a = b \neq c$ (2) $a \neq b = c$ (3) $a \neq b \neq c$ (4) $a = b = c$

(70) The A.M., G.M. and H.M. between two positive numbers a and b are equal then

(1) $a = b$ (2) $ab = 1$ (3) $a > b$ (4) $a < b$

(71) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \dots$ is valid for

(1) $-1 < x < 1$ (2) $-1 \leq x \leq 1$ (3) all real x (4) $x > 0$

(72) $e^{\log x}$ is equal to

(1) x (2) 1 (3) e (4) $\log_e x$

(73) The equation of x-axis is

(1) $x = 0$ (2) $x = 0, y = 0$ (3) $y = 0$ (4) $x = 4$

(74) The slope of the straight line $2x - 3y + 1 = 0$ is

(1) $\frac{-2}{3}$ (2) $\frac{-3}{2}$ (3) $\frac{2}{3}$ (4) $\frac{3}{2}$

(75) The y – intercept of the straight line $3x + 2y - 1 = 0$ is

(1) 2 (2) 3 (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

(76) Which of the following has the greatest y-intercept in magnitude?

(1) $2x + 3y = 4$ (2) $x + 2y = 3$ (3) $3x + 4y = 5$ (4) $4x + 5y = 6$

(77) If the equation of the straight line is $y = \sqrt{3}\ x + 4$, then the angle made by the straight line with the positive direction of x-axis is

(1) 45° (2) 30° (3) 60° (4) 90°

(78) If the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then

(1) $\frac{a_1}{a_2} = -\frac{b_1}{b_2}$ (2) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ (3) $a_1\,a_2 = -\,b_1b_2$ (4) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(79) Which of the following is a parallel line to $3x + 4y + 5 = 0$?

(1) $4x + 3y + 6 = 0$ (2) $3x - 4y + 6 = 0$

(3) $4x - 3y + 9 = 0$ (4) $3x + 4y + 6 = 0$

(80) Which of the following is the equation of a straight line that is neither parallel nor perpendicular to the straight line given by $x + y = 0$

(1) $y = x$ (2) $y - x + 2 = 0$ (3) $2y = 4x + 1$ (4) $y + x + 2 = 0$

(81) The equation of the straight line containing the point $(-2, 1)$ and parallel to $4x - 2y = 3$ is

(1) $y = 2x + 5$ (2) $y = 2x - 1$ (3) $y = x - 2$ (4) $y = \frac{1}{2}x$

(82) Equation of two parallel straight lines differ by

(1) x term (2) y term (3) constant term (4) xy term

(83) If the slope of a straight line is $\frac{2}{3}$, then the slope of the line perpendicular to it, is

(1) $\frac{2}{3}$ (2) $-\frac{2}{3}$ (3) $\frac{3}{2}$ (4) $-\frac{3}{2}$

(84) The graph of $xy = 0$ is

(1) a point (2) a line

(3) a pair of intersecting lines (4) a pair of parallel lines

(85) If the pair of straight lines given by $ax^2 + 2hxy + by^2 = 0$ are perpendicular then

(1) $ab = 0$ (2) $a + b = 0$ (3) $a - b = 0$ (4) $a = 0$

(86) When $h^2 = ab$ the angle between pair of straight lines

$ax^2 + 2hxy + by^2 = 0$ is

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{2}$ (4) $0°$

(87) If $2x^2 + 3yx - cy^2 = 0$ represents a pair of perpendicular lines then $c =$

(1) -2 (2) $-\frac{1}{2}$ (3) 2 (4) $\frac{1}{2}$

(88) If $2x^2 + kxy + 4y^2 = 0$ represents a pair of parallel lines then $k =$

(1) ± 32 (2) $\pm 2\sqrt{2}$ (3) $\pm 4\sqrt{2}$ (4) ± 8

(89) The condition for $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is

(1) $abc + 2fgh - bf^2 - ag^2 - ch^2 = 0$ (2) $abc - 2fgh - ag^2 - bf^2 - ch^2 = 0$

(3) $abc + 2fgh - ah^2 - bg^2 - cf^2 = 0$ (4) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(90) The length of the diameter of a circle with centre (2, 1) and passing through the point (– 2, 1) is

(1) 4 (2) 8 (3) $4\sqrt{5}$ (4) 2

(91) Given that (1, – 1) is the centre of the circle $x^2 + y^2 + ax + by - 9 = 0$. Its radius is

(1) 3 (2) $\sqrt{2}$ (3) $\sqrt{11}$ (4) 11

(92) The equation of a circle with centre (0, 0) and passing through the point (5, 0) is

(1) $x^2 + y^2 - 10x = 0$ (2) $x^2 + y^2 = 25$

(3) $x^2 + y^2 + 10x = 0$ (4) $x^2 + y^2 - 10y = 0$

(93) The radius of the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ is

(1) 1 (2) 2 (3) 3 (4) 4

(94) The centre of the circle $x^2 + y^2 + 2x - 4y - 4 = 0$ is

(1) (2, 4) (2) (1, 2) (3) (– 1, 2) (4) (– 2, – 4)

(95) If $2x + 3y = 0$ and $3x - 2y = 0$ are the equations of two diameters of a circle, then its centre is

(1) $(1, -2)$ (2) $(2, 3)$ (3) $(0, 0)$ (4) $(-3, 2)$

(96) If the line $y = 2x - c$ is a tangent to the circle $x^2 + y^2 = 5$, then the value of c is

(1) ± 5 (2) $\pm\sqrt{5}$ (3) $\pm 5\sqrt{5}$ (4) $\pm 5\sqrt{2}$

(97) The length of the tangent from $(4, 5)$ to the circle $x^2 + y^2 = 25$ is

(1) 5 (2) 4 (3) 25 (4) 16

(98) If the circle has both x and y axes as tangents and has radius 1 unit then the equation of the circle is

(1) $x^2 + (y - 1)^2 = 1$ (2) $x^2 + y^2 = 1$

(3) $(x - 1)^2 + (y - 1)^2 = 1$ (4) $(x - 1)^2 + y^2 = 1$

(99) Which of the following point lies inside the circle $x^2 + y^2 - 4x + 2y - 5 = 0$

(1) $(5, 10)$ (2) $(-5, 7)$ (3) $(9, 0)$ (4) $(1, 1)$

(100) The number of tangents that can be drawn from a point to the circle is

(1) 1 (2) 2 (3) 3 (4) 4

(101) If two circles touch each other externally then the distance between their centres is

(1) $r_1 - r_2$ (2) $\frac{r_1}{r_2}$ (3) $\frac{r_2}{r_1}$ (4) $r_1 + r_2$

(102) The number of points in which two circles touch each other internally is

(1) 1 (2) 2 (3) 0 (4) 3

(103) One radian is equal to (interms of degree)

(1) $\frac{180^\circ}{11}$ (2) $\frac{\pi}{180^\circ}$ (3) $\frac{180}{\pi}$ (4) $\frac{11}{180^\circ}$

(104) An angle between 0° and -90° has its terminal side in

(1) I quadrant (2) III quadrant (3) IV quadrant (4) II quadrant

(105) $\frac{1}{360}$ of a complete rotation clockwise is

(1) $-1°$ (2) $-360°$ (3) $-90°$ (4) $1°$

(106) If the terminal side is collinear with the initial side in the opposite direction then the angle included is

(1) $0°$ (2) $90°$ (3) $180°$ (4) $270°$

(107) Area of triangle ABC is

(1) $\frac{1}{2}\ ab\cos C$ (2) $\frac{1}{2}\ ab\sin C$ (3) $\frac{1}{2}\ ab\cos C$ (4) $\frac{1}{2}\ bc\sin B$

(108) The product $s(s-a)(s-b)(s-c)$ is equal to

(1) Δ (2) Δ^2 (3) 2Δ (4) $\frac{\Delta}{s}$

(109) In any triangle ABC, Δ is

(1) abc (2) $\frac{abc}{4R}$ (3) $\frac{abc}{2R}$ (4) $\frac{abc}{R}$

(110) In triangle ABC, the value of $\sin A \sin B \sin C$ is

(1) $\frac{\Delta}{2R}$ (2) $\frac{\Delta}{4R}$ (3) $\frac{\Delta}{2R^2}$ (4) $\frac{\Delta}{4R^2}$

(111) $\cos B$ is equal to

(1) $\frac{c^2+a^2-b^2}{2ca}$ (2) $\frac{c^2+b^2-a^2}{2bc}$ (3) $\frac{a^2+b^2-c^2}{2ab}$ (4) $\frac{a^2+b^2+c^2}{2ab}$

ANSWERS

EXERCISE 1.1

(1) (i) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(2) $x = 0,\ y = 7,\ z = 3$ (3) $x = \frac{3}{2}$, $y = \frac{5}{2}$, $z = 1$, $w = \frac{1}{2}$

(4) (i) $\begin{bmatrix} -2 & -7 \\ -6 & 10 \end{bmatrix}$ (ii) $\begin{bmatrix} -12 & 4 \\ 2 & -12 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & -4 \\ 6 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 4 & -4 \\ 6 & 4 \end{bmatrix}$

(v) $\begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$ (vii) $\begin{bmatrix} 9 & 0 \\ 14 & -16 \end{bmatrix}$ (viii) $\begin{bmatrix} 0 & 8 \\ 18 & -7 \end{bmatrix}$

(6) $X = \begin{bmatrix} 2 & 2 & -1 \\ -2 & 3 & -3 \\ -2 & 1 & -3 \end{bmatrix}$, $Y = \begin{bmatrix} -2 & -5 & -1 \\ -1 & 1 & 3 \\ 0 & -7 & 2 \end{bmatrix}$ (8) $k = 1$

(10) $x = 1, -3$ (11) $x = 2, -5$ (13) $\frac{1}{5}\begin{bmatrix} 9 & 3 \\ -6 & 7 \end{bmatrix}$ (14) $x = 1$, $y = 4$

EXERCISE 1.2

(1) 0 (2) (i) non-singular (ii) singular (3) (i) $x = \frac{27}{8}$ (ii) $x = 9$

(4) (i) 0 (ii) 0 (6) $a^3 + 3a^2$

EXERCISE 1.3

(3) $x = 0, 0, -(a + b + c)$ (4) $(a - b)(b - c)(c - a)(ab + bc + ca)$

EXERCISE 2.1

(1) $\overrightarrow{AC} = \vec{a} + \vec{b}$, $\overrightarrow{BD} = \vec{b} - \vec{a}$

EXERCISE 2.2

(1) $5\vec{i} + 5\vec{j} + 5\vec{k}$, $5\sqrt{3}$ (2) $\sqrt{185}$

(3) $\overrightarrow{AB} = -3\vec{i} - \vec{j} - 5\vec{k}$; $\overrightarrow{BC} = 4\vec{i} - 7\vec{j} + 7\vec{k}$;

$\overrightarrow{CA} = -\vec{i} + 8\vec{j} - 2\vec{k}$

$AB = \sqrt{35}$, $BC = \sqrt{114}$, $CA = \sqrt{69}$

(5) $m = 9$ (6) $\dfrac{\vec{i} + \sqrt{3}\,\vec{j}}{2}$ (7) $\pm\dfrac{3\vec{i} - 7\vec{j} + 6\vec{k}}{\sqrt{94}}$

(8) $\pm\dfrac{17\vec{i} - 3\vec{j} - 10\vec{k}}{\sqrt{398}}$ (12) $2\left(\vec{i} + \vec{j} + \vec{k}\right)$

(13) $\overrightarrow{PQ} = 4\vec{i} - 5\vec{j} + 11\vec{k}$; $\left(\dfrac{4}{9\sqrt{2}}, \dfrac{-5}{9\sqrt{2}}, \dfrac{11}{9\sqrt{2}}\right)$

(16) non-coplanar vectors.

EXERCISE 3.1

(1) $\dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$ (2) $\dfrac{20}{x-3} - \dfrac{13}{x-2}$

(3) $\dfrac{3}{2(x-1)} - \dfrac{7}{x-2} + \dfrac{13}{2(x-3)}$

(4) $\dfrac{1}{9(x+1)} - \dfrac{1}{9(x+2)} - \dfrac{1}{3(x+2)^2}$

(5) $\dfrac{-4}{9(x+2)} + \dfrac{4}{9(x-1)} - \dfrac{1}{3(x-1)^2}$

(6) $\dfrac{2}{25(x-2)} + \dfrac{3}{5(x-2)^2} - \dfrac{2}{25(x+3)}$

(7) $\dfrac{-7}{2x} + \dfrac{1}{x^2} + \dfrac{9}{2(x+2)}$

(8) $\dfrac{2}{(x-2)} + \dfrac{3}{(x-2)^2} - \dfrac{9}{(x-2)^3}$

(9) $\dfrac{1}{5(x+2)} + \dfrac{4x-8}{5(x^2+1)}$

(10) $\frac{1}{2(x+1)} - \frac{(x-3)}{2(x^2+1)}$

(11) $\frac{x-5}{x^2-2x-1} + \frac{4}{3x-2}$

(12) $1 - \frac{1}{x+1} + \frac{1}{(x+1)^2}$

EXERCISE 3.2

(1) 378 (2) 42 (3) 600 (4) 1320 (5) 42840

(6) 512 (7) 153 (8) (i) 27216 (ii) 90000 (9) $5 \times 5!$ (10) 21

(11) 2^5 (12) 9000 (13) (i) 125 (ii) 60 (14) 2^5

EXERCISE 3.3

(1) (i) 60 (ii) 2730 (iii) 120 (iv) $\frac{25!}{5!}$ (v) 15120

(2) 23 (3) 4

(4) 41 (7) 172800 (8) 5040 (9) 60 (10) 93324 (11) 34650

(12) (i) 840 (ii) 20 (13) 9000 (14) 4^5 (15) (i) 8! (ii) 7! (16) $\frac{19!}{2}$

EXERCISE 3.4

(1) (i) 45 (ii) 4950 (iii) 1 (2) 23 (3) 3 (4) 45

(5) (i) 12 (ii) 8 (6) 19 (7) 7

EXERCISE 3.5

(1) 66 (2) 200 (3) 210 (4) 425 (5) (i) ${}_{15}C_{11}$ (ii) ${}_{14}C_{10}$ (iii) ${}_{14}C_{11}$

(6) 780 (7) (i) 40 (ii) 116 (8) 1540 (9) 817190

EXERCISE 3.7

(1) (i) $243a^5 + 2025a^4b + 6750a^3b^2 + 11250a^2b^3 + 9375ab^4 + 3125\,b^5$

(ii) $a^5 - 10a^4b + 40a^3b^2 - 80a^2b^3 + 80ab^4 - 32b^5$

(iii) $32x^5 - 240x^6 + 720x^7 - 1080x^8 + 810x^9 - 243x^{10}$

(iv) $x^{11} + \frac{11x^{10}}{y} + \frac{55x^9}{y^2} + \frac{165x^8}{y^3} + \frac{330x^7}{y^4} + \frac{462x^6}{y^5} + \frac{462x^5}{y^6} + \frac{330x^4}{y^7}$

$+ \frac{165x^3}{y^8} + \frac{55x^2}{y^9} + \frac{11x}{y^{10}} + \frac{1}{y^{11}}$

(v) $x^{12} + 12x^{10}y^3 + 60x^8y^6 + 160x^6y^9 + 240x^4y^{12} + 192x^2y^{15} + 64y^{18}$

(vi) $x^4y^2 + 4x^{7/2}\, y^{3/2} + 6x^3y^3 + 4x^{5/2}y^{7/2} + x^2\ \ y^4$

(2) (i) $58\sqrt{2}$ (ii) 152 (iii) 352

(iv) $128a^3 + 4320a^2 + 9720a + 1458$ (v) $5822\sqrt{3}$

(3) (i) 1030301 (ii) 970299 (4) 0.9940

(5) (i) $8C_4\, 2^4x^{12}$ (ii) $16C_8$ (iii) $\frac{16C_8\, .a^8}{x^4}$

(iv) $13C_6\, .2^6x^7y^6$ and $-\, 13C_7\, .\, 2^7x^6y^7$

(v) $17C_8.2^8\frac{1}{x^7}$ and $17C_9.2^9\frac{1}{x^{10}}$

(7) $-\,165$ (8) (i) 7920 (ii) 2268 (iii) ${}_{12}C_4 \left(\frac{b}{c}\right)^4 9^8$

(9) $r = 3$ (10) 7, 14

EXERCISE 4.1

(1) (i) 25, – 125, 625, – 3125 and 15625 (ii) $\frac{3}{2}$, $\frac{9}{2}$, $\frac{21}{2}$, 21, $\frac{75}{2}$

(iii) – 1, – 12, – 23, – 34, – 45 (iv) $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$

(v) $\frac{2}{3}$, 0, $\frac{2}{3}$, 0, $\frac{2}{3}$ (vi) $\frac{1}{3}$, $\frac{4}{9}$, $\frac{1}{3}$, $\frac{16}{81}$, $\frac{25}{243}$

(2) (i) $\frac{11}{5}$, $\frac{15}{7}$ (ii) 1, 0 (iii) $\frac{64}{7}$, $\frac{121}{10}$ (iv) 64, – 512

(3) 0, $\frac{5}{2}$, 8, $\frac{17}{2}$, 24, $\frac{37}{2}$

(4) (i) 2, 2, 1, 0, –1 (ii) 1, 2, 3, 5, 8

(iii) 1, 2, 6, 24, 120 (iv) 1, 1, 5, 13, 41

(5) $\frac{1}{2}\left[1-\frac{1}{3^n}\right]$ (6) $\frac{5}{4}\ [5^n - 1]$ (7) $\frac{1}{2^{200}}\ (2^{100} - 1)$

EXERCISE 4.2

(1) (i) 4, 7, 10, 13, 16 (ii) 5, 7, 9, 11, 13, 15 (2) (i) 10 (ii) 1 (iii) p

(4) $\frac{1}{19}$ (6) 6, 24 (10) 2, 3, 6 (or) 6, 3, 2

EXERCISE 4.3

(1) (i) $\frac{1}{x^5}\left[x - 8 + \frac{40}{x} - \frac{160}{x^2} + \dots\right]$ (ii) $\frac{1}{\sqrt[3]{6}}\left[1 + \frac{x}{6} + \frac{x^2}{18} + \frac{7x^3}{324} + \dots\right]$

(2) (i) 10.01 (ii) 0.2 (5) $\frac{11.9.7.5}{4!}\ x^{12}$

(6) $\frac{(r+1)\ (r+2)\ (r+3)}{1.2.3}\,x^r$

EXERCISE 5.1

(1) $x^2 + y^2 - 2x + 8y - 19 = 0$ (2) $3x + y = 2$

(3) (i) $t = 1$ (ii) P(1, 2) (4) $y^2 - 24x^2 = 0$

(7) (i) $x^2 + y^2 + x - 3y + 2 = 0$ (ii) $15x^2 + 15y^2 + 66x - 96y + 207 = 0$

EXERCISE 5.2

(1) $4x - 7y - 10 = 0$ (2) $y = 3x + 4$ (3) $x - y = 6$

(4) $11x - y = 27$ (5) $2x + y = 6$; $x + 2y = 6$ (6) $x + 3y = 8$

(7) $3x - 2y = 0$; $2x - y = 0$ and $5x - 3y = 0$ (8) $\frac{14}{\sqrt{13}}$ units

(9) $2x - 3y + 12 = 0$ (10) $9x - 8y + 10 = 0$; $2x - y = 0$

(11) $2x - 3y = 6$; $3x - 2y = 6$ (12) x intercept $\frac{6}{7}$; y intercept 2

(13) (8, 0) and (– 2, 0) (14) $3\sqrt{2}$ units

EXERCISE 5.3

(1) $\frac{\pi}{4}$ (3) $3x + 2y + 1 = 0$ (4) $x - y - 1 = 0$ (5) (1, 2)

(6) $k = -9$ (7) $\frac{5\sqrt{5}}{2}$ units (8) $p = 1$; $p = 2$ (9) $28x + 7y - 74 = 0$

(10) $5x + 3y + 8 = 0$ (11) $x + y = 1$ (12) $5x + 3y + 5 = 0$ (14) $\frac{\pi}{4}$; $\frac{\pi}{4}$

(17) $a = 5$ (18) $a = \frac{16}{9}$ (19) $\left(2 , \frac{5}{3}\right)$ (21) (1, 12) (22) $(-4, -3)$

EXERCISE 5.4

(1) $a = 2$; $c = -3$ (2) $\pi/3$ (4) 1 (6) $2x^2 - 3xy - 2y^2 = 0$

(7) $3x^2 + 7xy + 2y^2 - 4x + 7y - 15 = 0$

(8) $k = -1$; $4x - 3y + 1=0$ and $3x + 4y - 1=0$; $\pi/2$

(9) C = 2 ; $6x - 2y + 1 = 0$ and $2x - y + 2 = 0$; $\tan^{-1}(1/7)$

(10) $k = -10$; $3x - 2y + 1 = 0$ and $4x + 5y + 3 = 0$

EXERCISE 5.5

(1) (i) (0, 0) ; 1 (ii) (2, 3) ; $\sqrt{22}$ (iii) (4, 3) ; 7

(iv) $\left(-\frac{2}{3}, \frac{2}{3}\right)$; $\frac{2\sqrt{5}}{3}$ (v) (4, 4) ; $\sqrt{10}$

(2) $a = 4$; $b = 2$; $2x^2 + 2y^2 + 4x + 4y - 1 = 0$

(3) $x^2 + y^2 - 4x - 6y + 11 = 0$

(4) $x^2 + y^2 - 6x - 4y - 12 = 0$

(5) $x^2 + y^2 - 14x + 6y + 42 = 0$

(6) $x^2 + y^2 + 8x - 10y + 25 = 0$

(7) $2\sqrt{10}\ \pi$ unit ; 10π square units

(8) $x^2 + y^2 - 12x + 11 = 0$; $x^2 + y^2 + 4x - 21 = 0$

(9) $x^2 + y^2 - 3x - 6y + 10 = 0$

(10) $x^2 + y^2 = 1$

(11) $x^2 + y^2 - 5x - y + 4 = 0$

(12) $x^2 + y^2 - 6x - 8y + 15 = 0$

(13) $x^2 + y^2 - 4x - 2y - 5 = 0$

(14) $16x^2 + 16y^2 = 1$

(15) $x = \frac{3}{2}\cos\theta$; $y = \frac{3}{2}\sin\theta$

EXERCISE 5.6

(1) $2\sqrt{5}$ units (3) $y - 1 = 0$ (4) outside

(5) (0, 0) and (4, – 3) lies inside; (– 2, 1) lies outside

(6) (0, 2) ; (2, 0) (7) $2x + y = \pm 3\sqrt{5}$ (8) $5\sqrt{2}$ units

(9) (i) $x^2 + y^2 - 10x - 12y + 25 = 0$ (ii) $x^2 + y^2 - 10x - 12y + 36 = 0$

(10) $4x + 3y + 6 = 0$ (11) (i) $x + y = \pm 4\sqrt{2}$ (ii) $x - y = \pm 4\sqrt{2}$

(12) $x - 5y + 19 = 0$ (13) ± 40 (14) $\left(-\frac{1}{4}, -\frac{5}{4}\right)$

EXERCISE 5.7

(3) $x^2 + y^2 - 2x - 6y - 39 = 0$ (4) $x^2 + y^2 - 8x + 12y - 49 = 0$

(6) (i) $x^2 + y^2 - 2x + 2y + 1 = 0$ (ii) $x^2 + y^2 - 6x - 4y - 44 = 0$

(7) $x^2 + y^2 - 16x - 18y - 4 = 0$ (8) $3x^2 + 3y^2 - 14x + 23y - 15 = 0$

EXERCISE 6.1

(1) (i) $\frac{\pi}{6}$ (ii) $\frac{5\pi}{9}$ (iii) $\frac{10\pi}{9}$ (iv) $\frac{-16\pi}{9}$ (v) $\frac{-17\pi}{36}$ (vi) $\frac{\pi}{24}$

(2) (i) $22° 30'$ (ii) $648°$ (iii) $-171°48'$ (app.) (iv) $105°$

(3) (i) Q_1 (ii) Q_3 (iii) Q_1

EXERCISE 6.2

(1) $\frac{-1331}{276}$ (2) (i) $-\sin 60°$ (ii) $-\cos 40°$ (iii) $\tan 10°$ (iv) $-\tan 60°$

(v) $\operatorname{cosec} 60°$ (vi) $-\sin 30°$ (vii) $\cos 30°$

(5) (i) – cosec A (ii) – sec A (iii) – cotA (iv) cosA (v) tan A

(8) (i) $-\frac{1}{\sqrt{2}}$ (ii) $-\frac{\sqrt{3}}{2}$ (iii) $-\sqrt{2}$ (iv) $-\frac{\sqrt{3}}{2}$

(v) – 1 (vi) $\frac{2}{\sqrt{3}}$ (vii) 1 (viii) $\frac{1}{\sqrt{3}}$

(10) (i) $\frac{13}{3}$ (ii) 1 (iii) 0 (iv) $\frac{\sqrt{2}-\sqrt{6}}{4}$ (v) 5 (vi) 2 (vii) $\frac{7}{3}$

(viii) 11 (ix) $\frac{25}{12}$ (x) $\frac{149}{120}$

EXERCISE 6.4

(1) (i) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (ii) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (iii) $2+\sqrt{3}$ (iv) $\frac{\sqrt{6}+\sqrt{2}}{4}$

(8) (i) $\frac{\sqrt{2}+1}{2}$, $\frac{\sqrt{6}+\sqrt{2}}{4}$ (ii) $\frac{\sqrt{2}-\sqrt{3}}{2}$, $\frac{\sqrt{6}+\sqrt{2}}{4}$

(12) $\sqrt{2}+1$ (14) $\frac{\sqrt{15}+2\sqrt{2}}{12}$

EXERCISE 6.5

(3) $\frac{1}{2}$ (6) (i) $\frac{9}{13}$ (ii) $\frac{117}{125}$

EXERCISE 6.6

(1) (i) $\sin 6\theta + \sin 2\theta$ (ii) $\cos 14\theta + \cos 2\theta$ (iii) $\sin 10\theta - \sin 4\theta$

(iv) cos2A – cos4 A (v) sin9A – sin3A (vi) $\frac{1}{2}$ $[\sin 13\theta + \sin 5\theta]$

(vii) $\frac{1}{2}$ [sin2A – sin A] (viii) $\frac{1}{2}$ [sin6A + sin A] (ix) $\frac{1}{2}$ $[\cos 3\theta + \cos \theta/3]$

(2) (i) 2sin9A cos4A (ii) 2 cos9A sin4A (iii) 2cos9A cos4A

(iv) – 2sin 9A sin4A (v) 2 cos42° sin10° (vi) 2 cos37° cos14°

(vii) 2 cos50° sin30° (viii) 2 sin30° cos20° (ix) 2 sin30° cos10°

(x) $2\cos\frac{53°}{2}\cos\frac{17°}{2}$

EXERCISE 6.7

(1) (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{\pi}{3}$ (iv) $\frac{\pi}{6}$ (v) $-\frac{\pi}{3}$ (vi) $-\frac{\pi}{6}$ (vii) $\frac{\pi}{3}$

(2) (i) $\frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$ (ii) $n\pi - \frac{\pi}{3}$ (iii) $\frac{2n\pi}{3} \pm \frac{\pi}{4}$

(3) (i) $n\pi \pm \frac{\pi}{4}$, $n\pi$ (ii) $\frac{n\pi}{3}$, $(2n + 1)\frac{\pi}{2}$ (iii) $n\pi$

(4) (i) $2n\pi \pm \frac{\pi}{3}$ (ii) $2n\pi \pm \pi$ (iii) $2n\pi \pm \frac{2\pi}{3}$, $(2n + 1)\frac{\pi}{4}$ (iv) $\frac{n\pi}{3}$, $2n\pi$

(5) (i) $2n\pi + \frac{\pi}{4}$ (or) $n\pi + (-1)^n \frac{\pi}{2} - \frac{\pi}{4}$

(ii) $2n\pi - \frac{\pi}{4}$ (iii) $2n\pi - \frac{\pi}{4}$ (iv) $2n\pi$, $2n\pi + \frac{2\pi}{3}$

EXERCISE 6.9

(1) (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{3}$ (iii) $-\frac{\pi}{2}$ (iv) $\frac{3\pi}{4}$ (v) $\frac{\pi}{3}$ (vi) $\frac{3\pi}{4}$

(3) (i) $\frac{12}{13}$ (ii) $\frac{4}{5}$ (iii) $\frac{15}{8}$ (iv) $\frac{\sqrt{3}}{2}$ (7) $x = \pm\frac{1}{\sqrt{2}}$

(8) $2 - \sqrt{3}$ (9) $x = \frac{1}{2}$

Objective Type Questions – Answers (Key)

(1) 1	(2) 4	(3) 2	(4) 1	(5) 2	(6) 4
(7) 1	(8) 3	(9) 1	(10) 1	(11) 1	(12) 2
(13) 1	(14) 4	(15) 1	(16) 4	(17) 2	(18) 1
(19) 3	(20) 3	(21) 2	(22) 1	(23) 4	(24) 3
(25) 1	(26) 3	(27) 2	(28) 2	(29) 2	(30) 3
(31) 1	(32) 4	(33) 4	(34) 1	(35) 1	(36) 3
(37) 3	(38) 2	(39) 2	(40) 4	(41) 4	(42) 1
(43) 1	(44) 3	(45) 1	(46) 2	(47) 1	(48) 3
(49) 1	(50) 1	(51) 3	(52) 4	(53) 1	(54) 3
(55) 3	(56) 3	(57) 1	(58) 2	(59) 3	(60) 1
(61) 2	(62) 3	(63) 2	(64) 3	(65) 3	(66) 2
(67) 1	(68) 1	(69) 4	(70) 1	(71) 3	(72) 1
(73) 3	(74) 3	(75) 3	(76) 2	(77) 3	(78) 3
(79) 4	(80) 3	(81) 1	(82) 3	(83) 4	(84) 3
(85) 2	(86) 4	(87) 3	(88) 3	(89) 4	(90) 2
(91) 3	(92) 1	(93) 3	(94) 3	(95) 3	(96) 1
(97) 2	(98) 3	(99) 4	(100) 2	(101) 4	(102) 1
(103) 3	(104) 4	(105) 1	(106) 3	(107) 2	(108) 2
(109) 2	(110) 3	(111) 1			